Assignment 4: Residues, NFAs

Solutions

Solved practice problems are numbered in red,
assigned problems and sub-problems in green.

1. (6%) What needs to be changed in the definition of product automata to obtain a correct definition of the product of ***partial***-automata?

Solution. Suppose the given partial-automata are $M_0 = (\Sigma, Q_0, s_0, A_0, \delta_0)$ and $M_1 = (\Sigma, Q_1, s_1, A_1, \delta_1)$.

The ***conjunctive product*** is then $(\Sigma,\ Q_0 \times Q_1,\ \langle s_0,\ s_1\rangle,\ A, \delta)$ where

- $\delta(\langle q_0, q_1\rangle,\ \sigma) = \langle \delta_0(q_0, \sigma),\ \delta_1(q_1, \sigma)\rangle$ if $\delta_i(q_i \sigma)$ is defined, and is undefined otherwise.
- $A = A_0 \times A_1$

For ***disjunctive*** product we need to refer explicitly to undefinability, say by using $\square$ as a token for "undefined". Write $Q_0^\square$ for $Q_0 \cup \{\square\}$, and similarly for other sets of states. Also let $\delta_i^\square(q, \sigma) = \delta_i(q, \sigma)$ if defined, and $= \square$ otherwise.

The ***disjunctive product*** is then $(\Sigma,\ Q_0^\square \times Q_1^\square,\ \langle s_0,\ s_1\rangle,\ A, \delta)$ where

- $\delta(\langle q_0, q_1\rangle,\ \sigma)$ is undefined if if both $\delta_0^\square(q_0, \sigma)$ and $\delta_1^\square(q_1, \sigma)$ are not $\square$,
  and is $\langle \delta_0^\square(q_0, \sigma),\ \delta_1^\square(q_1, \sigma)\rangle$ otherwise.
- $A = (A_0 \times Q_1) \ \cup \ (Q_0 \times A_1)$.

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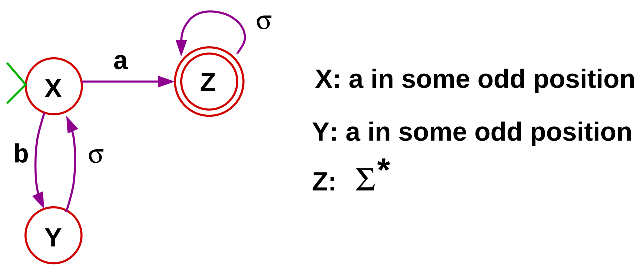
2. (7+7+10%)

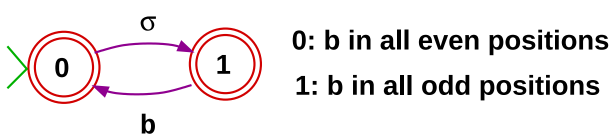
(a) Let $L_{a1} \subseteq \{\mathtt{a}, \mathtt{b}\}^*$ consist of the strings with $\mathtt{a}$ in ***some*** odd position. Identify the residues of L and build a DFA M_{a1} from them.

Solution.



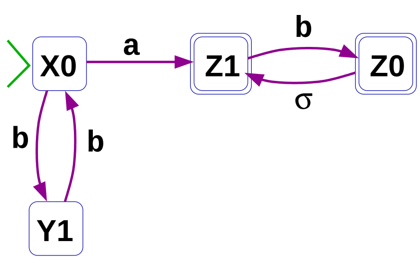
(b) Let $L_{b0} \subseteq \{\mathtt{a}, \mathtt{b}\}^*$ consist of the strings with $\mathtt{b}$ in ***all*** even positions. Identify the residues of L and build a DFA M_{b0} from them.

Solution.



(c) Using your answer to the problem above, construct the conjunctive product of M_{a1} and M_{b0} to obtain a DFA that recognizes $L_{a1} \cap L_{b0}$.

Solution.



3. (20%) A CFA (conjunctive NFA) C (over alphabet Σ) is like an NFA, except that a string w is accepted if ***every*** state p such that $s \xrightarrow{w} p$ is accepting.

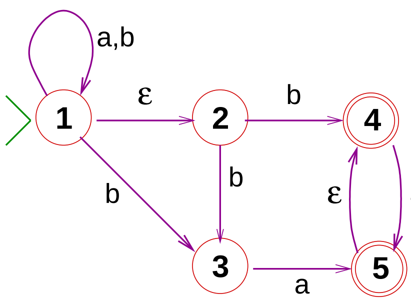
(a) Prove that a language is recognized by a CFA iff its complement is recognized by an NFA.

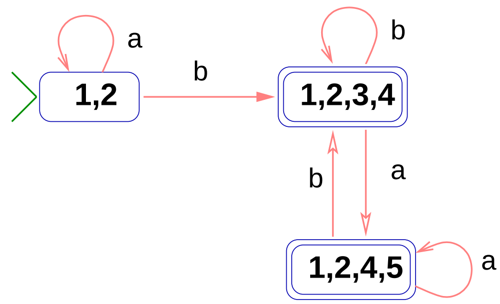
Solution. Given an NFA $N = (\Sigma, Q, s, A, \Delta)$ define its *dual* to be the CFA $\widehat{N} = (\Sigma, Q, s, Q - A, \Delta)$. N accepts a string w iff there is some $a \in A$ such $s \xrightarrow{w} a$, which happens iff it is not the case that $s \xrightarrow{w} b$ for all $b \in Q - A$, i.e. exactly when $\widehat{N}$, as a CNFA, does not accept w. Thus $\mathcal{L}(N) = \Sigma^* - \mathcal{L}(\widehat{N})$.

(b) Conclude that every language recognized by a CFA is recognized by a DFA.

Solution. $L \subseteq \Sigma^*$ is regular iff $\bar{L} \equiv \Sigma^* - L$ is regular, i.e. iff $\bar{L}$ is recognized by some NFA N. By (a) that is equivalent to L being recognized by the CNFA $\widehat{N}$.

A. Convert the following NFA into an equivalent DFA.

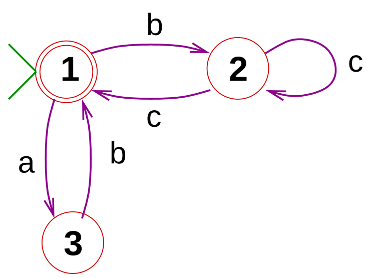




4. (15%)

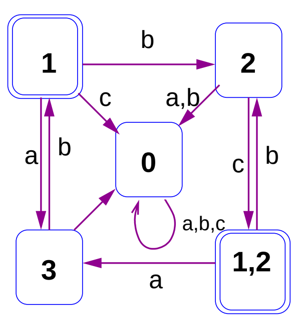
i. Construct an NFA N with three states that recognizes the language $L = \mathcal{L}((\mathtt{ab} \cup \mathtt{bc}^+)^*)$.

Solution.



(a) Convert the NFA N above into an equivalent DFA. (Place any sink in the center of your diagram.)

Solution.



5. (20%) Use ***residues*** to show that $L \subseteq \{\mathtt{a}, \mathtt{b}\}^*$ defined by $L = \{x \cdot \mathtt{a}^n \mid n > 0,\ |x| = n\}$ is not regular.

Solution. The residues $L/\mathtt{b}^n$ for $n > 0$ are all different, because if $i < j$ then

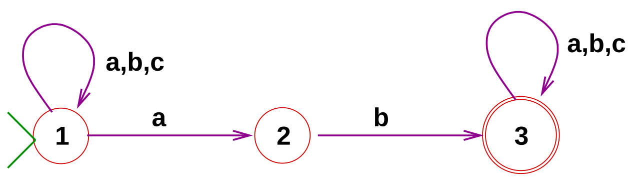
(a) $\mathtt{a}^i \in L/\mathtt{b}^i$ because $\mathtt{b}^i\mathtt{a}^i \in L$; but

(b) $\mathtt{a}^i \notin L/\mathtt{b}^j$ because when $j > i$ we have $\mathtt{b}^j\mathtt{a}^i \notin L$, since if $\mathtt{b}^j\mathtt{a}^i = x \cdot \mathtt{a}^n$ for some x and n, then $|x| = n < i < j$, and so $|x \cdot \mathtt{a}^n| = 2n < j + i$.

Thus L has infinitely many residues, and cannot be regular.

6. (15%) Convert the following NFA into an equivalent DFA.

(Note that two states here have the same residue, so this is not a minimal DFA for L.)



Solution.