B501, Fall 2023

Assignment 1: Sets and relations

(Due by EOD W Aug 31)

This assignment contains solved practice problems, numbered in red. The assigned problems and sub-problems are numbered in green.

1. (15%) The operation $\bar{\cap}$ of ***co-intersection*** is defined for subsets of U by $A \bar{\cap} B = U - (A \cap B)$.

- **i.** Define the complement in terms of co-intersection.
  Solution. $\overline{A} = A \bar{\cap} A$
- **(a)** Define intersection in terms of co-intersection. (Together with the previous part this shows that all basic set operations can be defined in terms of just one operation: co-intersection!)

2. (15%) A set $A \subseteq \mathbb{N}$ is ***co-finite*** if its complement $\mathbb{N} - A$ is finite.

- **i.** Give two examples of a co-finite subset of $\mathbb{N}$.
  Solution. $\mathbb{N}$, $\{x \in \mathbb{N} \mid x > 7\}$.
- **(a)** Give two examples of subsets of $\mathbb{N}$ that are neither finite nor co-finite.
- **(b)** Exhibit an infinite sequence $A_1 \supset A_2 \supset A_3 \cdots$ of subests of $\mathbb{N}$, each one a proper subset of its predecessor in the list.

3. (20%) Prove the following statements. You may use the Separation Principle in your proofs.

- **(a)** There is no "set of all sets."
- **(b)** There is no such thing as the "set of all singletons".

4. (20%) For each of the following sets S and collections $C \subseteq \mathcal{P}(S)$ determine whether C is a partition of S.

i. $S = \mathbb{N}$; C consists of three sets: the prime numbers, the composite numbers, the singleton $\{1\}$.

Solution. This is a partition, as every natural number is in exactly one of the three sets.

(a) $S =$ the adult population of Indiana. $\mathcal{C}$ consists of the three sets: speakers of English, speakers of Spanish, people who speak neither English nor Spanish.

(b) $S = \mathbb{R}$. $\mathcal{C}$ consists of the three sets: the rational numbers, the irrational numbers, and the integers. (Search on line for definitions, if you need them.)

5. (30%) The following problem shows that partitions and equivalence relations are two sides of the same coin.

Let S be a set.

(a) Suppose that $\mathcal{C}$ is a partition of S . Define the relation $E \subseteq S \times S$ to hold between $x, y, \in S$ iff they are both in the same $A \in \mathcal{C}$. Show that E is an equivalence relation.

(b) Suppose that E is an equivalence relation on S . Show that the collection of equivalence classes of E is a partition of S .