Assignment 9: Undecidability

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (10%) Which of the following is true? Explain!

(a) There are two undecidable languages whose intersection is decidable.

Solution. True. Consider an undecidable L. So $\bar{L}$ is also undecidable. But the intersection of the two is $\emptyset$, which is decidable.

(b) There are two undecidable languages whose union is finite.

Solution. False. Since every finite language is decidable, undecidable language are infinite, and so must be their union.

(i) There are two undecidable languages whose concatenation is decidable.
Solution. True. Let $U \subseteq \Sigma^*$ be an undecidable language. Take $X = U \cup \{\varepsilon\}$ and $Y = \bar{U} \cup \{\varepsilon\}$. Then both X and Y are undecidable, but $X \cdot Y = \Sigma^*$, because every $w \in \Sigma^*$ is either in U , and then $w = w \cdot \varepsilon \in X \cdot Y$, or is in $\bar{U}$, in which case $w = \varepsilon \cdot w \in X \cdot Y$.

(c) There are two decidable languages whose concatenation is undecidable.

Solution. False. The concatenation of decidable languae is decidable.

2. (25%) For each of the following problems about Turing acceptors determine whether it is decidable, SD but not decidable, or not SD. You may use any method, including Rice's Theorem and Shapiro's Theorem.

(i) Does acceptor M accept the string `01`?

Solution. This is a non-trivial scope-property, so it is undecidable by Rice's Theorem.

The relation $\vdash$, where $c \vdash M^{\#}$ iff c is an accepting trace of M for input `01`, is a decidable certification for this problem, so it is SD.

(ii) Does acceptor M accept at least two different strings?

Solution. This is a non-trivial scope-property, so it is undecidable by Rice's Theorem.

The relation $\vdash$, where $c \vdash M^{\#}$ iff c is a pair of accepting traces of M for two different input strings, is a decidable certification for this problem, so it is SD.

(Note that the two strings on their own are ***not*** an adequate certificate, because checking that they are accepted may not be decidable!)

(a) Does a given Turing acceptor M accept ε within 10^{10} steps?

Solution. Decidable. Just run M on input ε for up to 10^{10} steps, and accept M if and when acceptance is reached, and reject otherwise.

(b) Does Turing acceptor M accept ***some*** string within 10^{10} steps?
[Hint: How long are the strings that M can actually read within 10^{10} steps?]

Solution. Decidable. In 10^{10} steps M can scan at most 10^{10} symbols of the input. So if the input is accepted, the string consisting of its first 10^{10} symbols must also be accepted. There is a finite number of strings of length $\leqslant 10^{10}$ and we can therefore decide the problem for M by cycling through all these strings, and checking whether M accept the current string within 10^{10} steps.

(c) Does acceptor M accept fewer than 10 different strings?

Solution. This is a non-trivial scope-problem, so it is undecidable by Rice's Theorem. It complemenet is SD: a certificate for an instance M of the complement is a list of accepting traces of M for 10 different strings. So the problem is not SD, for else it would be both SD and co-SD, and therefore decidable.

(d) Does acceptor M recognize a decidable language? [Hint: Shapiro's Theorem]

(e) Does a given acceptor M accept any string? [Caution: It is undecidable whether M accepts a string w.]

A. Prove that if $L \subseteq \Sigma^*$ has a ***semi-decidable*** certification, then L is SD.

Solution. Suppose $\vdash_L$ is a SD certification for a language $L \subseteq \Sigma^*$. The relation $\vdash_L$ being SD, there is a decidable certification $\vdash_{cert}$ for it. Consider the decidable binary relation $\vdash$ that holds between a string $d\#c$ and $w \in \Sigma^*$ iff $d \vdash_{cL} (c, w)$. ($\# \notin \Sigma$). Since $\vdash_{cert}$ is a certification for $\vdash_L$, we have $d\#c \vdash w$, i.e. $d \vdash_{cert} (c, w)$. iff $c \vdash_L w$, i.e. iff $w \in L$.

3. (25%) Let $L \subseteq \Sigma^*$, where $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Prove:

(i) L is decidable iff $L = \{w \mid f(w) = \varepsilon\}$ for some computable function $f : \Sigma^* \to \Sigma^*$.

Solution. If $L = \mathcal{L}(M)$ for some decider M then the following function $f : \Sigma^* \to \Sigma^*$ is computable. On input w f outputs ε if M accepts w and $\mathtt{a}$ otherwise. Then $L = \{w \mid f(w) = \varepsilon\}$.

Conversely, if $L = \{w \mid f(w) = \varepsilon\}$ where $f : \Sigma^* \to \Sigma^*$ is computable by a transducer T, then L is recognized by an acceptor that, on input w, simulates T on w and accepts if and when an output ε is obtained.

(a) L is SD iff $L = \{w \mid fw = \varepsilon\}$ for some computable ***partial-function*** $f : \Sigma^* \rightharpoonup \Sigma^*$.

Solution. If $L = \mathcal{L}(M)$ for some acceptor M then the following partial-function $f : \Sigma^* \rightharpoonup \Sigma^*$ is computable. On input w f outputs ε if and when M accepts w, and is undefined otherwise. Then $L = \{w \mid f(w) = \varepsilon\}$.

Conversely, if $L = \{w \mid f(w) = \varepsilon\}$ where $f : \Sigma^* \rightharpoonup \Sigma^*$ is computable by a transducer T, then L is recognized by an acceptor that, on input w, simulates T on w and accepts if and when an output ε is obtained.

(b) We know that a language $L \subseteq \Sigma^*$ is recognized iff it is computably enumerated, i.e. is the target of a computable functions $f : \mathbb{N} \to \Sigma^*$.

Prove that the same remains true if we take for f computable ***partial-*** functions.

4. (20%) The problem **COMMON-ACCEPT** asks whether a given pair (M_0, M_1) of Turing acceptors accept a common string.

(a) Prove that **COMMON-ACCEPT** is SD.

Solution. The following relation $\vdash$ is a decidable certification for **COMMON-ACCEPT**: $c \vdash (M_0, M_1)$ iff c is a pair (c_0, c_1) with c_0 an accepting trace of M_0 for a string w, and c_1 an accepting trace of M_1 for the same w.

(b) Define a computable reduction of **ε-ACCEPT** to **COMMON-ACCEPT**. Conclude that **COMMON-ACCEPT** is not decidable. (This cannot be proved by invoking Rice's Theorem as we stated it, because the instances are here ***pairs*** of acceptors.)

Solution. Let ρ map an instance M of **ε-ACCEPT** to the instance (E, M) of **COMMON-ACCEPT**, where E is an acceptor for the singleton language $\{\varepsilon\}$. Then M accepts ε iff $\{\varepsilon\} = \mathcal{L}(E) \subseteq \mathcal{L}(M)$, i.e. ρ is a reduction. ρ is computable trivially.

5. (20%) The problem **SUBLANGUAGE** asks whether a given pair (M, M') of Turing acceptors satisfies $\mathcal{L}(M) \subseteq \mathcal{L}(M')$.

(i) Define a computable reduction of **ε-ACCEPT** to **SUBLANGUAGE**.
Solution. Fix an acceptor E for the singleton language $\{\varepsilon\}$. Let ρ be a function that maps an instance M of **ε-ACCEPT** to the instance (E, M) of **SUBLANGUAGE**. ρ is clearly computable, as a purely syntactic program modification. M accepts ε iff $\{\varepsilon\} \subseteq \mathcal{L}(M)$, that is iff $\rho(M^{\#}) = (E, M) \in$ **SUBLANGUAGE**, so ρ is a reduction. It is trivially computable.

(a) Define a computable reduction of **ε-NONACCEPT** to **SUBLANGUAGE**.

Solution. Let ρ map an instance M of **ε-NONACCEPT** to the instance (M, P) of **SUBLANGUAGE**, where P is an acceptors recognizing Σ^+. The M fails to accept ε iff $\mathcal{L}(M) \subseteq \Sigma^+$, i.e. iff (M, P) satisfies **SUBLANGUAGE**. ρ is trivially computable.

(b) Conclude that neither **SUBLANGUAGE** nor its complement are SD.

Solution. We have shown in class that **ε-NONACCEPT** is not SD. By (a) **SUBLANGUAGE** is not SD. And by (i) the complement of **SUBLANGUAGE** reduces to **ε-NONACCEPT**, and so that complement is not SD either.