Assignment 3: Languages, Clipping

Solved practice problems are numbered in red,
assigned problems and sub-problems in green.

1. (20%) Refer to the definition of regular languages given in class ("generated from finite languages by set operations, concatenation, and star"). Show that if L is a regular language then so are the following:

(a) $even(L) = \{w \in L \mid |w| \text{ is even }\}$

Solution. $even(L) = L \cap (\Sigma \cdot \Sigma)^*$. L and Σ are regular and, by definition, the concatenation, star and intersection of regular languages are also regular. So $even(L)$ is regular.

(b) $\tilde{L} = \{x_1 \cdot y_1 \cdot \cdots \cdot x_n \cdot y_n \mid n \geqslant 0, x_i \in L,\ y_i \notin L\}$

Solution. $\tilde{L} = (L \cdot (\Sigma^* - L))^*$. L and Σ are regular and, by definition, the star, difference and concatenation of regular languages are also regular. So $\tilde{L}$ is regular.

i. $\{u\#v \mid u \in L,\ v \notin L\}$,
where $\#$ is a fresh symbol (not in the alphabet of L).

Solution. Since L is regular, its complement $\bar{L}$ is regular. The language $\{\#\}$ is regular since it is finite. So the given language, $L \cdot \{\#\} \cdot \bar{L}$, is regular as the concatenation of regular languages.

2. (20%) Let $L = \mathcal{L}(\alpha)$ where α is a regular expression for the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.

For each of the following languages M explain how to convert α into a regular expression β that denotes M. No proof is necessary.

i. $M = \{f(w) \mid w \in L\}$, where $f(w)$ is w with every a doubled,
e.g. $f(\mathsf{baaca}) = \mathsf{baaaacaa}$.

Solution. Take β to be α with each a replaced by $(\mathsf{a} \bullet \mathsf{a})$

(a) $M = L \cdot L$

Solution. $\beta = \alpha \bullet \alpha$.

(b) $M = L^R = \{w^R \mid w \in L\}$, where w^R is the reverse of w.

Solution. Define β as the mirror-image of α:

- For α one of $\varepsilon, \sigma, \emptyset$ let $\beta = \alpha$.

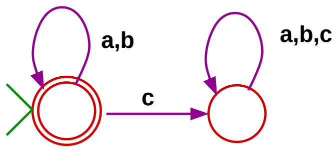
If β_0, β_1 are the mirror images of α_0 and α_1, then

- $\beta_1 \bullet \beta_0$ is the mirror-image of $\alpha_0 \bullet \alpha_1$
- $\beta_1 \cup \beta_0$ is the mirror-image of $\alpha_0 \cup \alpha_1$, and
- $\beta_0^\star$ is the mirror-image of $\alpha_0^\star$.

3. (30%) For each of the following languages build an automaton that recognizes it.

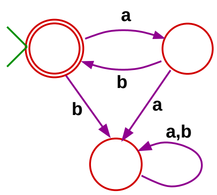
i. $\{a, b\}^*$, where $\Sigma = \{a, b, c\}$

Solution.



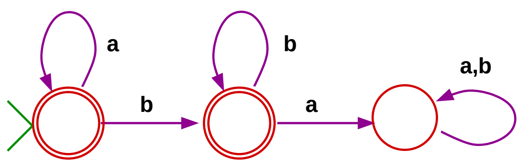
(a) $\{ab\}^*$, where $\Sigma = \{a, b\}$

Solution.



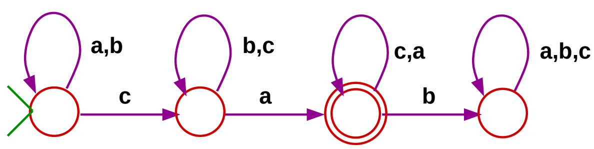
(b) $\{a\}^* \cdot \{b\}^*$, where $\Sigma = \{a, b\}$

Solution.



(c) $\{x \cdot c \cdot y \cdot a \cdot z \mid x \in \{a, b\}^*,\ y \in \{b, c\}^*,\ z \in \{c, a\}^*\}$, where $\Sigma = \{a, b, c\}$.

Solution.



4. (30%) Prove that the following languages are not recognized by any automaton.

i. $\{w \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}^* \mid \#_a(w) + \#_b(w) = \#_c(w)\}$.

Solution. We show that L fails the Clipping Property.
Let $w = \mathtt{a}^k\mathtt{c}^k$ (no $\mathtt{b}$'s), and u the substring $\mathtt{a}^k$ of w. We have $w \in L$ and $|u| \geqslant k$.

If $y = \mathtt{a}^p$ is a non-empty substring of u then the string w' obtained from w by clipping y is $\mathtt{a}^{k-p}\mathtt{c}^k$, which is not in L. So L fails the Clipping Property, and is not recognized by any automaton.

(a) $\{\mathtt{a}^p\mathtt{b}^q \mid p < q\}$.

Solution. We show that the language fails the Clipping Property.

Let $k > 0$. Consider $w = \mathtt{a}^k\mathtt{b}^{k+1}$ and $u = \mathtt{b}^{k+1}$ the suffix of w. We have $w \in L$ and $|u| \geqslant k$.

For any non-empty substring $y = \mathtt{a}^\ell$ of u clipping the reduct w' obtained from w by removing y is of the form $\mathtt{a}^k\mathtt{b}^\ell$ with $\ell \leqslant k$, which is not in L.

So L fails the clipping property, and cannot be recognized by any automaton.

(b) $\{x \cdot x^R \mid x \in \{\mathtt{a},\mathtt{b}\}^*\}$ (x^R is the reverse of x.)

Solution. We show that L fails the Clipping Property.
Let $k > 0$. Take $w = \mathtt{a}^k\mathtt{bba}^k$ and u the initial substring $\mathtt{a}^k$ of w. We have $w \in L$ and $|u| \geqslant k$.

If $y = \mathtt{a}^p$ is any non-empty substring of u,
the string w' obtained from w by clipping y is of the form $\mathtt{a}^{k-p}\mathtt{bba}^k$, Such a string cannot be a palindrome, because its first half has two $\mathtt{b}$'s and its second half has none. So L fails the Clipping Property, and is not recognized by any automaton.

(c) $\{\mathtt{a}^{2^n} \mid n \geqslant 0\} = \{\mathtt{a},\ \mathtt{aa},\ \mathtt{a}^4,\ \mathtt{a}^8, \ldots\}$

Solution. L fails the clipping property: Given $k > 0$ let $w = \mathtt{a}^{2^{k+1}}$ and $u = \mathtt{a}^k$. We have $w \in L$ and $|u| \geqslant k$. If y is a non-empty substring of u of length ℓ then $0 < \ell \leqslant k$. The reduct w' of w over y is of the form $\mathtt{a}^{2^{k+1}-\ell}$, which is not in L because $2^k < 2^{k+1} - \ell < 2^{k+1}$.