B501, Fall 2023

Assignment 10: PTime reductions and NP-completeness

A. For each of the following determine whether it is true, and explain your answer.

(i) If L is NP and $L \leqslant_p L'$ then L' is NP.

Solution. False. Let K be a non-NP language (for example an undecidable language). Take $L' = L \cdot \# \cdot K$. Then $L \leqslant_p L'$. But also $K \leqslant_p L'$, so L' is non-NP.

(ii) If L is NP-hard and $L \leqslant_p L'$ then L' is NP-hard.

Solution. True. If L is NP-hard then (by dfn) every NP-problem is $\leqslant_p L$. By transitivity of $\leqslant_p$, every NP problem is $\leqslant_p L'$.

(iii) If L' is NP-hard and $L \leqslant_p L'$ then L is NP-hard.

Solution. False. The extra assumption implies that there is a PTime-decidable problem $\mathcal{P}$ that is not NP-hard, for otherwise $\mathcal{P}$ would be reducible to any non-trivial problem L' and yet not be NP-hard.

1. (20%) Consider the following decision problems, and the corresponding claims that they are NP. For each claim determine whether it is valid. (If b is a binary string then we write $[b]_2$ for its numeric value as a binary numeral.)

(a) *Given a Turing acceptor* M, *does it accept some string* w *within* $\leqslant |w|^2$ *steps.*

Claim: We can take as PTime certificate for M a string w accepted by M in $\leqslant |w|^2$ steps.

(b) *Given a Turing acceptor* M *and a binary numeral* b,
M accepts some w of length $\leqslant [b]_2$ in $\leqslant |w|^2$ steps.

Claim: We can take as PTime-certificate for an instance (M, b) a string w accepted by M in $\leqslant |w|^2$ steps.

(c) *Given a Turing acceptor* M *and a unary numeral* I^n,
M accepts some w of length $\leqslant$ in $\leqslant n^2$ steps

Claim: As PTime-certificate for (M, v) we can take a string w of length $\leqslant n$ accepted by M in $\leqslant |w|^2$ steps.

(d) *Given a Turing acceptor* M, *does it accept* ε.

Claim: We can take as PTime-certificate for M an accepting trace of M for input ε.

(e) *Given a boolean expression* E, *is* E *satisfied by a majority of all valuations for* E*'s variables?*

Claim: We can take as PTime certificate for E a list of valuations that satisfy E.

B. Define **INTEGER-MATCH**: *Given two finite sets* S,T *of positive integers, are there non-empty subsets* $P \subseteq S$ *and* $Q \subseteq T$ *such that* $\sum P = \sum Q$.

Given that **EXACT-SUM** is NP-hard, show that **INTEGER-MATCH** is NP-complete.

Solution. **INTEGER-MATCH** has a PTime certification, with instance (S,T) certified by sets $P \subseteq S$ and $Q \subseteq T$ such that $\sum P = \sum Q$.

The certificate's size is bounded by the size of (S,T) and its correctness can be verified in PTime. So **INTEGER-MATCH** is NP.

To show NP-hardness we define a reduction $\rho :$ **EXACT-SUM** $\leqslant_p$ **INTEGER-MATCH**. ρ maps an instance (S,t) of **EXACT-SUM** to the instance $(S,\{t\})$ of **INTEGER-MATCH**.

ρ is clearly computable in PTime. It is a reduction: If (S,t) satisfies **EXACT-SUM** with a subset P then $(S,\{t\})$ satisfies **INTEGER-MATCH** with the given P and $Q = \{t\}$.

Conversely, if **INTEGER-MATCH** is satisfied with subsets P,Q then $Q = \{t\}$, since Q can't be empty, and so (S,t) satisfies **EXACT-SUM** with that same P.

Given that **EXACT-SUM** is NP-hard, it follows that **INTEGER-MATCH** is NP-hard as well, and since it is NP, **EXACT-SUM** is NP-complete.

2. (20%) **ZERO-SUM**: *Given a finite set* S *of integers (not necessarily positive), is there a non-empty subset* $Z \subseteq S$ *that adds up to 0, i.e.* $\sum Z = 0$.

Given that **EXACT-SUM** is NP-hard, prove that **ZERO-SUM** is NP-complete. [Hint: For the reduction from **EXACT-SUM** add to the set one entry.]

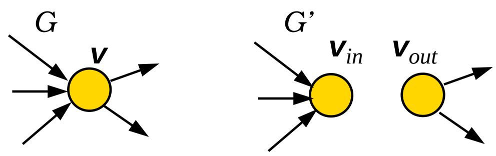
3. **BISAT**: *Given a boolean expression* E, *is it satisfied by at least two different valuations.*

Given that **BOOL-SAT** is NP-hard, prove that **BISAT** is NP-complete.

4. Recall that the **HAMILTONIAN-PATH (HP)** problem asks, given a directed graph G, whether it has a Hamiltonian-path (H-path), i.e. a path visiting every vertex once. The **HAMILTONIAN-CYCLE (HC)** problem asks the same question for a ***cycle***, i.e. a closed loop.

(i) Define a reduction $\rho :$ HC $\leqslant_p$ HP.

Solution. Given a digraph $G = (V, E)$ Chose a vertex $v \in V$. Let $G' = \rho(G)$ be G with v split into two vertices v_{in} and v_{out}. v_{in} inherits the incoming edges of v, and v_{out} the outgoing edges of v.



ρ is computed in PTime trivially.

Suppose G has a H-cycle. $v \to v_1 \cdots \to v_k \to v$.

Then $v_{out} \to v_1 \cdots \to v_k \to v_{in}$ is a H-path in G'.

Conversely, if G' has a H-path then the path's first vertex must be v_{out} (which has no incoming edges) and it must end at v_{in} (which has no outgoing edges). So $v \to v_1 \cdots \to v_k \to v$ is a H-cycle in G.

(a) Define a reduction $\rho :$ HP $\leqslant_p$ HC.

[Hint: For the reduction add a vertex]

5. (20%) A simple graph $G = (V, E)$ is ***subgraph*** of $G' = (V', E')$ if $V \subseteq V'$ and E consists of the edges in E' between vertices in V.

SUBGRAPH: *Given simple graphs* $G = (V, E)$ *and* $G' = (V', E')$,
is G *a subgraph of* G'.

Given that **CLIQUE** is NP-hard, show that **SUBGRAPH** is NP-complete.