B501 Fall 2023

Assignment 3: Languages, Clipping

Solved practice problems are numbered in red,
assigned problems and sub-problems in green.

1. (20%) Show that if L is a regular language then so are the following:

- **i.** $\mathit{even}(L) = \{w \in L \mid |w| \text{ is even }\}$

  Solution. $\mathit{even}(L) = L \cap (\Sigma \cdot \Sigma)^*$. L and Σ are regular and, by definition, the concatenation, star and intersection of regular languages are also regular. So $\mathit{even}(L)$ is regular.

- (a) $\tilde{L} = \{x_1 \cdot y_1 \cdots \cdots x_n \cdot y_n \mid n \geqslant 0, x_i \in L,\ y_i \notin L\}$

- (b) $\{u\#v \mid u \in L,\ v \notin L\}$,
  where $\#$ is a fresh symbol (not in the alphabet of L).

2. (20%) Let $L = \mathcal{L}(\alpha)$ where α is a regular expression for the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.

For each of the following languages M explain how to convert α into a regular expression β that denotes M. No proof is necessary.

- **i.** $M = \{f(w) \mid w \in L\}$, where $f(w)$ is w with every a doubled, e.g. $f(\mathsf{baaca}) = \mathsf{baaaacaa}$.

  Solution. Take β to be α with each a replaced by $(\mathsf{a} \bullet \mathsf{a})$

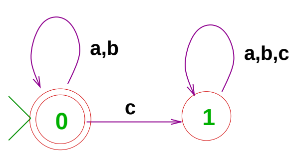
- (a) $M = L \cdot L$

- (b) $M = L^R = \{w^R \mid w \in L\}$, where w^R is the reverse of w .

3. (30%) For each of the following languages build an automaton that recognizes it.

- **i.** $\{\mathsf{a}, \mathsf{b}\}^*$, where $\Sigma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$

  Solution.



- (a) $\{\mathsf{ab}\}^*$, where $\Sigma = \{\mathsf{a}, \mathsf{b}\}$

- (b) $\{\mathsf{a}\}^* \cdot \{\mathsf{b}\}^*$, where $\Sigma = \{\mathsf{a}, \mathsf{b}\}$

- (c) $\{x \cdot \mathsf{c} \cdot y \cdot \mathsf{a} \cdot z \mid x \in \{\mathsf{a}, \mathsf{b}\}^*,\ y \in \{\mathsf{b}, \mathsf{c}\}^*,\ z \in \{\mathsf{c}, \mathsf{a}\}^*\}$,
  where the alphabet is $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.

4. (30%) Prove that the following languages are not recognized by any automaton.

i. $\{w \in \{\mathsf{a},\mathsf{b},\mathsf{c}\}^* \mid \#_a(w) + \#_b(w) = \#_c(w)\}$.

Solution. We show that L fails the Clipping Property.
Let $w = \mathsf{a}^k\mathsf{c}^k$ (no b's), and u the substring a^k of w. We have $w \in L$ and $|u| \geqslant k$.
If $y = \mathsf{a}^p$ is a non-empty substring of u then the string w' obtained from w by clipping y is $\mathsf{a}^{k-p}\mathsf{c}^k$, which is not in L. So L fails the Clipping Property, and is not recognized by any automaton.

Remark. The argument above uses directly the failure of the Clipping Theorem. It is common to give such arguments referring to an assumed automaton. In that alternative style, also acceptable from you, the proof above would be rendered as follows.

Towards contradiction suppose that L *is recognized by an automaton* M, *and let* k *be its number of states. Let* $w = \mathsf{a}^k\mathsf{c}^k = \mathsf{a}^k\mathsf{b}^0\mathsf{c}^{k+0}$, *which is* L. *Take* $u = \mathsf{a}^k$, *the initial substring of* w.
By the Clipping Theorem u *has a non-empty substring* $y = \mathsf{a}^\ell$ ($\ell > 0$) *so that the string* w' *obtained from* w *by removing* y *is accepted by* M. *But* w' *is* $\mathsf{a}^{k-\ell}\mathsf{c}^k$, *which is not in* L *since* $\ell > 0$, *and therefore not accepted by* M.
Since this is a contradiction, the assumption that an automaton M *recognizing* L *exists is false.*

(a) $\{\mathsf{a}^p\mathsf{b}^q \mid p < q\}$.

(b) $\{x \cdot x^R \mid x \in \{\mathsf{a},\mathsf{b}\}^*\}$ (x^R is the reverse of x.)

(c) $\{\mathsf{a}^{2^n} \mid n \geqslant 0\} = \{\mathsf{a},\ \mathsf{aa},\ \mathsf{a}^4,\ \mathsf{a}^8, \ldots\}$