MATHEMATICAL MACHINES

Computing

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  - The data is textual
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- Most computing consists in actions that modify data:
  - The data is textual
  - The actions are discrete: well-defined and single-step.
- The data is textual because discrete data has textual representation. (Though not all computing is discrete, eg Analog Computing is not.)

Acceptors

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- Two main options: acceptors and transducers.
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- An ***acceptor*** is an algorithm that takes a textual input (representing input data) and upon termination may or may not issue ***accept*** as output.
- An acceptor that terminates for all input is a ***decider.***
- When a decider terminate for an input without accepting we say that it ***rejects*** the input.
- A decider is thus a solution for a decision problem.

Transducers

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- A transducer computes a ***partial-function*** (i.e. univalent mapping).
- An acceptor can be viewed as a transducer with ***accept*** as the only possible output;

  and a decider as a total transducer with ***accept*** and ***reject*** as the only possible outputs.

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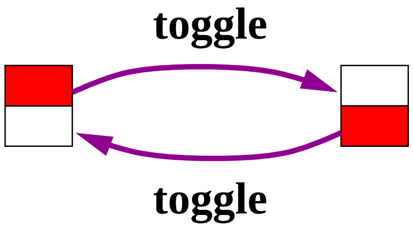
  - Random-access, or sequential reading?
- We start with the ***automaton,***
  an acceptor with no external memory that reads its input sequentially!
- This model captures the behavior of
  many familiar physical devices.
  Let’s look at a couple of very simple ones.

The electric switch

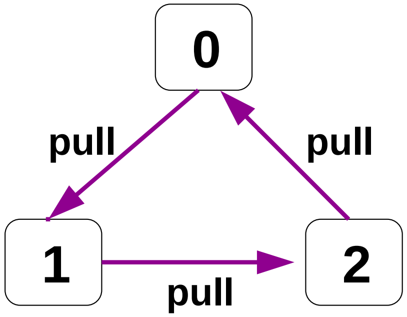


- The position of the switch is inverted
  after an odd number of toggles,
  and remains unchanged after an even number.

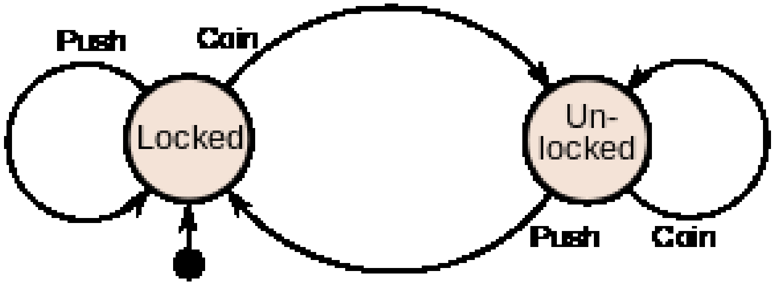
The ceiling fan

- A ceiling fan with manual cord-controlled:

  The speed is incremented (mod 2) with each pull.



The toll-turnstile



- The turnstile can be in one of two states: locked or unlocked.
- The action *insert token*
  changes the state *locked* into *unlocked*.
- The action *push and pass*
  changes the state *unlocked* into *locked*.

States

- A core concept of mathematical machines is the ***state.***
- E.g. a state of an elevator might consist of
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- A core concept of mathematical machines is the ***state.***
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  its position, motion (up, down, rest), upcoming destinations, time idle, etc.
- States are often labeled, for convenience, but don't have to be.
- Given a practical problem, deciding what are the relevant "states"
  often requires careful analysis.
- But once a mathematical model is distilled,
  the ***states*** become an abstraction,
  which we can represent graphically, e.g. by a circle.

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- A pair of states related by a transition-rule `a` is an ***action*** of `a`.
- For the toll-turnstile and the stopwatch the transition-rules are determined by certain human actions.

Textual form of transitions

- Since all finite discrete structures have simple textual codes,
  we can assume that:

1. All input data is textual
2. Each transition is coded by a single reserved letter
3. The action of the transition labeled `a`
   is the reading (i.e. consumption) of `a`,
   much like the movement of a cursor.

$$\begin{tikzcd}
\underline{\texttt{a}}\texttt{bracadabra} \arrow[d, "\texttt{a}"] \\
\underline{\texttt{b}}\texttt{racadabra}
\end{tikzcd}$$

A transition system

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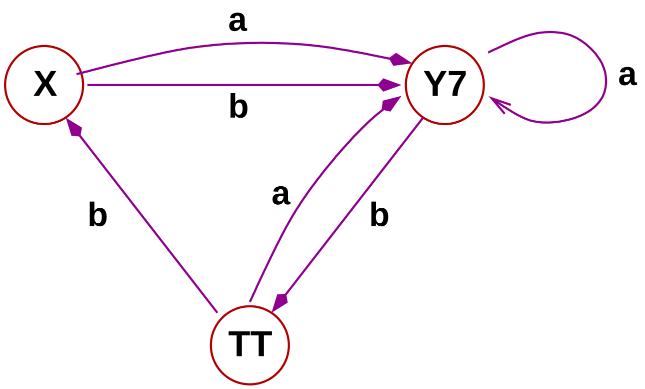
- So a transition-system can be represented as a labeled di-graph: The nodes are the states, and the the actions are labeled edges.
- When all transition-rules are functions, there is exactly one edge for each state and action:

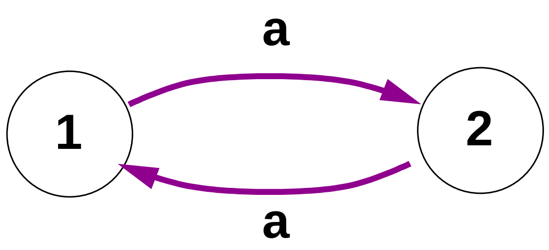


Example: Detecting an odd number of actions

- Consider the switch.

  We represent the transition “toggle” by the letter `a` ,

  and label the states as 1 and 2:



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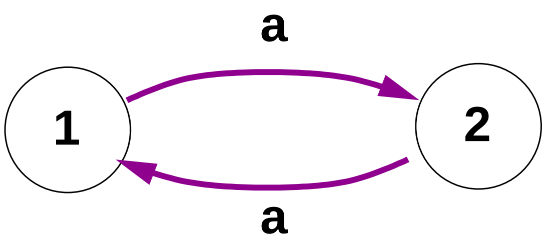
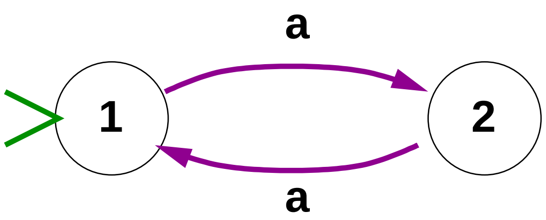


- The device reads strings of `a`’s,
  and with each letter read it switch state.
- Reading odd number of `a` ’s leads to the opposite state.
- The physical nature of the toggle action is no longer present,
  and is indeed irrelevant.

Start state and accepting states

- We intend to start at a particular state, so we single out one state as the ***initial*** (starting) state, indicated graphically by an incoming arrow.



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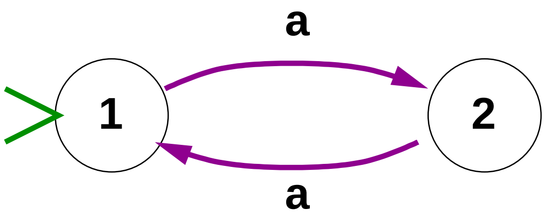

Where do the strings of length 1,3,... odd n lead?

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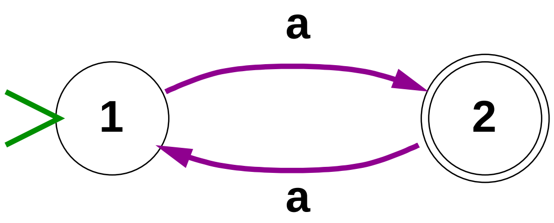
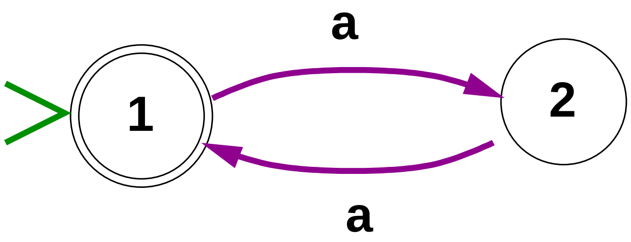


- The strings of odd length leads to state 2,
  so to accept just those strings we'd set 2
  as the unique accepting state.
- We do this graphically by doubling the contour of state 2.
- In general there can be several accepting states.

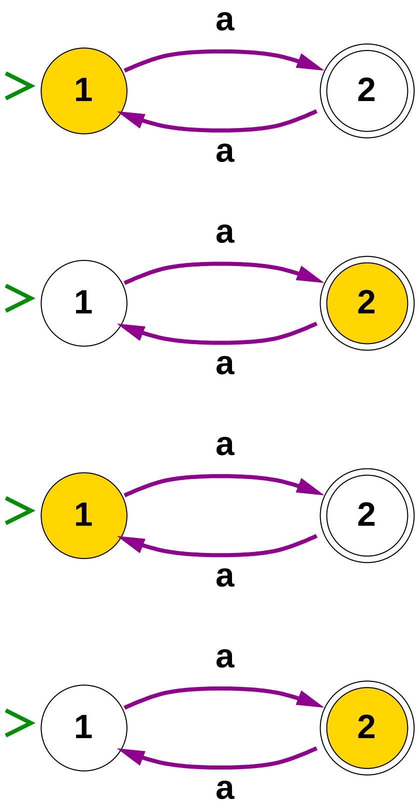
Initial state can be accepting

- It is possible that the initial state is accepting.
- To accept the strings of even length
  set **1** as the only accepting state:



The device in action

- Device accepting odd length:



READING

a

aa

aaa

string accepted IFF has odd #a
aaa accepted

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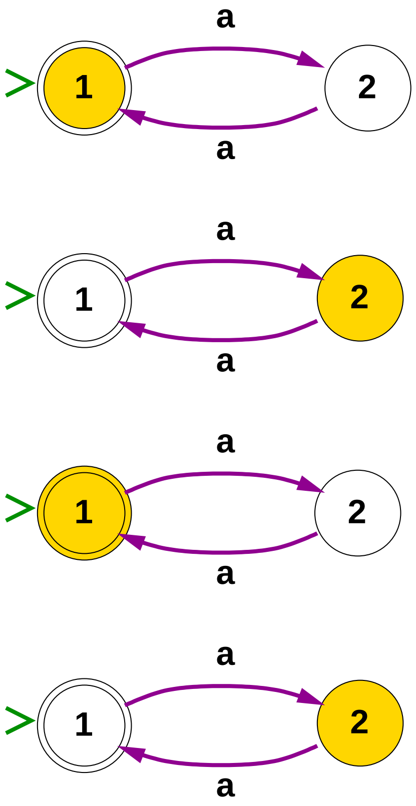

READING

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aa

aaa

string accepted IFF has even #a
aaa not accepted

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  - A set $A \subseteq S$ of states singled out as **accepting states**.
  - A **transition function** $\delta : Q \times \Sigma \to Q$.
    Given state $q \in Q$ and input-symbol σ
    $\delta(q, \sigma)$ is the new (target) state.
- We also write $q \xrightarrow{\sigma} p$ for $\delta(q, \sigma) = p$.
  Note: p may be the same as q.

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  We don't mention Σ when irrelevant or clear.
- *Automaton* is of Greek origin:
  auto = self, *matos* = move.
  Plural: *automata* or *automatons*. *Automat***a** is never singular.
- Since automata play a central role,
  they've acquired over time several alternative names, in particular *deterministic finite automaton (DFA)*.which we'll frequently use.

Some practical applications of automata

Textual applications

- Pattern matching, search engines
- Lexical analysis for compilation
- Data compression
- Automatic translation

Some practical applications of automata

Software systems

- Cyber-security
- System planning
- Information streaming
- Bio-informatics

Some practical applications of automata

Hardware systems

- Circuit design
- Robotics

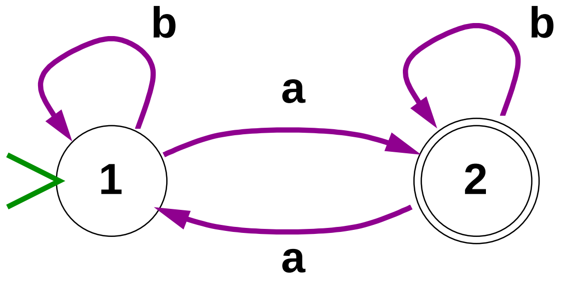
Some practical applications of automata

Verification

- System modeling
- Verification of communication protocols
- Verification of embedded systems
- Model checking

Example of a formal description

- Here's an automaton M over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ that accepts strings with an odd number of $\mathtt{a}$'s (and no others).



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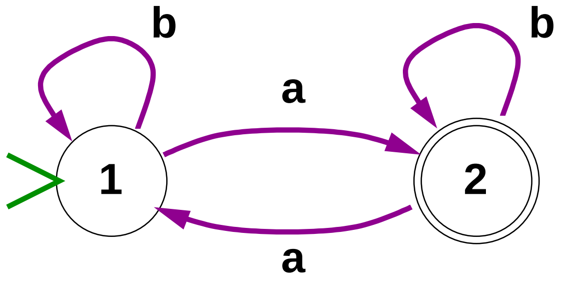

- Its formal definition: $M = (\Sigma, Q, s, A, \delta)$ where
  - ⋆ $\Sigma = \{\mathtt{a}, \mathtt{b}\}$
  - ⋆ $Q = \{1, 2\}$
  - ⋆ $s = 1$
  - ⋆ $A = \{2\}$

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  there is exactly one next-state for each symbol read.
- Computation terminates iff the end of the input string is reached.
- The essence of a DFA is in its being an ***online acceptor***.

Traces

- If $w = \sigma_1 \cdots \sigma_n$ then we write $q \xrightarrow{\sigma_1 \cdots \sigma_n} p$ to state that $q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$ for some states $r_1, \ldots, r_{n-1}$.

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- The sequence of states $q, \; r_1, \; r_2, \; \cdots \; r_{n-1}, \; p$ is a ***state-trace*** of the automaton.

Inductive definition of traces

- The ternary relation $q \xrightarrow{w} p$ can be defined inductively, by recurrence on w :
  - ► $q \xrightarrow{\varepsilon} q$
  - ► If $\delta(q,\sigma) = p$ that is $q \xrightarrow{\sigma u} r$, and $p \xrightarrow{u} r$ then $p \xrightarrow{\sigma} q$.

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  - If $\delta(q,\sigma) = p$ that is $q \xrightarrow{\sigma u} r$,
    and $p \xrightarrow{u} r$ then $p \xrightarrow{\sigma} q$.
- This definition invokes no auxiliary data
  that might be modified during execution.
- No mathematical machine we'll encounter (except NFAs)
  has such a definition:
  They all are based on a notion of ***configuration***,
  which combines the machine's states with modifiable data.

Accepted strings, recognized languages

- For $A \subseteq Q$ let's write $q \xrightarrow{w} A$

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- M $\boxed{\textit{accepts}}$ w when $s \xrightarrow{w} A$.

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- We re-use here the notation $\mathcal{L}(\cdots)$ that we used for regular expressions.
- Two automata are ***equivalent*** if they recognize the same language.

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Only two are crucial: violating them changes computing's nature:

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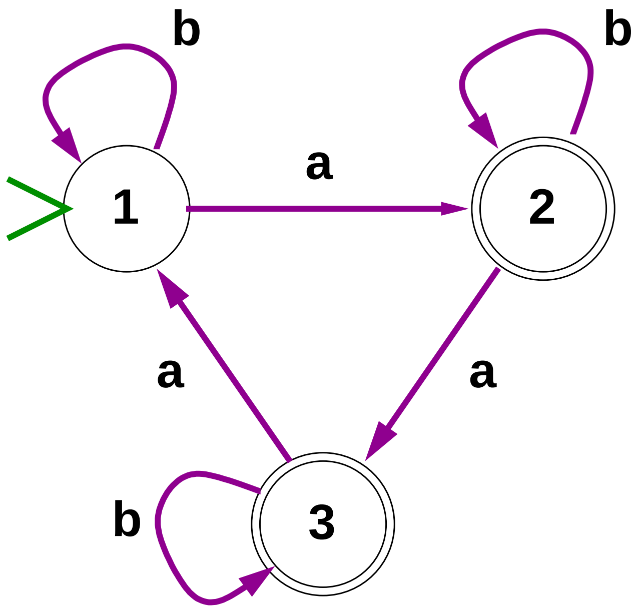
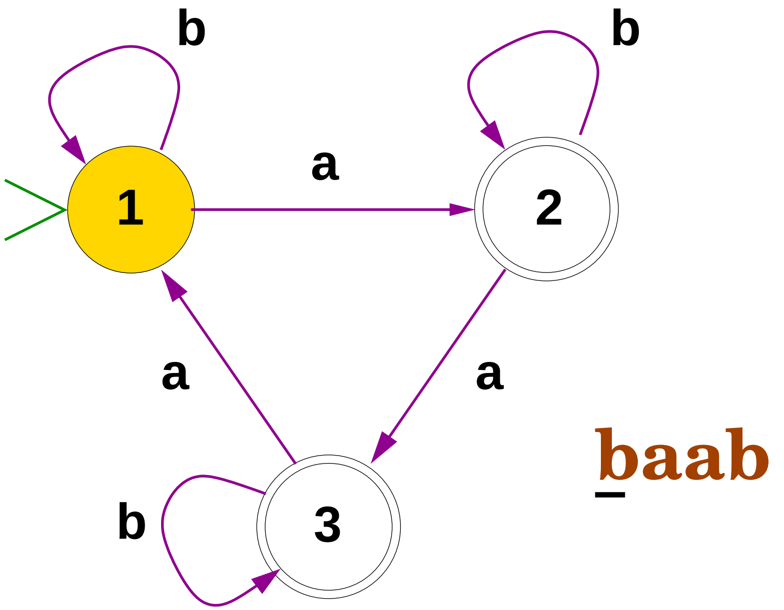
Example: An automaton for Mod 3



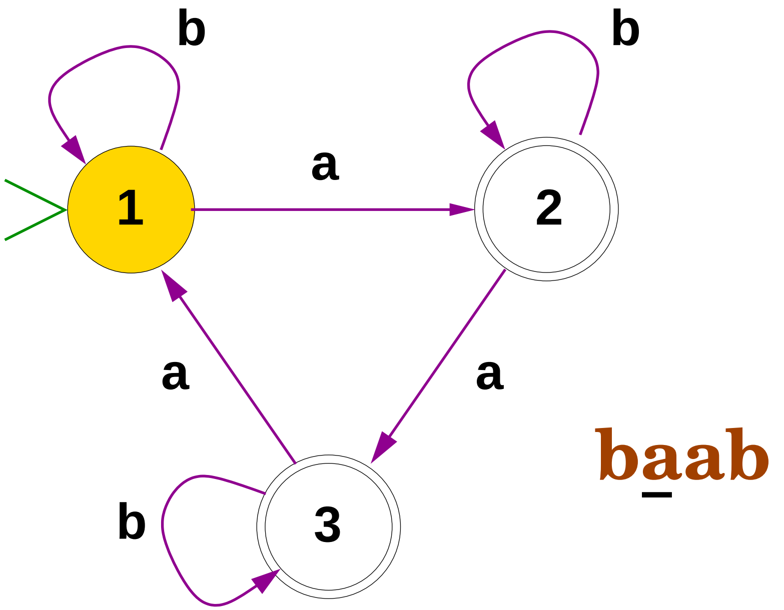
- $w \in \{\mathtt{a},\mathtt{b}\}^*$ accepted iff $\#_a(w) \neq 0 \pmod 3$

Example of an accepted string



- State 1 (initial). Nothing read yet.

An accepted string



- Still state 1. Initial b read.

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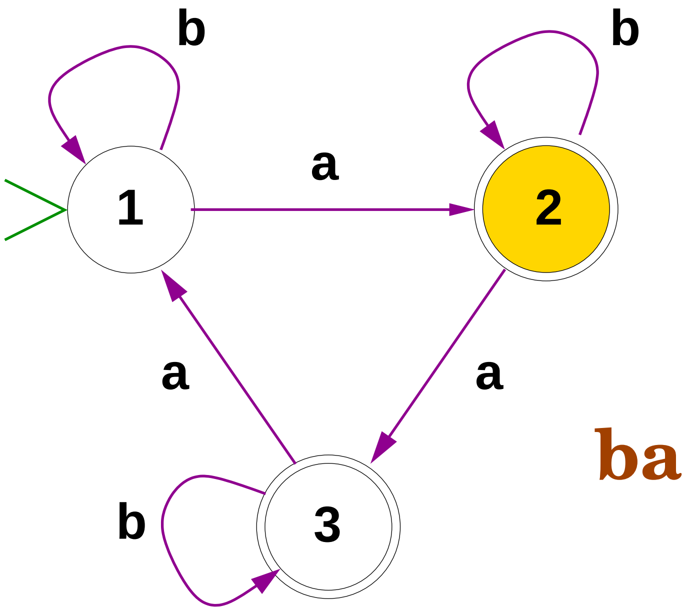
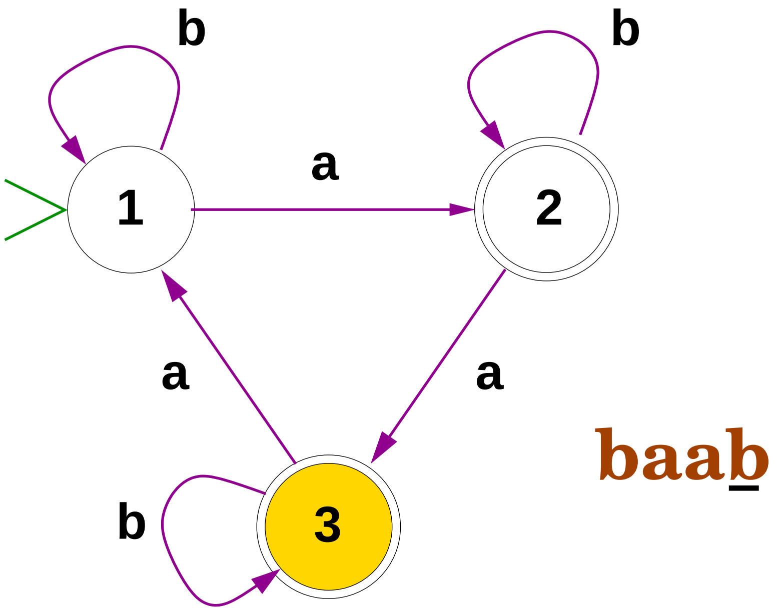


baab

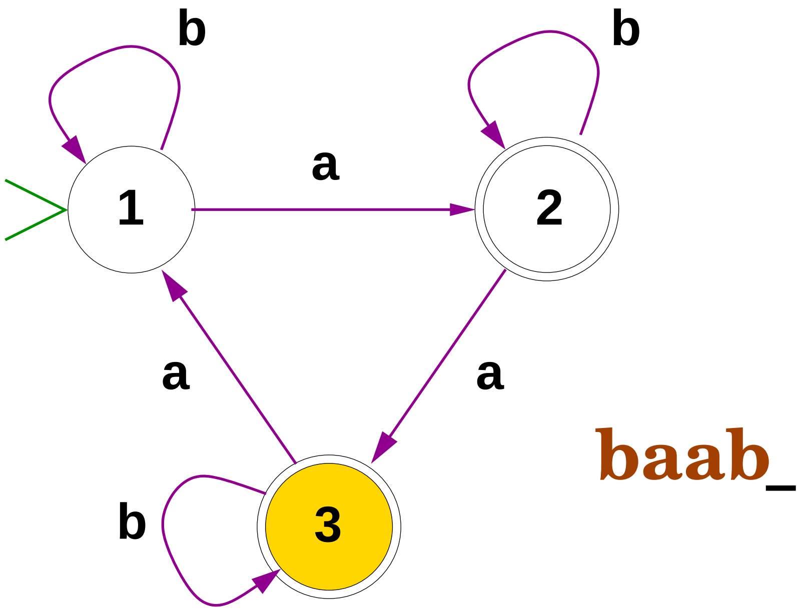
- Read `ba`, state 2.

An accepted string



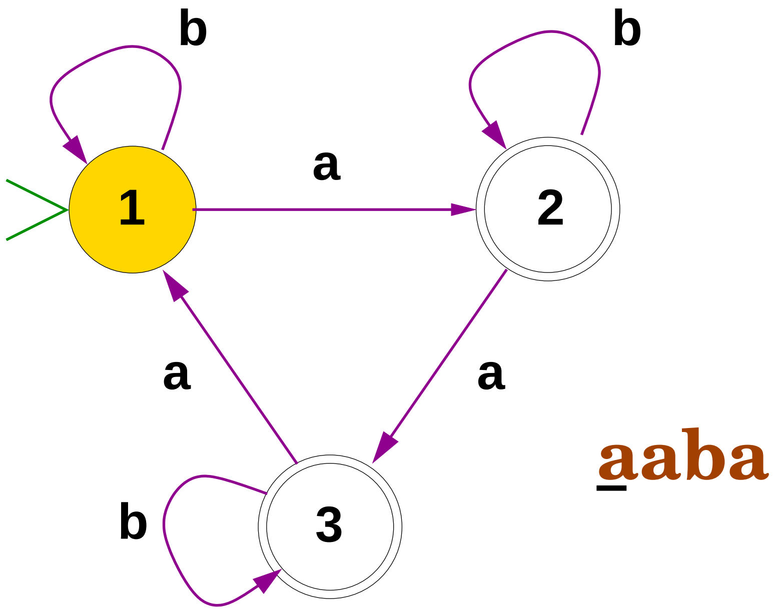
- Read baa, state 3.

An accepted string



- Finished reading *baab*, state 3, accepted.

A non-accepted string



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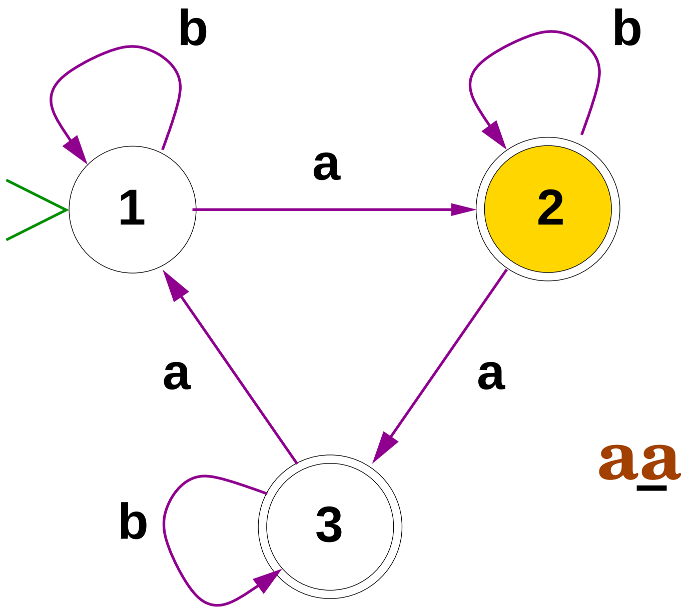

$\mathbf{a\underline{a}ba}$

- Read a, State 2.

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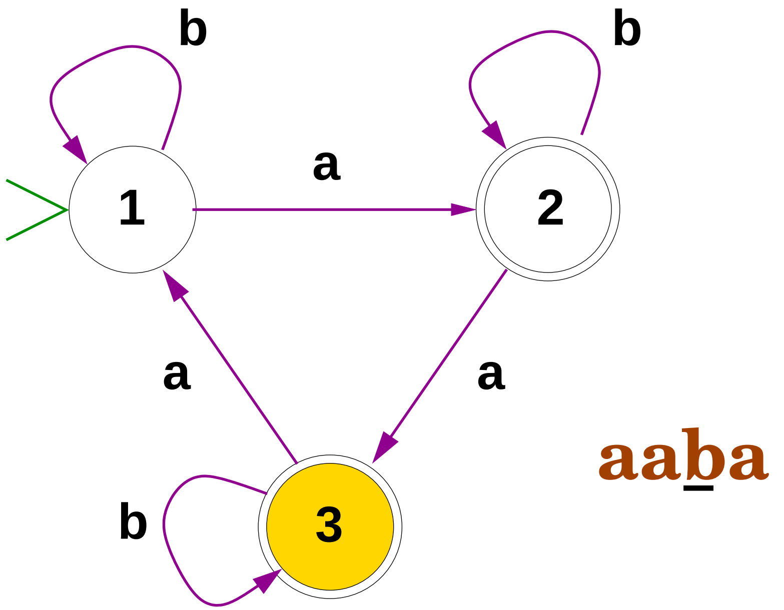

- Read `aa`, state 3.

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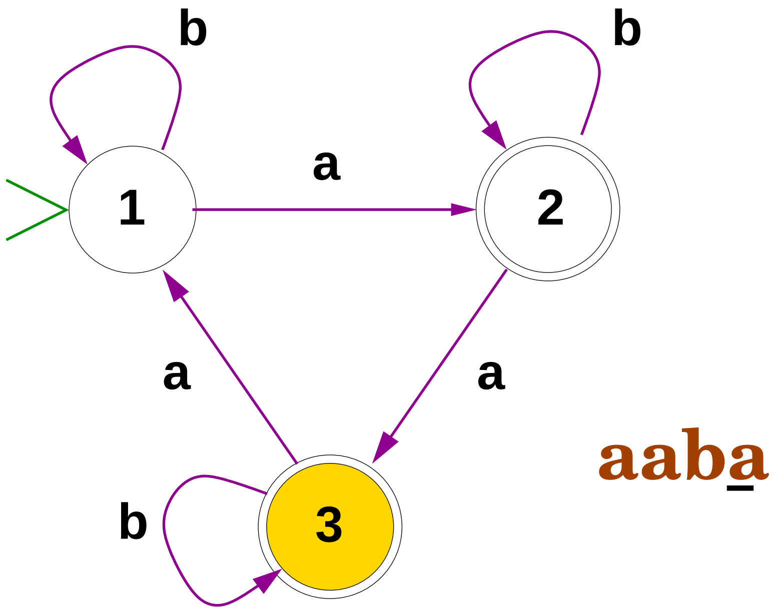

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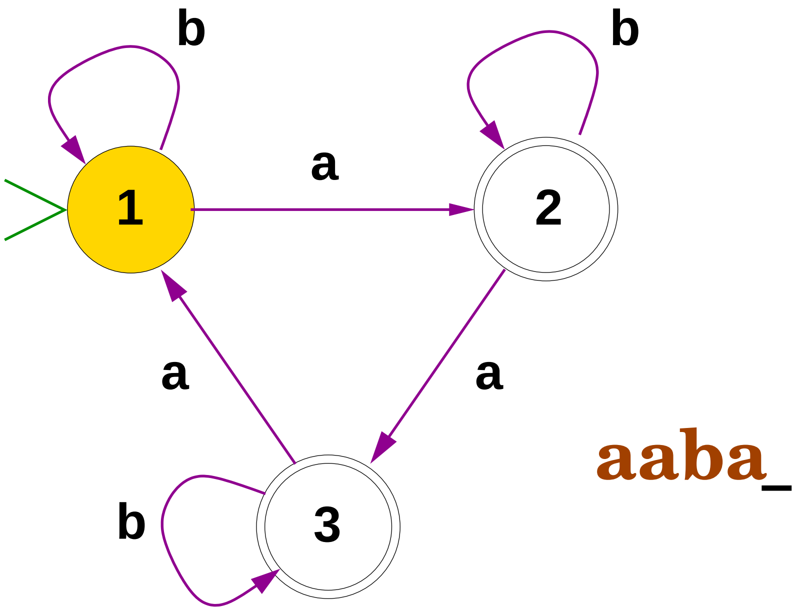


- Finished reading aaba, state 1, not accepted.

A computation trace

- For our example above, the computation for the string **`baab`** is

  $$1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 3.$$

  Abbreviated notation: $1 \xrightarrow{\texttt{baab}} 3$

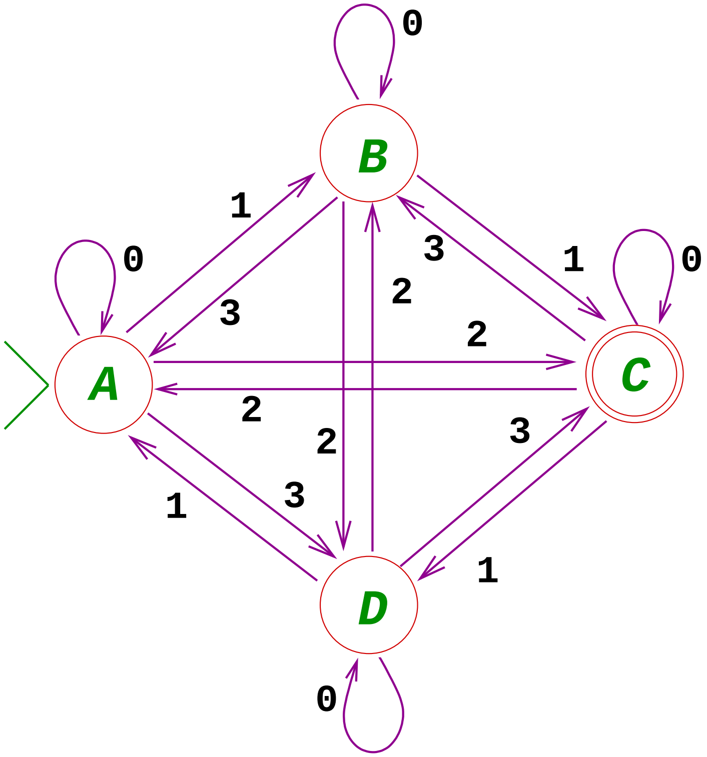
- The computation for the string **`aaba`** is

  $$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 3 \xrightarrow{\texttt{a}} 1.$$

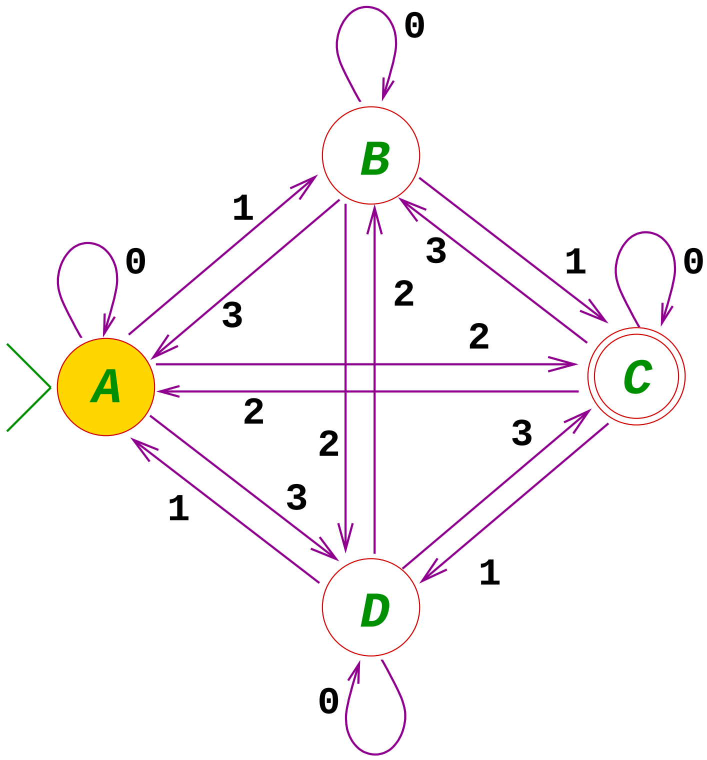
  Abbreviated notation: $1 \xrightarrow{\texttt{aaba}} 3$

Example: Addition mod 4

- The following automaton is over the alphabet {0, 1, 2, 3}
- It accept a string of digits iff they add up to 2 modulo 4.

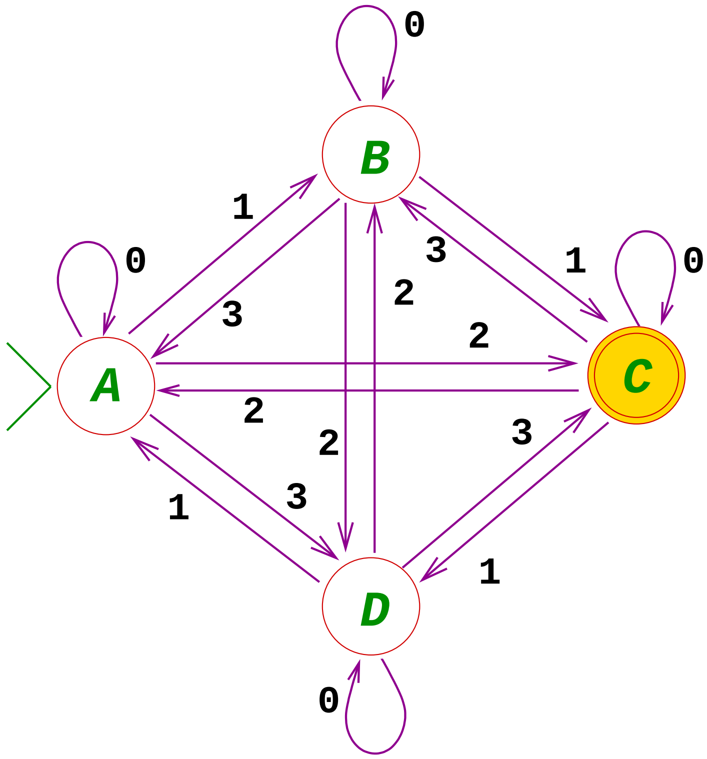


- Reading input 21032 from initial state A:



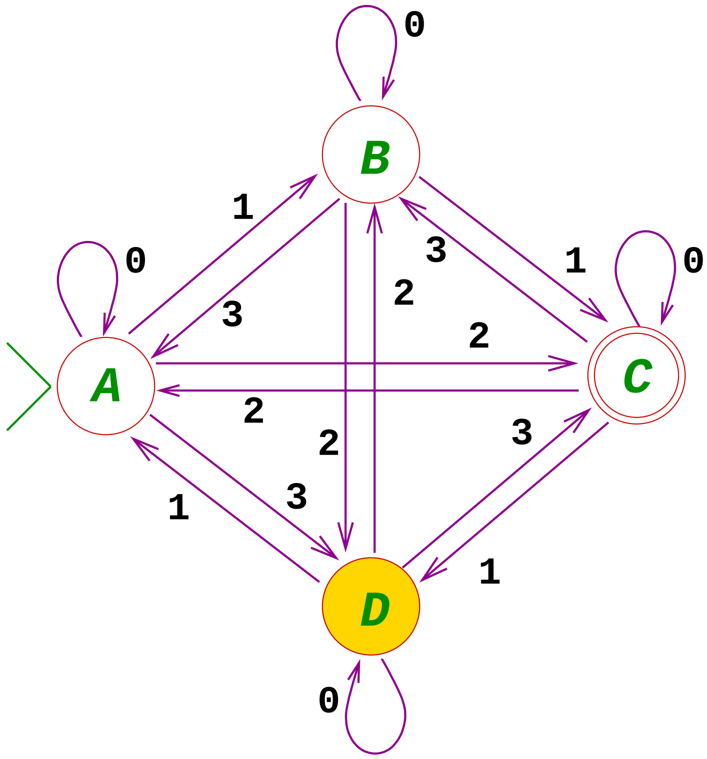
A 21032

- Reads remaining string 1032:



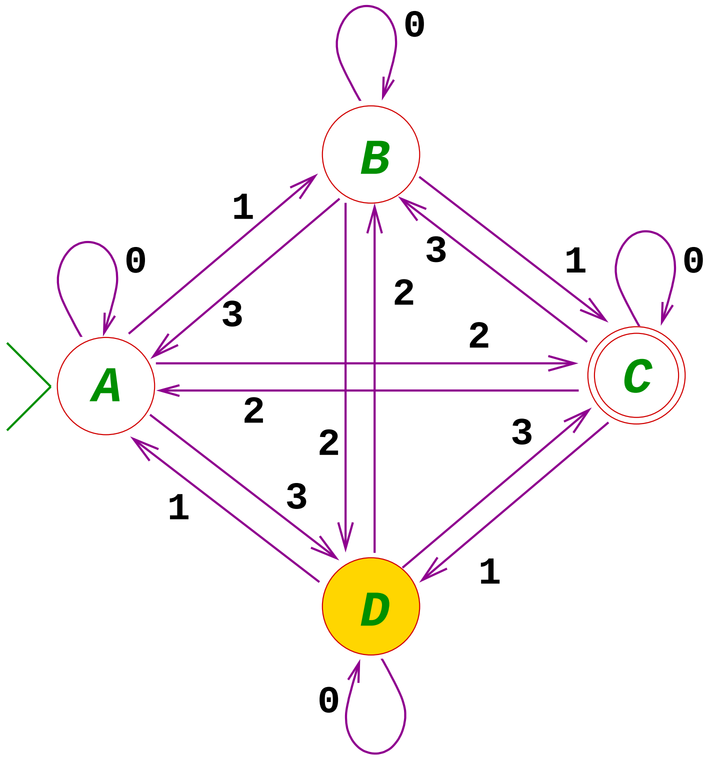
C **1032**

- Reads remaining string 032:



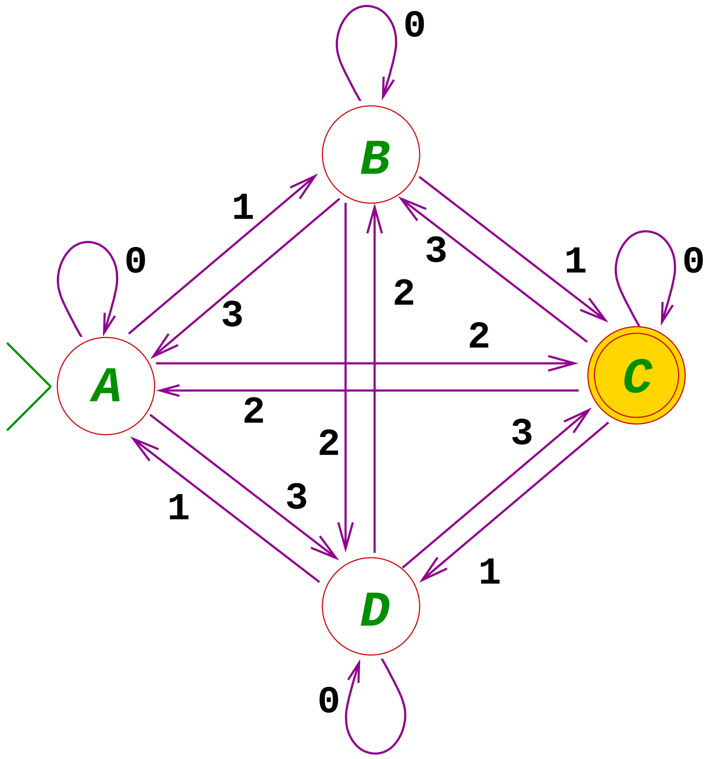
D 032

- Reads remainder 32:



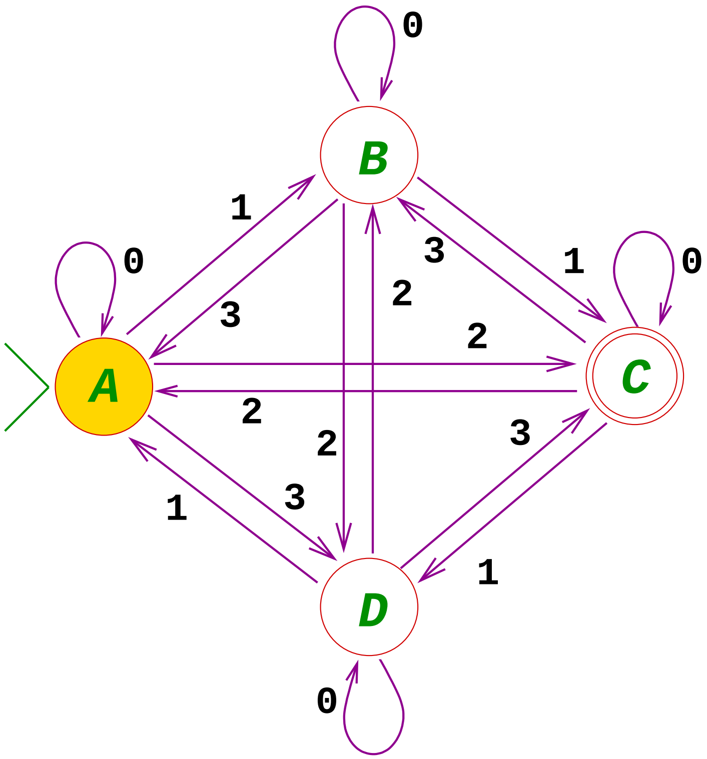
D 32

- Reads remainder 2:



C 2

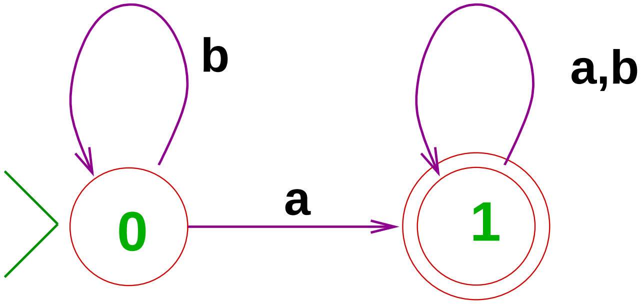
- Reads remainder ε (empty string):



A ε

- Ends reading. A not an accept-state, 21032 not accepted.

Additional examples

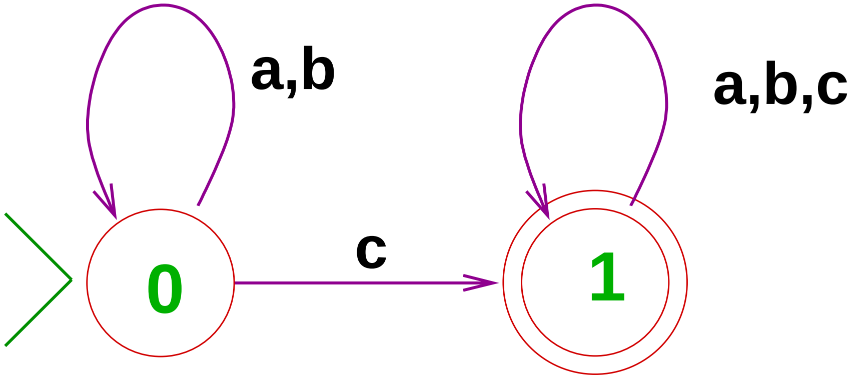


$$0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{a}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1$$

$$0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0 \xrightarrow{\texttt{b}} 0$$

What is the language recognized?

Three letter example

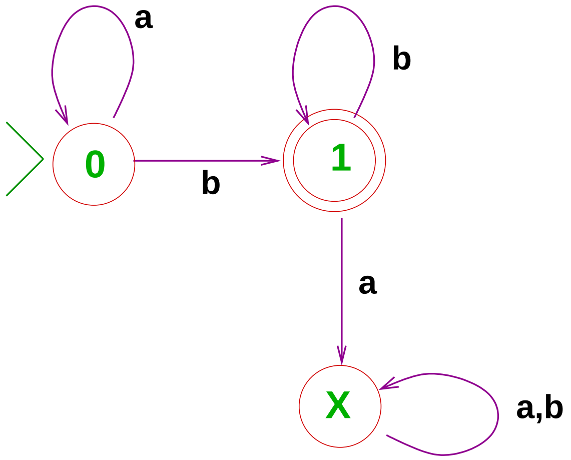


$$0 \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{b}} O \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{c}} 1 \xrightarrow{\texttt{b}} 1$$

$$0 \xrightarrow{\texttt{c}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} 1$$

What are the language accepted?

An automaton with a sink



$0 \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{a}} 0 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{b}} 1$

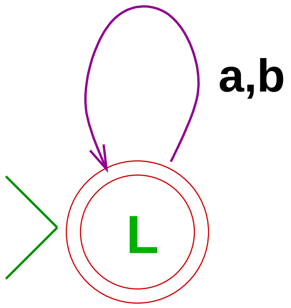
$0 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{b}} 1 \xrightarrow{\texttt{a}} X \xrightarrow{\texttt{b}} X \xrightarrow{\texttt{a}} X$

Note: Every state has exactly one arrow for every $\sigma \in \Sigma$.

- A ***sink*** is a non-accepting state with all outgoing transitions pointing to itself.

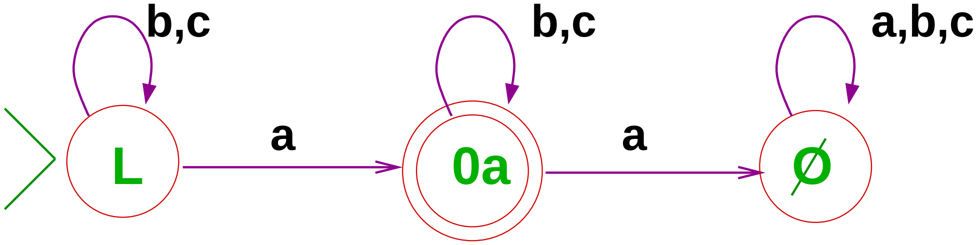
Example

Here is a trivial automaton with a single state:



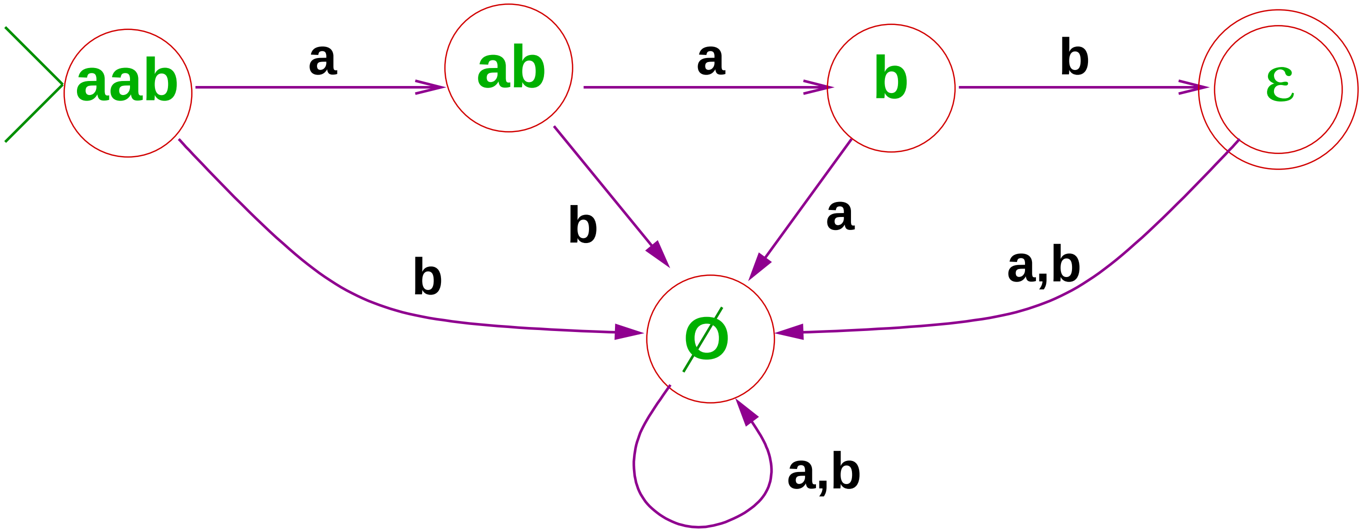
What strings are accepted?

Example



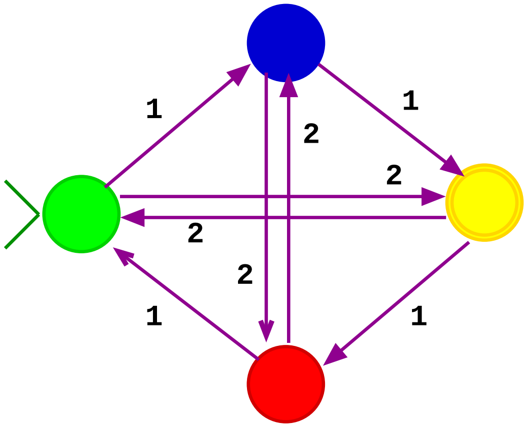
accepts the strings with exactly one a , and no other.

Example

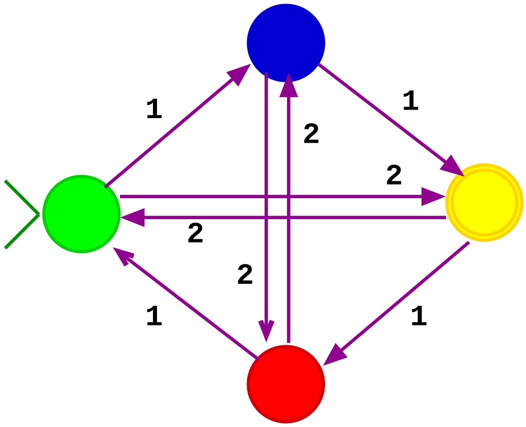


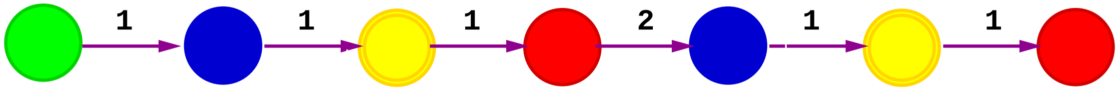
accepts the string `aab` and no other.

AUTOMATA ARE REPETITIVE

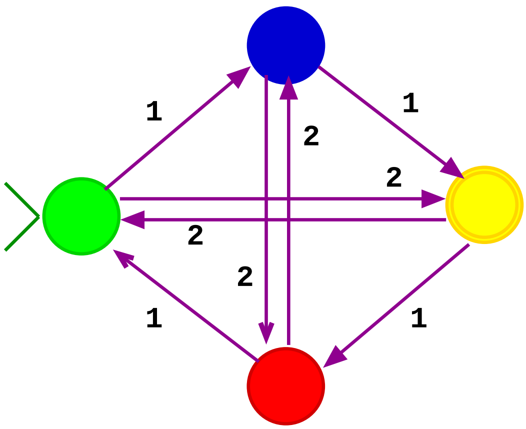


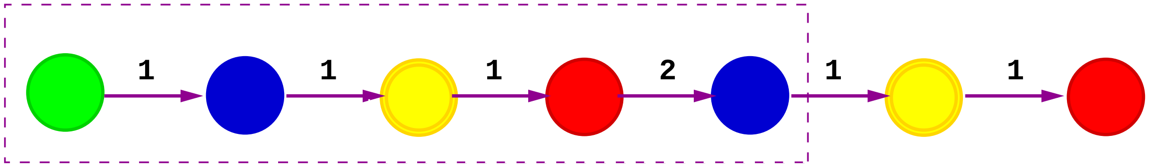
- Here's an automaton that accepts a string $w \in \{1,2\}^*$ iff the sum of the digits in w is $2 \mod (4)$.



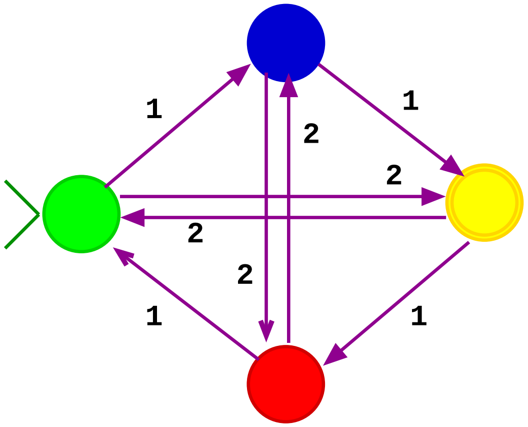


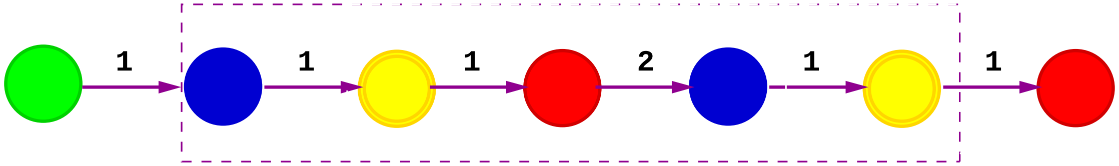
- This is its trace for input 111212.
  The input has 6 symbols, so the trace lists 7 states.



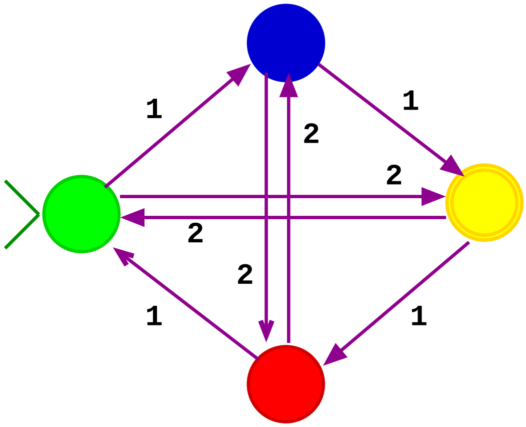


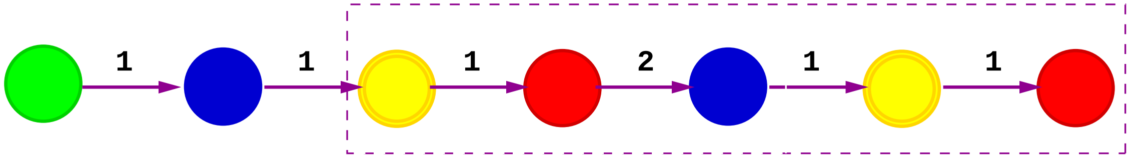
- Looking at the first 5 of the 7, we must have a state repeating, because there are only 4 states.





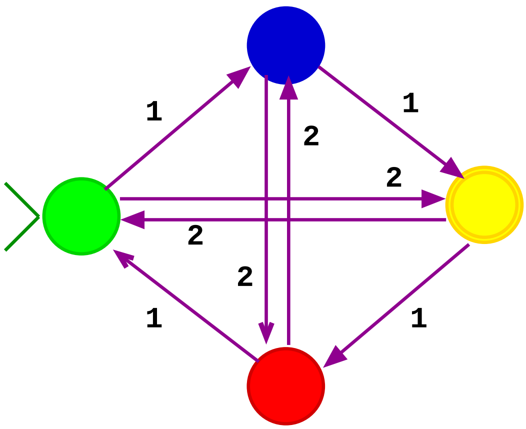
The same happens for the next stretch of 5 states (i.e. 4 input symbols)

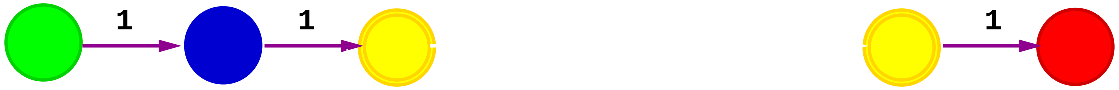




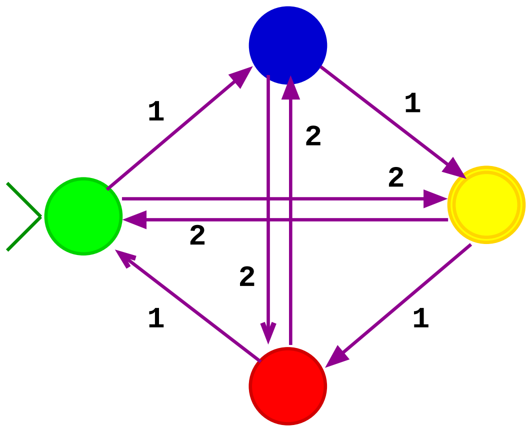
And the next one.

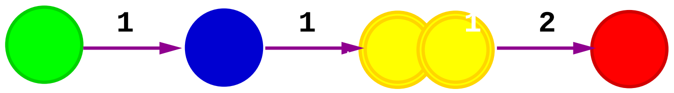
No matter which window of 5 states we take there will be a state repeating!





We can short-circuit the steps from the yellow state to itself, and the result will still be a legit trace, but for 112.





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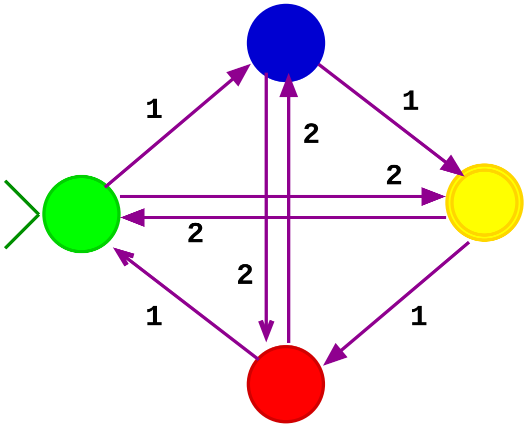
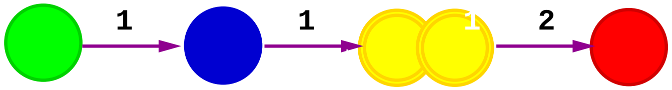
Shortcuts in traces

- We observed:

  Let M *be a* k *-state DFA.*
  If $q \xrightarrow{u} p$ *and* $|u| \geqslant k$ *then*
  $q \xrightarrow{u'} p$ *where* u' *is* u *with some*
  substring $y \neq \varepsilon$ *clipped off, i.e. removed.*





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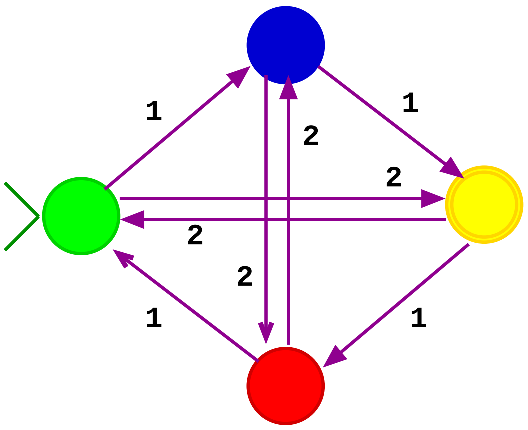

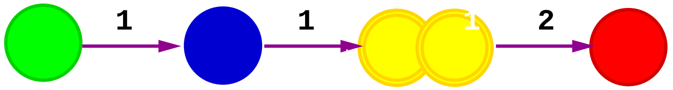

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$$s \xrightarrow{w_0} p \xrightarrow{u} q \xrightarrow{w_1} A$$

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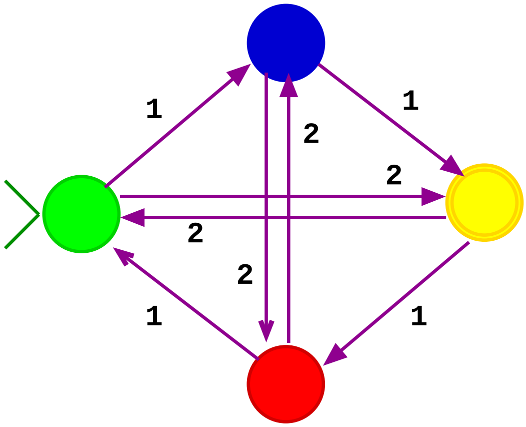

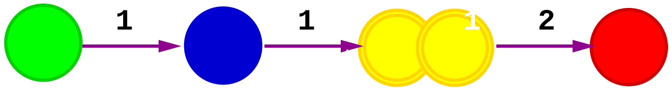

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The Clipping Theorem

- **Theorem.** *If a k -state DFA accepts a string w , and u is a substring of w of length $\geqslant k$, then u has a substring $y \neq \varepsilon$ such that w with y removed is also accepted.*

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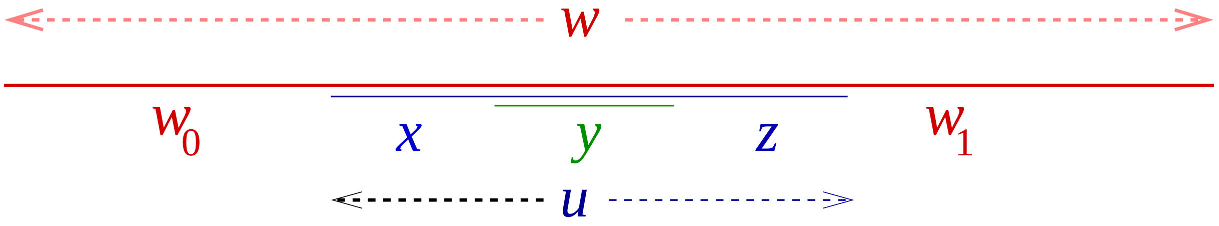

An application: the shortest string accepted

- If M is a 10 state automaton that accepts some string. What is the length ℓ of the ***shortest*** string accepted?

  1. $\ell \in [30..100]$
  2. $\ell \in [10..25]$
  3. $\ell \in [0..9]$
  4. Can’t tell, could be anything.

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  This contradicts the assumed minimality of $|w|$.

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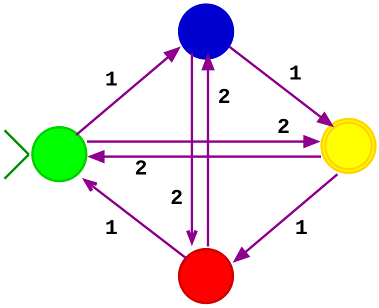
- Example: What is the shortest string accepted by



The dual question

- I want a DFA that accepts exactly the strings of length $\geqslant 100$.
- What's the smallest number ℓ of states I need?

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  1. $\ell \in [1..9]$
  2. $\ell \in [10..99]$
  3. $\ell \in [100..999]$
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- Answer: 101:

  A DFA with 100 states will accept some string of length < 100.

On not being an insect

- How do you tell that the critter on your desk is not an insect?

On not being an insect

- How do you tell that the critter on your desk is not an insect?
- Check that it violates some property of insects, e.g. it has eight rather than six legs.
- How do you tell that a given language L is not recognized by any automaton?
- Refer to a property that all recognized languages have, but L does not.

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The Clipping Property

- The Clipping Theorem says that

  Every language L recognized by a DFA has the following **Clipping Property:**

  ⋆ There is a k (the number of states in an acceptor for L),

  ⋆ so that for every $w \in L$

  ⋆ if u is a substring of w of length $\geqslant k$,

  ⋆ then it has a “clippable” substring $y \neq \varepsilon$:
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- A language ***fails Clipping*** when

  ⋆ for any $k > 0$

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  and a substring u of w of length $\geqslant k$,

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    and a substring u of w of length $\geqslant k$,
  - ⋆ so that ***any*** clipping within u yields a $w' \notin L$.

- If L fails Clipping then it is not recognized.

Example: an-bn

- Let $L = \{\mathsf{a}^n\mathsf{b}^n \mid n \geqslant 0\}$
- L fails clipping:

  1. Let $k > 0$
  2. Choose $w = \mathsf{a}^k\mathsf{b}^k$ and $u = \mathsf{a}^k$.
     We have $w \in L$ and $|u| \geqslant k$.
  3. Any clipping in u yields from w
     a w' of the form $\mathsf{a}^p\mathsf{b}^k$ with $p < k$.
     So $w' \notin L$.

- Consequence: L fails the Clipping Property and cannot be recognized.

Example: Unary addition

- Consider the strings representing addition in unary:
  $$A = \{\mathbf{1}^p + \mathbf{1}^q = \mathbf{1}^{p+q} \mid p, q > 0\}.$$
- A fails the Clipping Property:

  1. Let $k > 0$.
  2. Choose $w = \mathbf{1}^k + \mathbf{1} = \mathbf{1}^{k+1}$
     and u the substring $\mathbf{1}^{k+1}$.
     $w \in A$ and $|u| \geqslant k$.
  3. Any clipping in u yields from w a string
     $w' = \mathbf{1}^\ell + \mathbf{1} = \mathbf{1}^{k+1}$ with $\ell < k$.
     $w' \notin A$.

- A fails Clipping, and so cannot be recognized.

Example: Perfect squares in unary

- Consider $L = \{\mathtt{1}^{n^2} \mid n \geqslant 0\}$.
- L fails the Clipping Property:

1. Let $k > 0$.
2. Choose $w = \mathtt{1}^{k^2}$ and $u = \mathtt{1}^{k}$.
   $w \in L$ and $|u| \geqslant k$.
3. For any clipped y we have $1 \leqslant |y| \leqslant |u| = k$,
   so for the reduced string $w' = \mathtt{1}^{\ell}$ where $k^2 - k \leqslant \ell < k^2$.
   $w' \notin L$ because ℓ cannot be a square: the largest square preceding k^2 is $(k-1)^2 = k^2 - 2k + 1$ which is $< k^2 - k \leqslant \ell$.

- So L fails Clipping, and cannot be recognized.

Example: The mahimahi language

- Consider $L = \{x \cdot x \mid x \in \{\mathtt{0}, \mathtt{1}\}^*\}$
- Idea: Take $w = x \cdot x$ with x that starts with a marker.

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1. Let $k > 0$.
2. Choose $w = \mathtt{0}\mathtt{1}^k\mathtt{0}\mathtt{1}^k$ and $u =$ left substring $\mathtt{1}^k$ in w.
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2. Choose $w = \mathtt{01}^k\mathtt{01}^k$ and $u =$ left substring $\mathtt{1}^k$ in w.
   $w \in L$ and $|u| \geqslant k$.
3. Any clipped y in u yields from w
   a reduced string $w' = \mathtt{01}^\ell\mathtt{01}^k$
   where $\ell < k$.
   Such w' cannot be of the form xx,
   because its first half starts with $\mathtt{0}$
   while its second half starts with $\mathtt{1}$.

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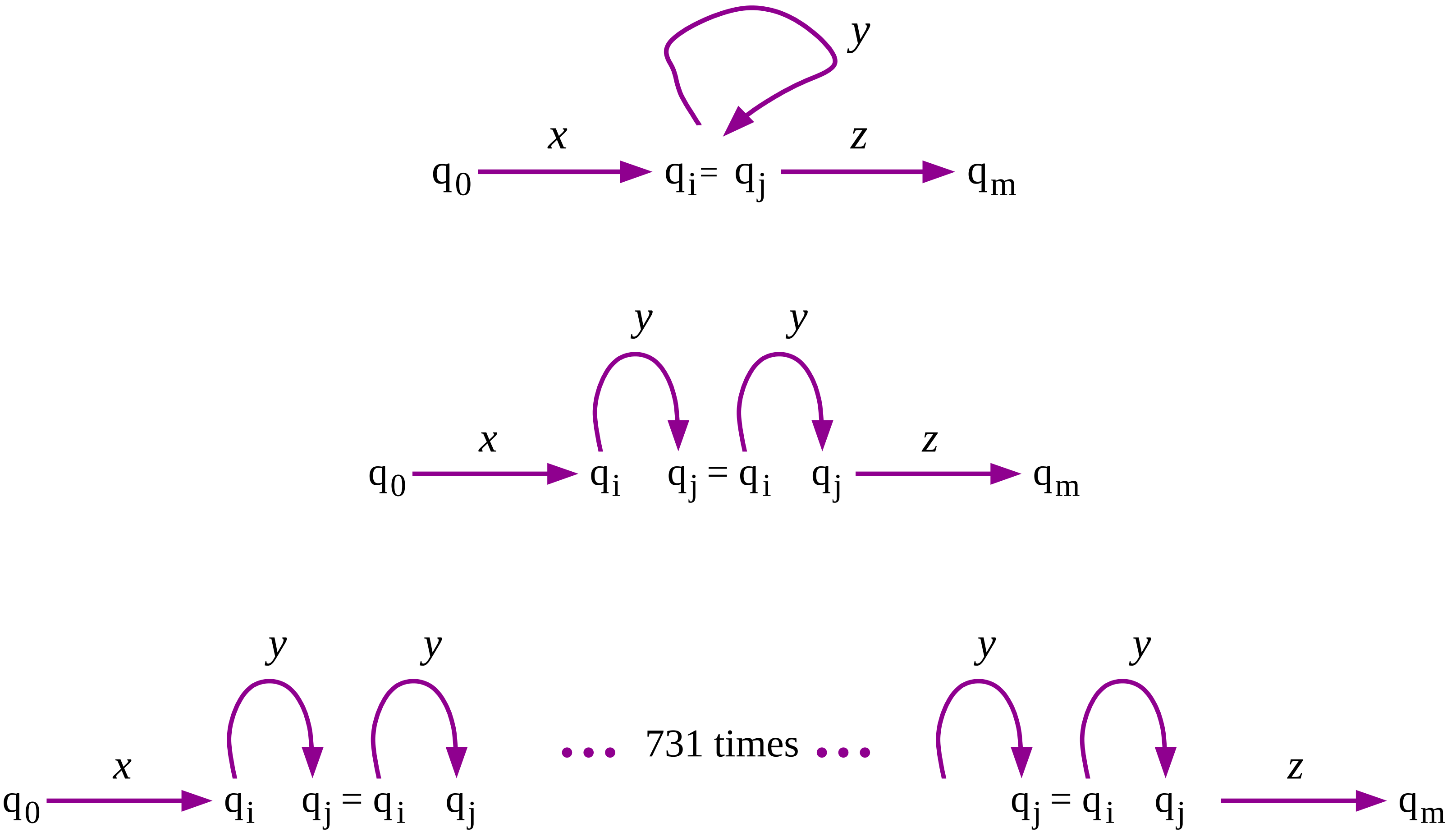
   a reduced string $w' = \mathtt{0}\mathtt{1}^\ell\mathtt{0}\mathtt{1}^k$
   where $\ell < k$.
   Such w' cannot be of the form xx,
   because its first half starts with $\mathtt{0}$
   while its second half starts with $\mathtt{1}$.

- L fails the Clipping Property, and cannot be recognized.

Pumping up rather than clipping



Pumping instances

- Let $w \in \Sigma^*$ and

  y a particular substring of w: $w = x \cdot y \cdot z$.

- The $\boxed{n\text{-th pumping instance}}$ of $w = x \cdot y \cdot z$

  over (the exhibited occurrence of) y

  is defined to be $x \cdot y^n \cdot z$.

The Pumping Theorem

- Let M be a k-state DFA over Σ, $L = \mathcal{L}(M)$.
- As for Clipping, choose $w \in L$ and a substring u of w of length $\geqslant k$.
- CONCLUDE: u has a non-empty substring y
  such that all pumping instances of w over y are in L.
- Recall: The n*-th pumping instance* of w over
  (a particular occurrence of) y
  is the result of replacing y by y^n.

Failing Pumping

A language ***fails Pumping*** when:

1. For any $k > 0$
2. there are $w \in L$
   and substring u of w of length $\geqslant k$
3. so that for ***every*** y within u
   there is a pumping instance w over y which is not in L.

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- Contradiction. M cannot exist.

Example: Necessary use of Pumping

- Show that the language

$$L = \{ w \cdot \mathtt{a}^n \mid w \in \{\mathtt{a}, \mathtt{b}\}^*,\ \#_a(w) = n \}$$

is not recognized.

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- Suppose L were recognized by a k-state DFA.
  Let $w = \mathtt{b}^k\mathtt{a}^k$, which is in L,
  and take $u = \mathtt{b}^k$, the prefix of w.

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  But this is impossible, because the second half of w' must have $\mathtt{b}$ ’s.

- Thus no DFA recognizing L exists.

Minimum states for finite language recognition

- Any ***finite*** language L is recognized by an automaton!
- But how many states are needed?

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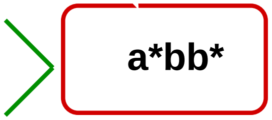
Minimum states for finite language recognition

- Any ***finite*** language L is recognized by an automaton!
- But how many states are needed?
- At least as many as the longest string-length in L.
- Proof: If M with k states recognizes a string longer than k, then Pumping applies, and L is infinite!

CONSTRUCTING AUTOMATA

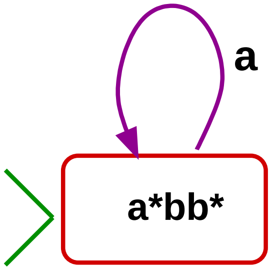
- We give a method that, given a language L, attempts to construct a DFA M recognizing L.
- If and when the process teminates, we obtain such an M.
- We start with a couple of non-trivial examples, before articulating the method and giving more examples.

Example: a ’s precede b ’s



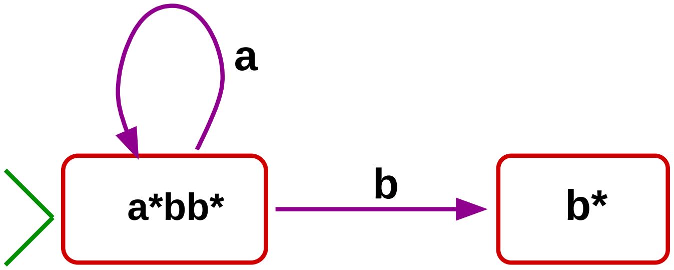
- Construct an automaton recognizing $\mathcal{L}(a^*bb^*)$. That is, accepting strings of a ’s followed by one or more b ’s, and ***only*** those.
- The initial state is the declaration of this goal.
- What will be an updated goal after reading an a ?

Reading an `a`



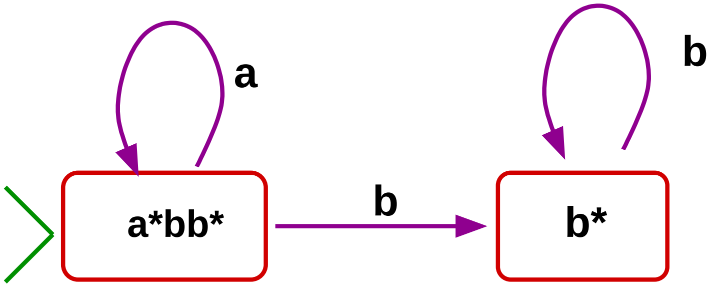
- The goal is unchanged!.
- But what happens if we read a `b` ?

Reading a b



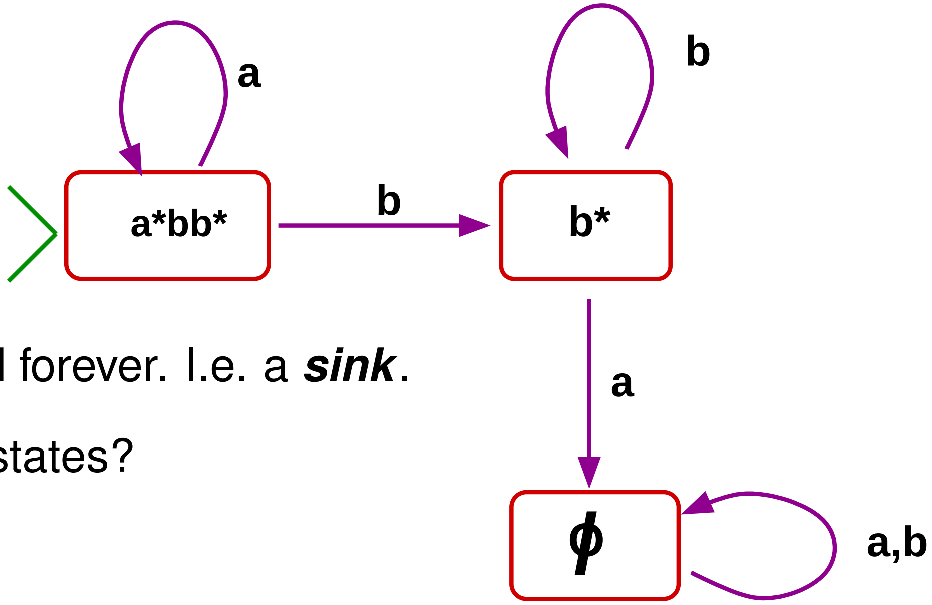
- A new goal: from now on only b ’s, any number.
- What if we read a b ***now***?

Reading another `b`



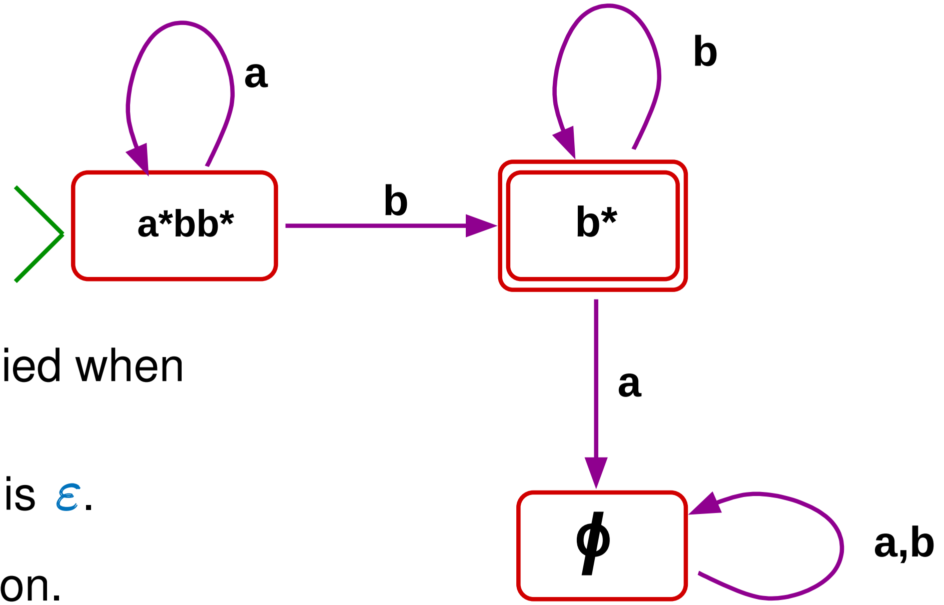
- No change.
- And what if, instead, we read an `a` ?

Reading an `a` *instead*



- This is a non-accept, now and forever. I.e. a ***sink***.
- And which are the accepting states?

What are the accepting states

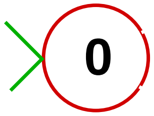


- Accept if current goal is satisfied when nothing left to read,
  i.e. when the current string is ε.
- This completes the construction.

Example: Ending as it starts

0 $\sigma w \sigma$

ε



*****

- Construct an automaton accepting strings $\sigma w \sigma$,

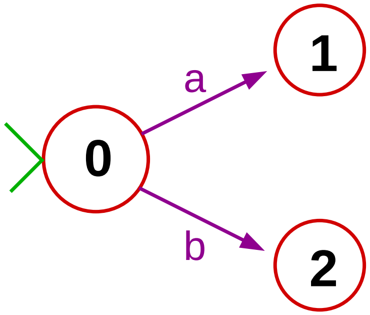
  i.e. with last letter identical to the first, and ***no others***.

- The initial state is the declaration of this goal.

- What will be the updated goals after reading the first letter?

Example: Ending as it starts

Reading the first letter:



0 $\sigma w \sigma$

1 $\varepsilon \mid w\mathrm{a}$

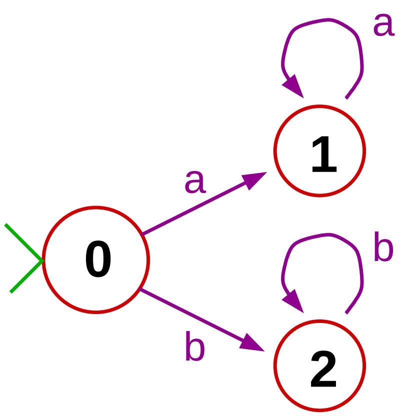
2 $\mid w\mathrm{b}$

*

- Either this is the last letter, or else it repeats at the end.
- What if we now read this letter again?

Example: Ending as it starts

Sought letter repeated:



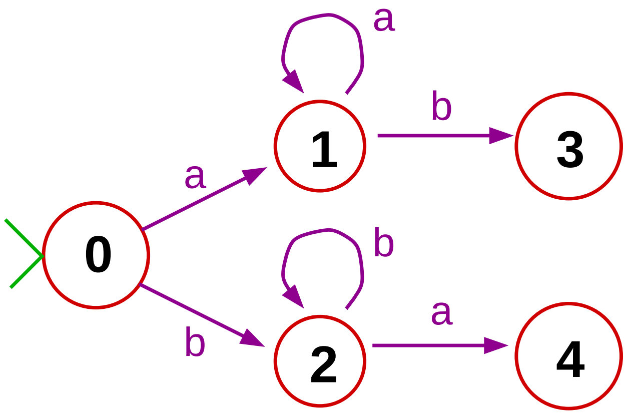
0	$\sigma w \sigma$
1	$\varepsilon \mid w\mathrm{a}$
2	$\varepsilon \mid w\mathrm{b}$

*

- The goal does not change.
- And what about the opposite letter ***now***?

Example: Ending as it starts

Reading opposite letter:



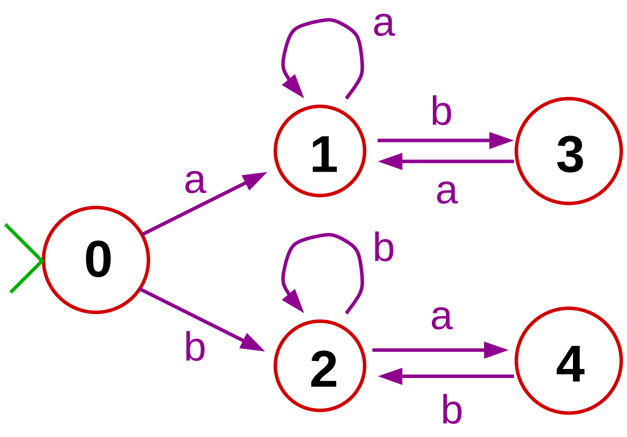
0	$\sigma w \sigma$
1	$\varepsilon \mid w$a
2	$\varepsilon \mid w$b
3	wa
4	wb

*

- The option of not reading further has been blocked.

Example: Ending as it starts

Opposite letter repeating:



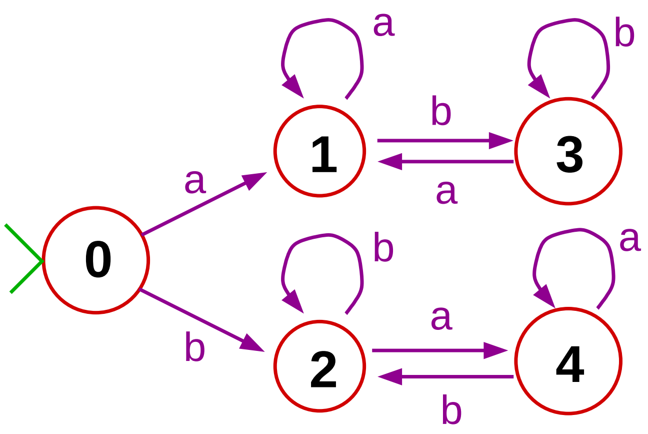
0	$\sigma w \sigma$
1	ε **|** wa
2	ε **|** wb
3	wa
4	wb

*

- But if the sought letter is read now,
  the previous goal is restored.
- And if we keep reading the wrong letter?

Example: Ending as it starts

Return to sought letter:



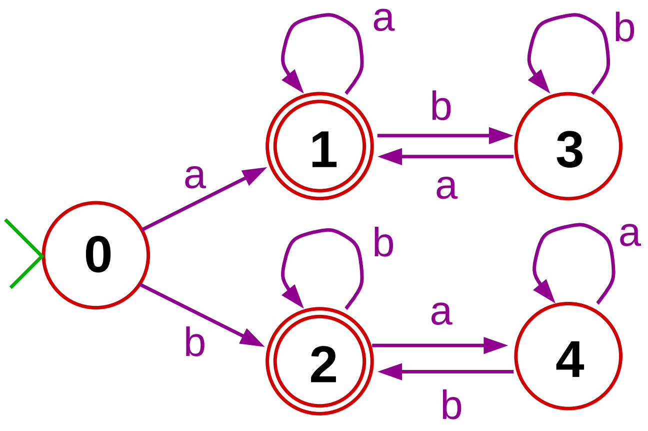
0	$\sigma w \sigma$
1	$\varepsilon \mid w\mathrm{a}$
2	$\varepsilon \mid w\mathrm{b}$
3	$w\mathrm{a}$
4	$w\mathrm{b}$

*

- No change of goal.
- What are the accepting states?

Example: Ending as it starts

The accepting states:



0	$\sigma w \sigma$
1	$\varepsilon \mid w\mathrm{a}$
2	$\varepsilon \mid w\mathrm{b}$
3	$w\mathrm{a}$
4	$w\mathrm{b}$

*

- Accept if current goal is satisfied when nothing left to read.
- This completes the construction.

Goal oriented automaton construction

- *When you head to an unfamiliar destination,
would you prefer the GPS map to display the road already covered,
or rather the road ahead?*

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- Programming is a ***goal oriented*** process.
  The relevant mission is to achieve a goal.
  The initial task of an acceptor for L is
  "accept the strings in L and no others"!

Goal oriented automaton construction

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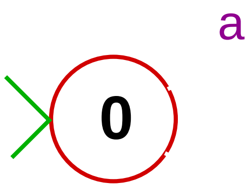
  or rather the road ahead?
- Programming is a ***goal oriented*** process.
  The relevant mission is to achieve a goal.
  The initial task of an acceptor for L is
  "accept the strings in L ***and no others"!***
- The tasks are adjusted as the input string is read.
  Each task is of the form

the string ahead leads into a string in L

Identifying accepting tasks

- The development above updates states (conditions) as required when symbols σ are read.
- A string $x = \sigma u$ satisfying the current condition (=state) leads to A iff u started at the next condition leads to A.
- So the accepting conditions are the ones that are satisfied when reading ends, i.e. when the string-ahead is ε.

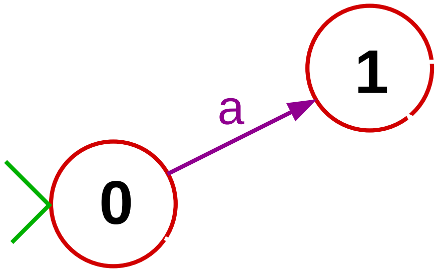
Example: Repeated last symbol



state dictionary

0	$w\sigma\sigma$

Example: Repeated last symbol



0	$w\sigma\sigma$
1	a **|** $w\sigma\sigma$

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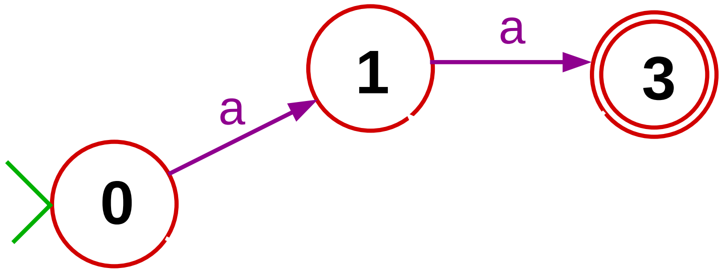

0 $w\sigma\sigma$

1 a **|** $w\sigma\sigma$

3 ε **|** a **|** $w\sigma\sigma$

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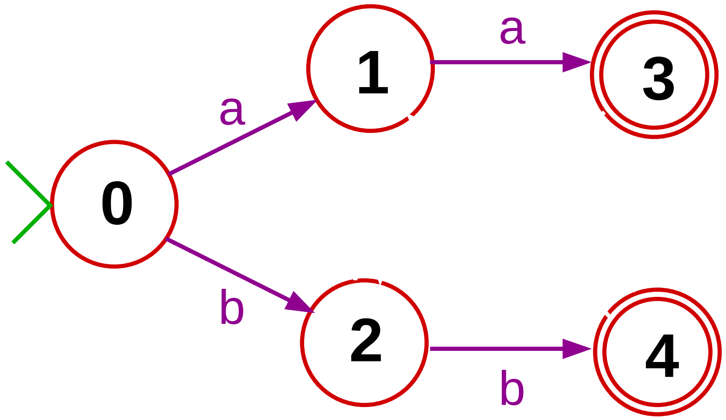
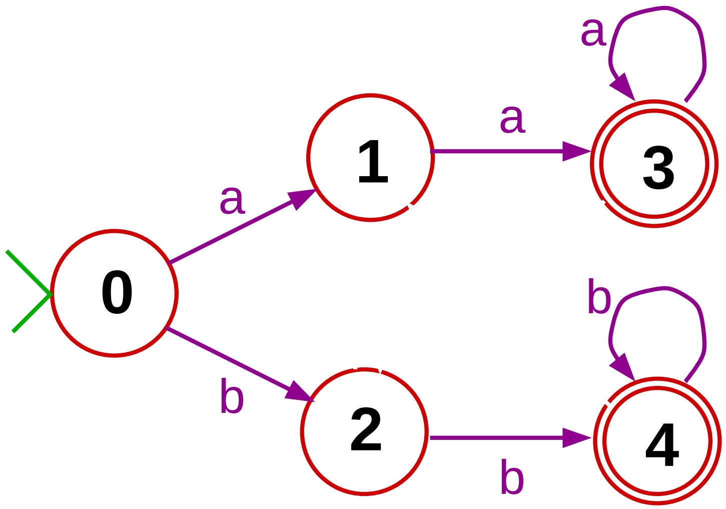


0	$w\sigma\sigma$
1	$\mathrm{a} \mid w\sigma\sigma$
2	$\mathrm{b} \mid w\sigma\sigma$
3	$\varepsilon \mid \mathrm{a} \mid w\sigma\sigma$
4	$\varepsilon \mid \mathrm{b} \mid w\sigma\sigma$

Example: Repeated last symbol



0	$w\sigma\sigma$
1	a **|** $w\sigma\sigma$
2	b **|** $w\sigma\sigma$
3	ε **|** a **|** $w\sigma\sigma$
4	ε **|** b **|** $w\sigma\sigma$

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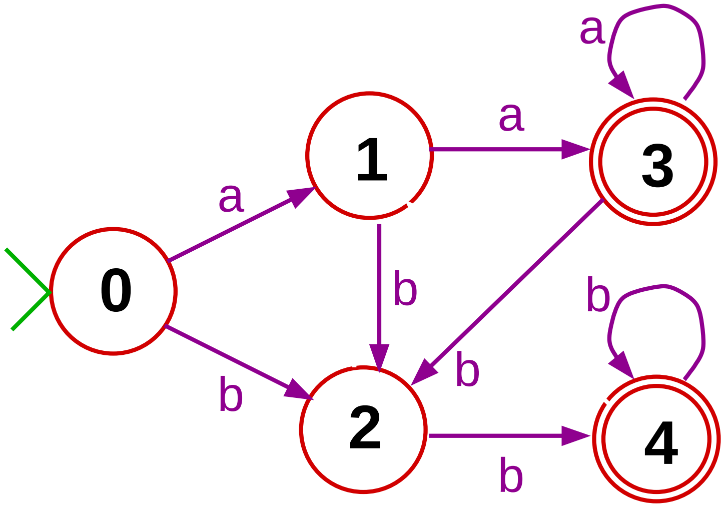

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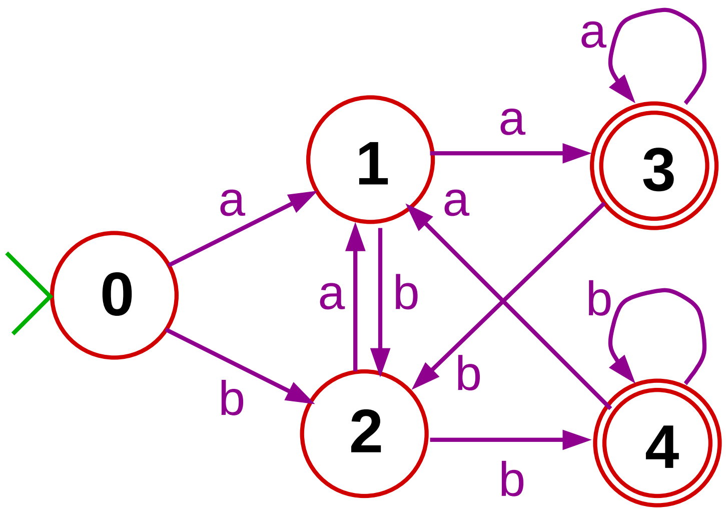
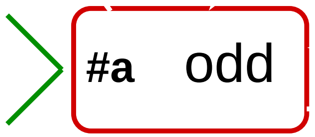


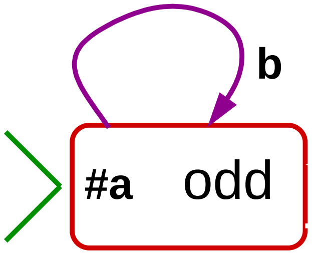
0	$w\sigma\sigma$
1	a $\mid$ $w\sigma\sigma$
2	b $\mid$ $w\sigma\sigma$
3	ε $\mid$ a $\mid$ $w\sigma\sigma$
4	ε $\mid$ b $\mid$ $w\sigma\sigma$

Example: Recognizing odd length



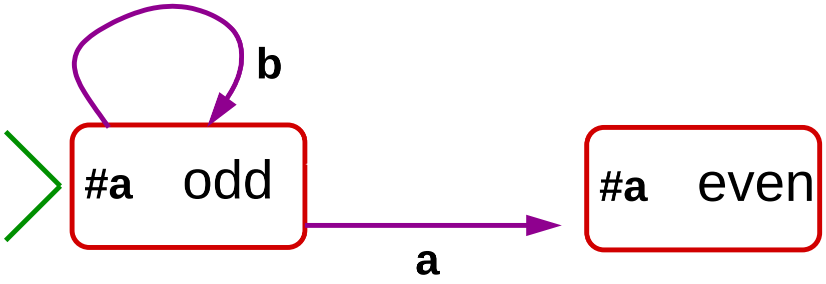
- Initial task: accept strings with an odd number of a’s

Example: Recognizing odd length



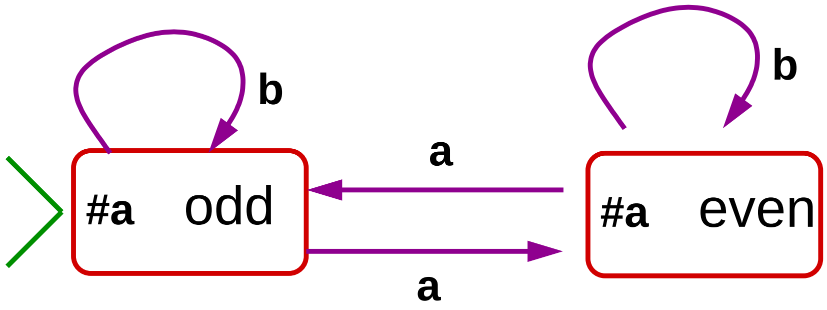
- Reading a b does not change the task

Example: Recognizing odd length



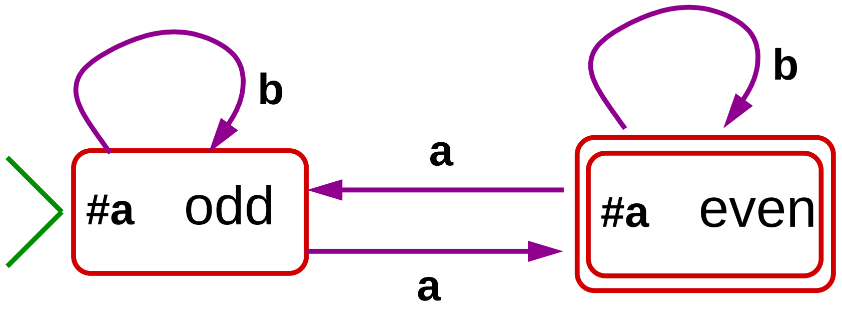
- Reading an a revises the task to:
  accept strings with an even number of a's

Example: Recognizing odd length



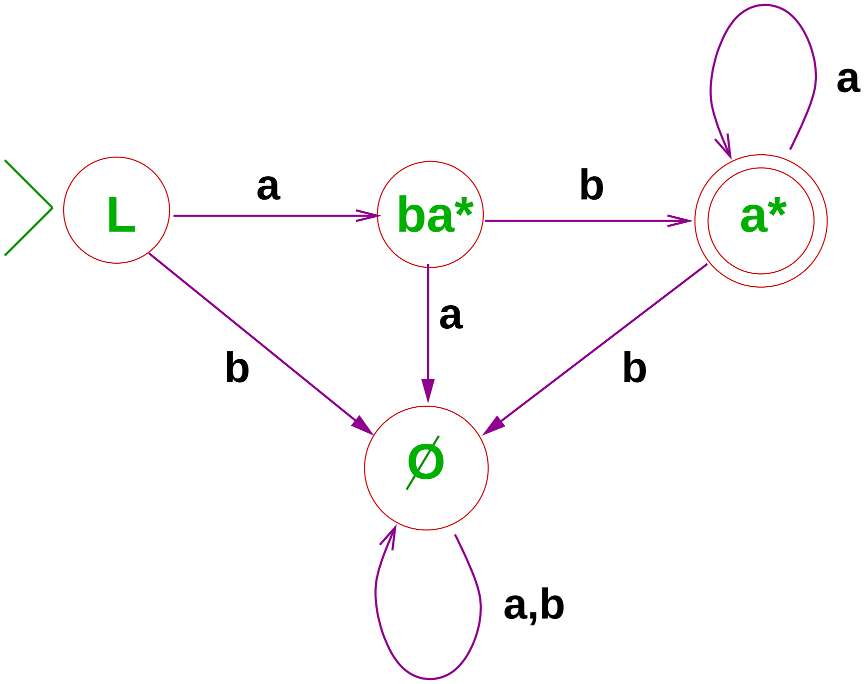
- Same reasoning for the “even” task

Example: Recognizing odd length



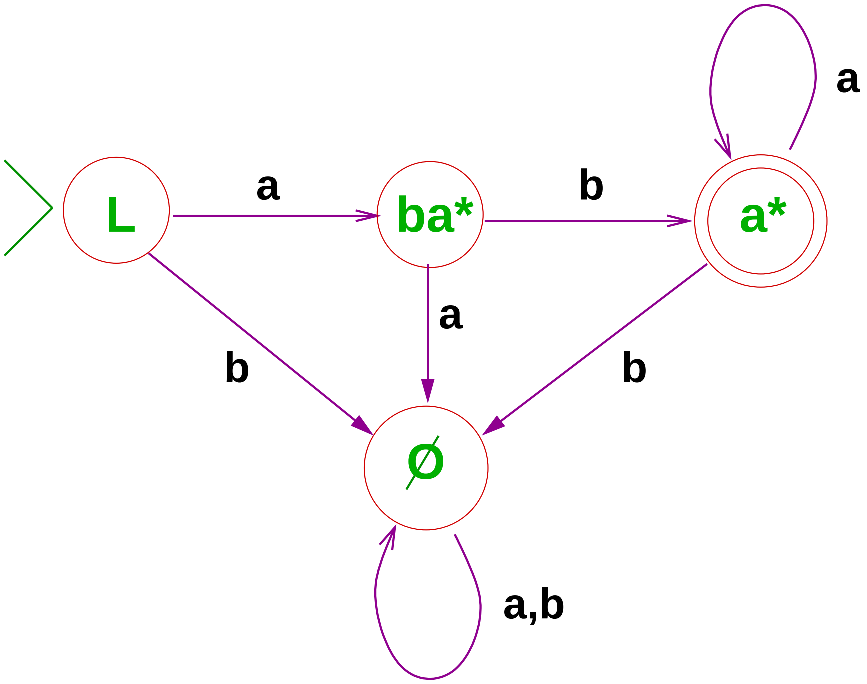
- Accept description fulfilled by ε.

Example: aba*



Accepts the strings of the form $\mathtt{aba}^n$ with $n \geqslant 0$, and no others.

Example: aba*

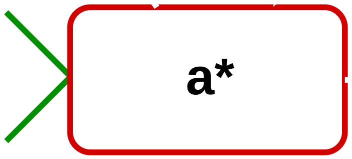


Accepts the strings of the form $\mathtt{aba}^n$ with $n \geqslant 0$, and no others.

- Note the sink at the bottom of the diagram.

A trivial example: Just $\mathtt{a}$ ’s

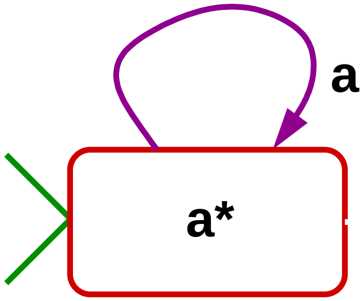
Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^{\star})$
as a sub-language of $\{\mathtt{a}, \mathtt{b}\}^{*}$



- Initial task: accept strings of $\mathtt{a}$’s

A trivial example: Just a *’s*

Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^{\star})$ as a sub-language of $\{\mathtt{a}, \mathtt{b}\}^{*}$



- Reading an a does not change the task

A trivial example: Just $\mathtt{a}$ *'s*

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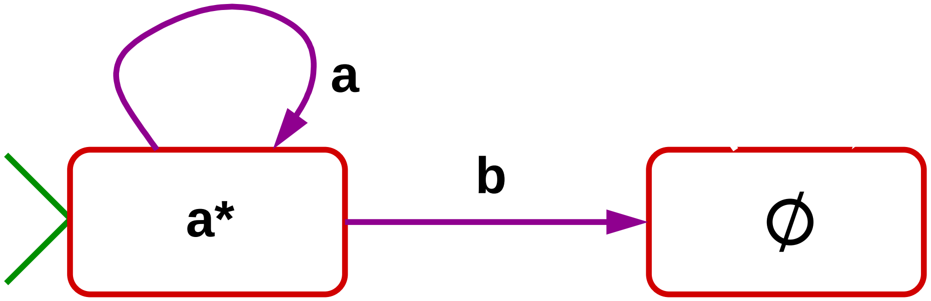
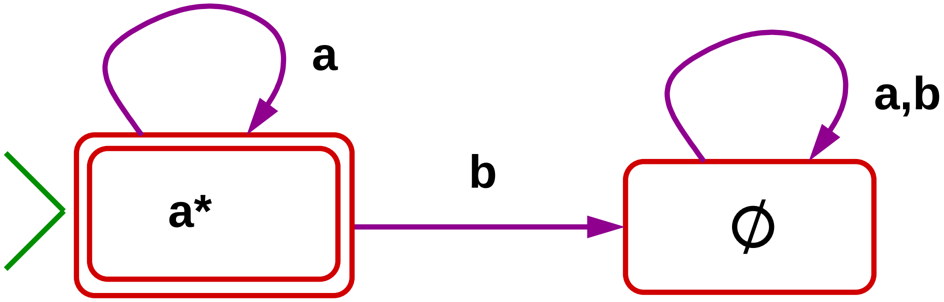


- Reading a $\mathtt{b}$ revises the task to not accepting anything. A ***sink.***

A trivial example: Just $\mathtt{a}$ *’s*

Construct an automaton recognizing $\mathcal{L}(\mathtt{a}^*)$ as a sub-language of $\{\mathtt{a},\mathtt{b}\}^*$

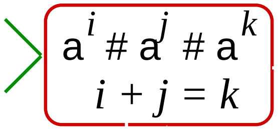


- No escape from the sink

Example: Addition mod 2

Automaton over $\{\mathsf{a}, \#\}$ recognizing

$$\{\mathsf{a}^i \# \mathsf{a}^j \# \mathsf{a}^k \;\mid\; i + j = k \text{ (mod 2) } \}$$



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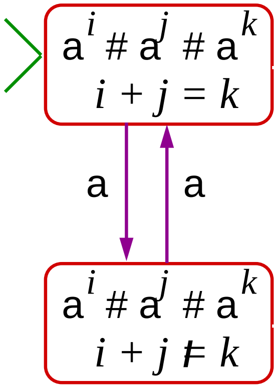
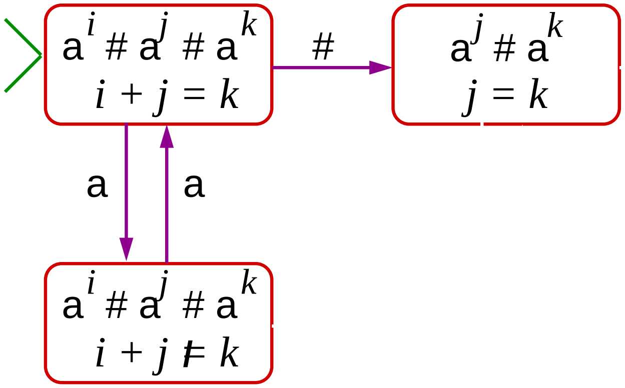

$\mathsf{a}^i \# \mathsf{a}^j \# \mathsf{a}^k$
$i + j = k$
a
a
$\mathsf{a}^i \# \mathsf{a}^j \# \mathsf{a}^k$
$i + j \neq k$

Reading a's toggles between equlity and inequality of parities.

Example: Addition mod 2

Automaton over $\{a, \#\}$ recognizing

$$\{a^i \# a^j \# a^k \mid i + j = k \text{ (mod 2) } \}$$



Reading the separator $\#$ means $i = 0$.

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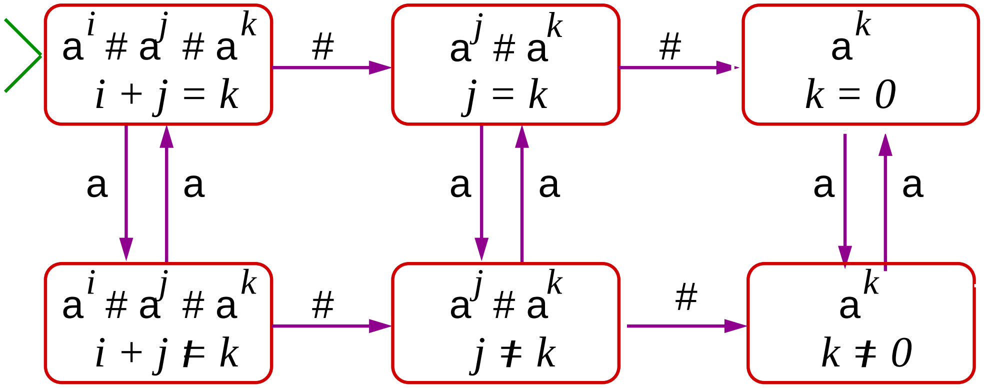
$$\{a^i \# a^j \# a^k \mid i + j = k \text{ (mod 2)} \}$$

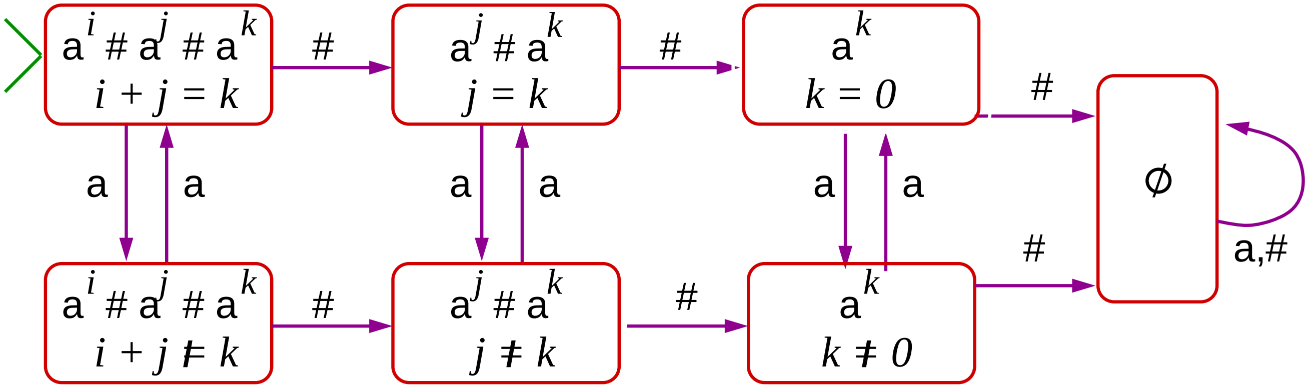


The same arguments are repeated

Example: Addition mod 2

Automaton over $\{a, \#\}$ recognizing

$$\{a^i \# a^j \# a^k \mid i + j = k \text{ (mod 2)} \}$$

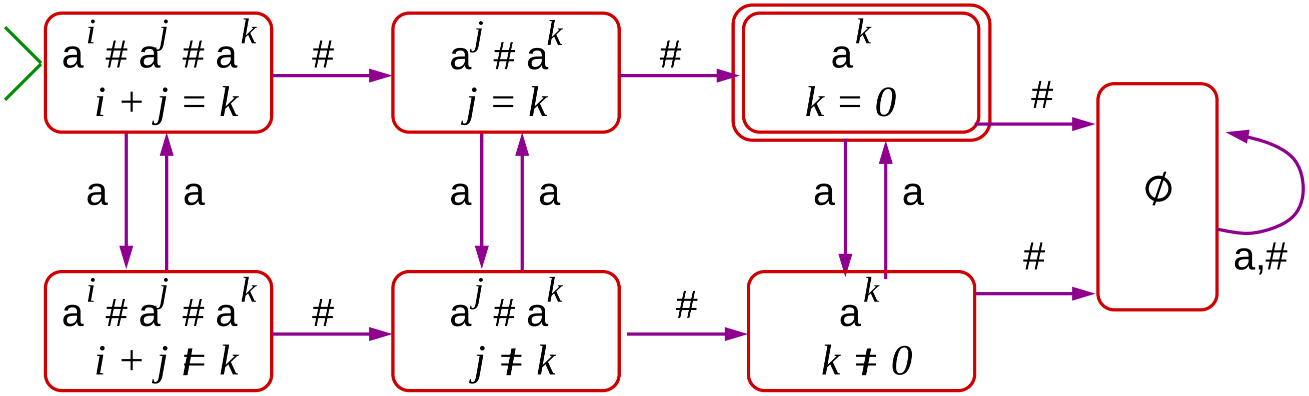


Encountering an extra separator leads to a sink

Example: Addition mod 2

Automaton over $\{a, \#\}$ recognizing

$$\{a^i \# a^j \# a^k \mid i + j = k \text{ (mod 2) } \}$$



The single one accepting state is the one satisfied by ε.

Summary of the method, again

- The initial acceptance-condition is the language to be recognized.
- Given a new acceptance-condition we calculate for each $\sigma \in \Sigma$ how reading σ leads to a new acceptance-condition.

  That is, a string $w = \sigma u$ satisfies the current acceptance condition iff u satisfies the acceptance-condition after σ is read.
- An acceptance-condition is an accepting state iff it is satisfied by ε.

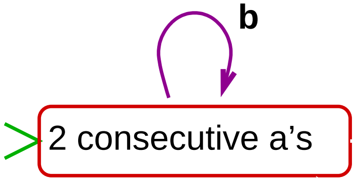
Example: Two consecutive $\mathtt{a}$*'s*

Construct an automaton recognizing $\mathcal{L}(\Sigma^* \cdot \mathtt{aa} \cdot \Sigma^*)$

2 consecutive a's

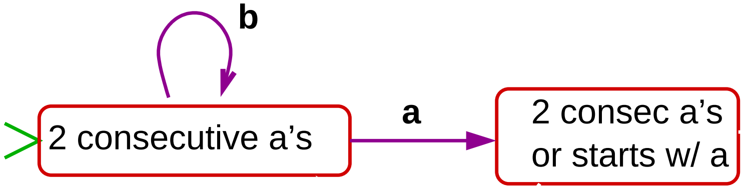
Example: Two consecutive `a`'s

Reading `b` leaves the task unchanged:



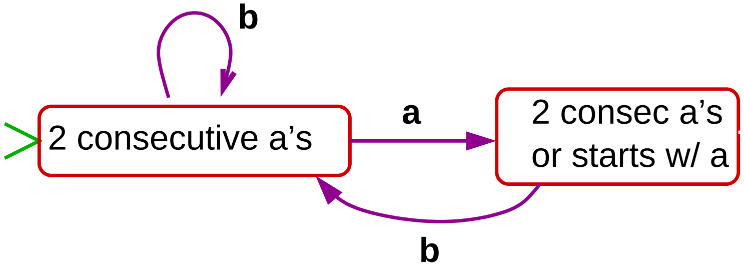
Example: Two consecutive a *’s*

But reading a opens two future options:



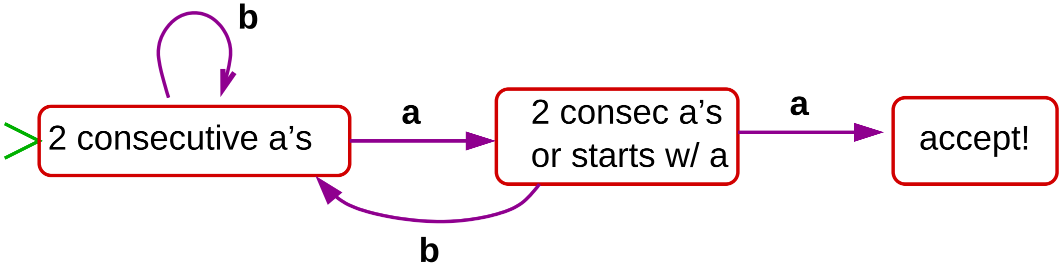
Example: Two consecutive a's

From these two options reading b kills the first:



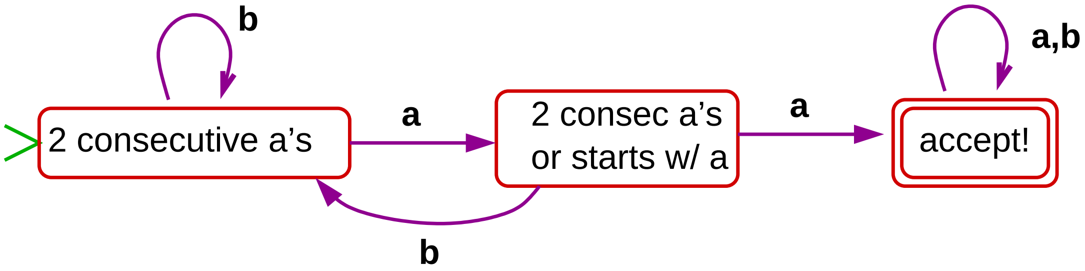
Example: Two consecutive a *'s*

But reading an a settles acceptance:

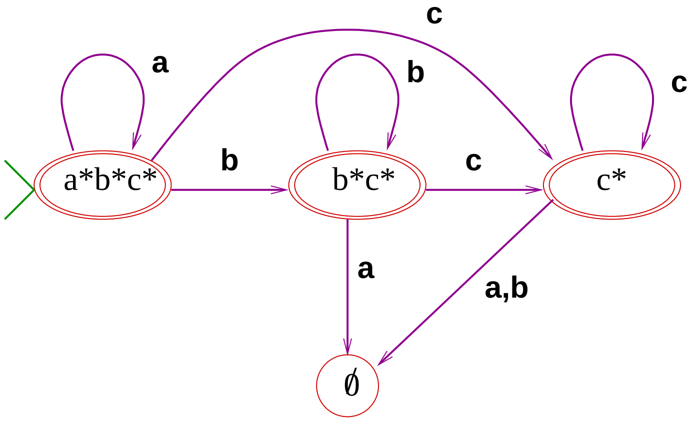


Example: Two consecutive `a` *’s*

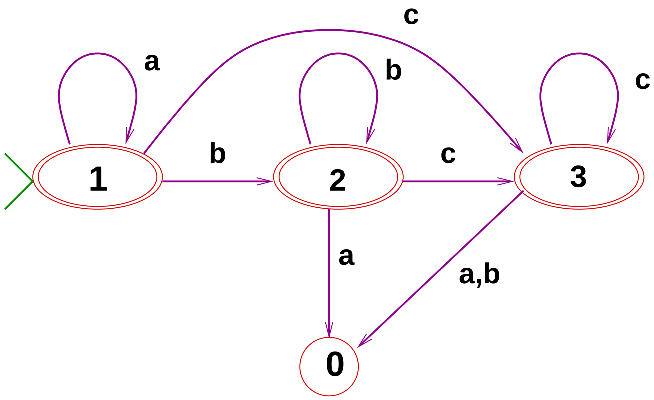
No further reading alterns that conclusion:



Example 7: `a*b*c*`

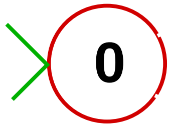


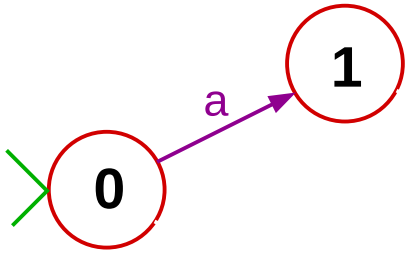
- Label states as we wish, with optional “dictionary.”



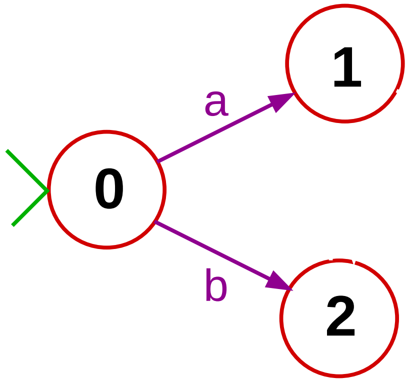
Example 8: Ends with two identical

0 $*\sigma\sigma$

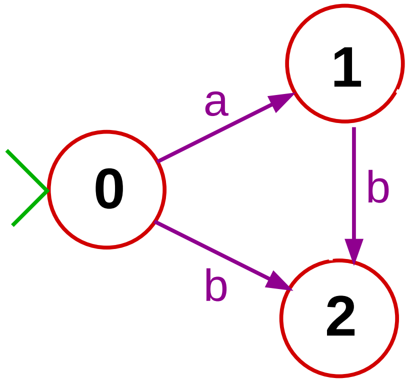




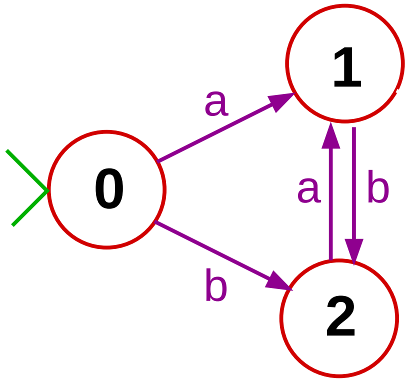
0	*σσ
1	a **|** *σσ



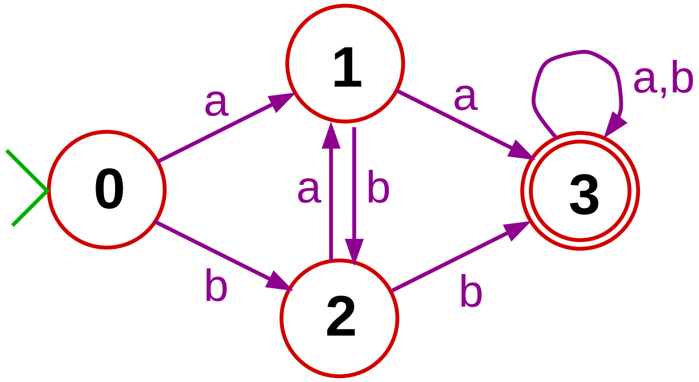
0	*σσ
1	a | *σσ
2	b | *σσ



0	*σσ
1	a **|** *σσ
2	b **|** *σσ

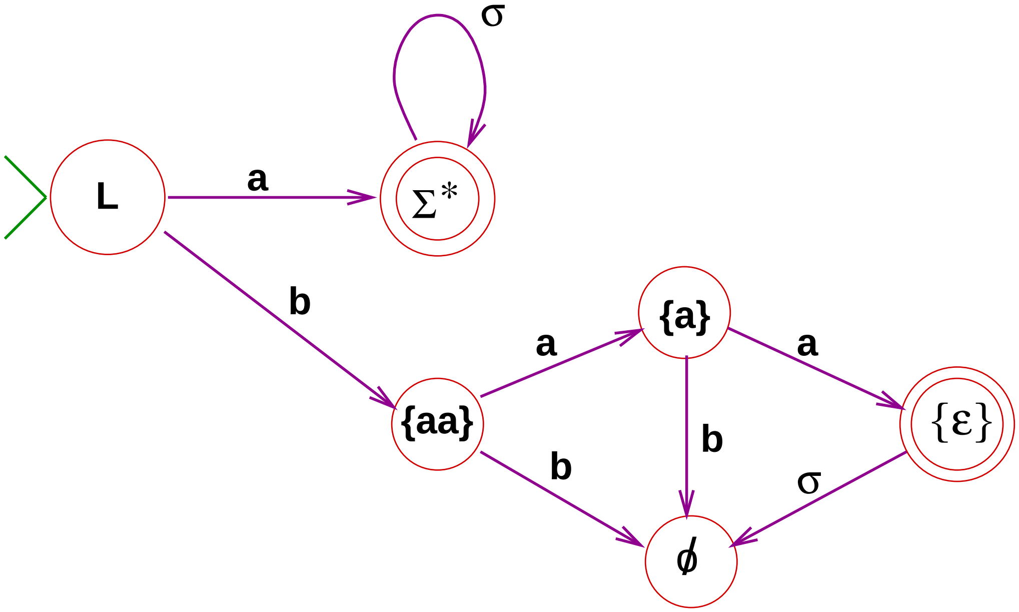


0 *σσ
1 a | *σσ
2 b | *σσ



0	*σσ
1	a | *σσ
2	b | *σσ
3	*

Example: Initial `a` *or the string* `baa`



Example: Symbolic binary addition

- The following example illustrates the use of compound data as “symbols” of an alphabet.

- Consider a long addition in binary, such as $$\begin{array}{cccccc} & 0 & 0 & 1 & 1 & 0 \\ + & 0 & 1 & 1 & 0 & 1 \\ \hline & 1 & 0 & 0 & 1 & 1 \end{array}$$

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- This table does not look like a string.
  But all such tables have height 3 we can consider each column as a “symbol” in the alphabet $\Sigma = \{0,1\}^3$, that is

$$\Sigma^3 = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$$

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- The long addition above can be consrued as the string $\begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\1\end{bmatrix}$

An automaton recognizing symbolic binary addition

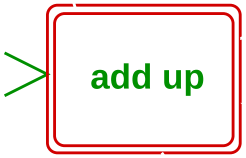
- Is there an automaton over Σ^3 that recognizes the correct symbolic binary additions?
- That is, can we construct an automaton M that accepts strings like

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

but not strings like

$$\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

An automaton recognizing symbolic binary addition



Start state is the goal that the table ***adds-up***:

remaining columns add up

An automaton recognizing symbolic binary addition

add up
carry over

Start state is the goal that the table ***adds-up***:

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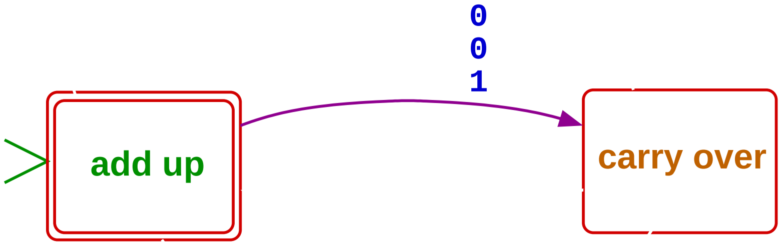
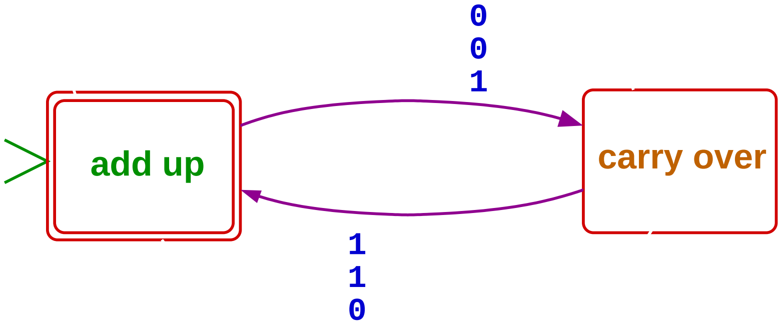
The main other state is *remaining columns yield* ***carry-over***

An automaton recognizing symbolic binary addition



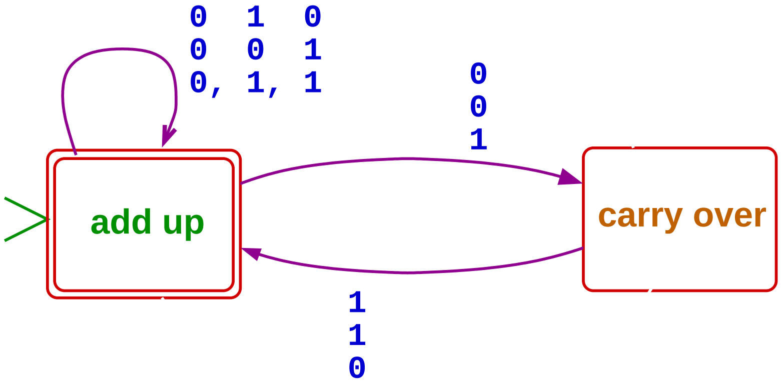
There is one column switching from ***add-up*** to ***carry-over***

An automaton recognizing symbolic binary addition



There is one column switching from ***add-up*** to ***carry-over***
and one column switching back from ***carry-over*** to ***add-up***

An automaton recognizing symbolic binary addition



Three columns leave the ***add-up*** goal unchanged

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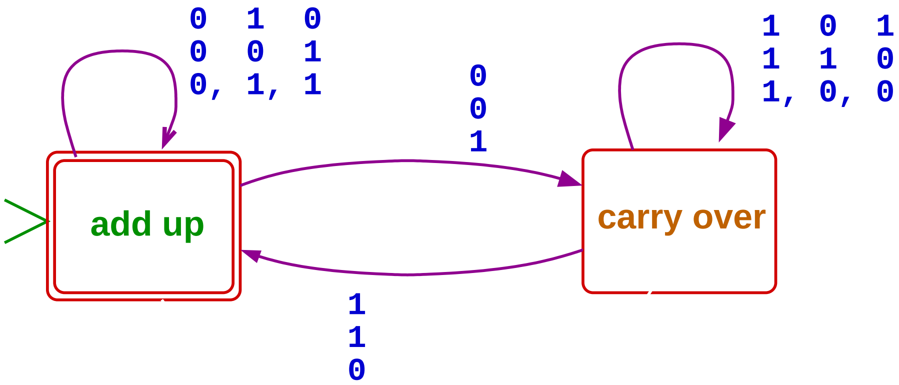

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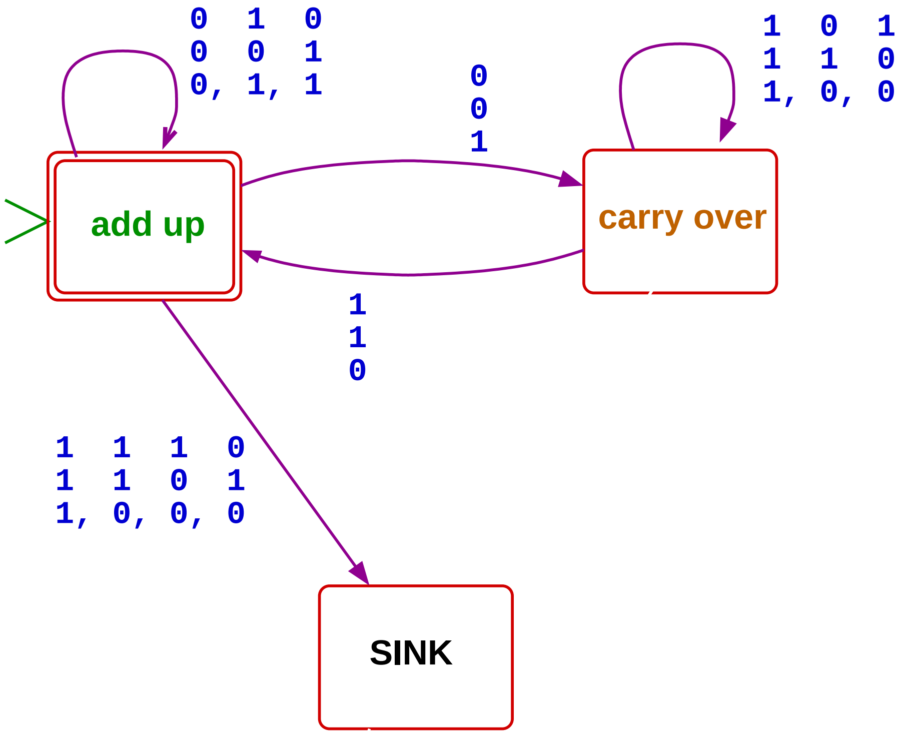

Four columns lead from ***add-up*** to a ***sink***

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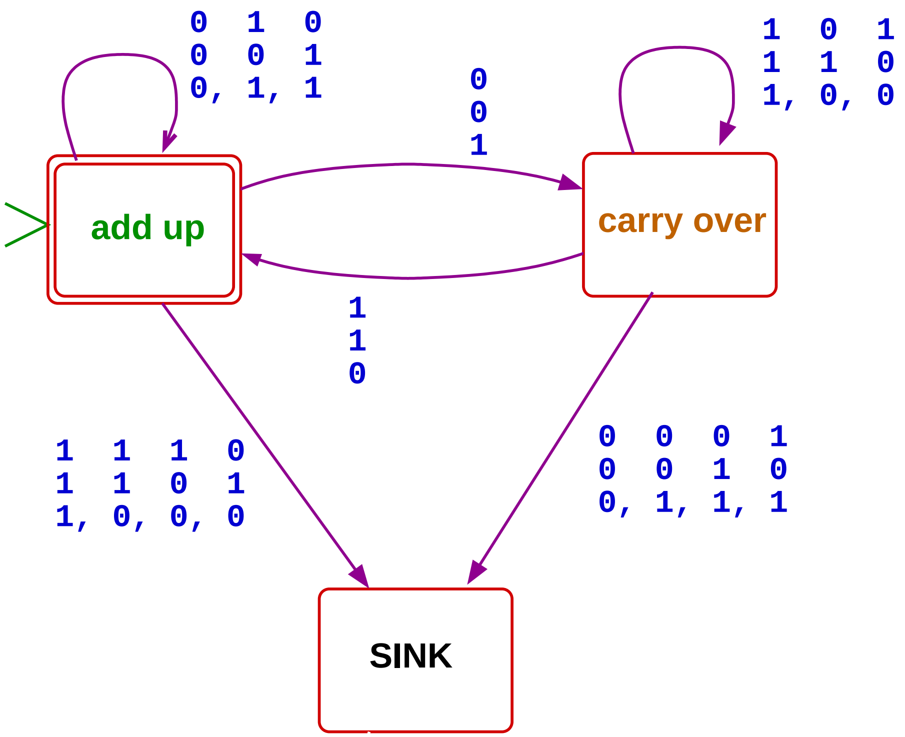
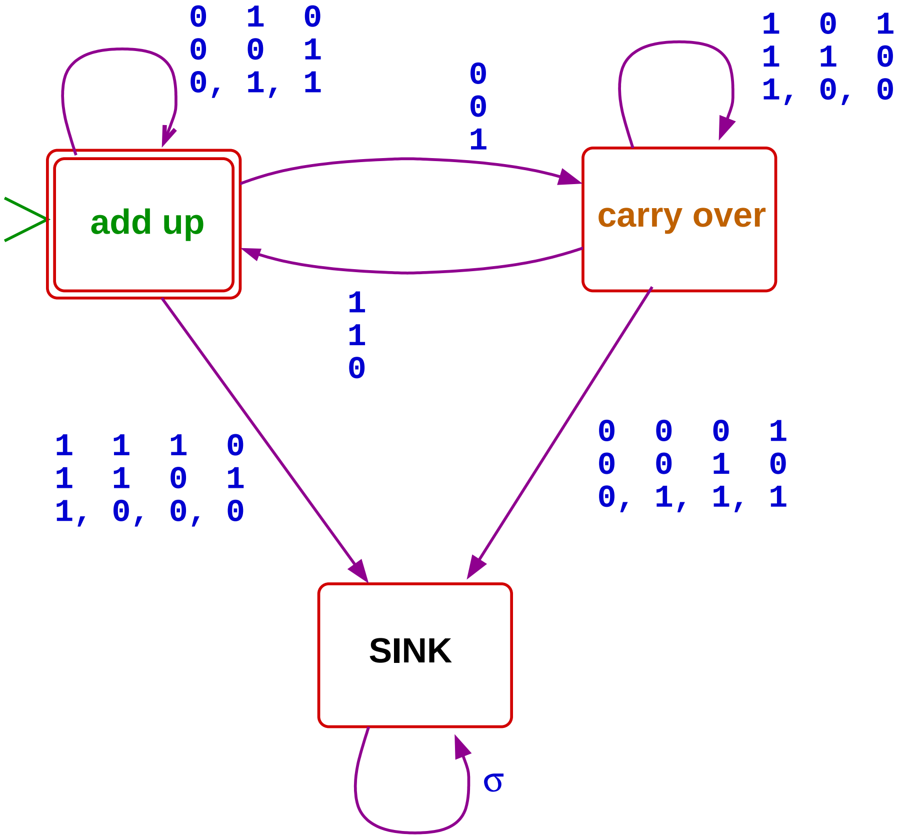


Four columns lead from ***add-up*** to a ***sink***
and four from ***carry-over*** to that ***sink***

An automaton recognizing symbolic binary addition



Finally, ***sink*** is a sink.

Example: Binary numerals divisible by 3

- Consider every string $w \in \{\mathtt{0}, \mathtt{1}\}^*$ to be a binary numerals.
- The ***numeric value*** $[w]_2$ of a string $w = d_k d_{k-1} \cdots d_0$ is $\Sigma_i\ 2^i$.
- The numerals divisible by 2 are those that end with $\mathtt{0}$.

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We have that $4^k =_3 1$, by induction on k.

  - ▸ $4^0 = 1$
  - ▸ If $4^k = 3x + 1$ then $4^{k+1} = 4(3x+1) = 13x + 1.$

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So $2^{2k} = 3x + 1$ for some x , and $2^{2k+1} = 2(3x+1) = 6x+2$.

$\therefore$ $2^n =_3 1$ for even n , and $=_3 2$ for odd n .

Example: Binary numerals divisible by 3

- For any input w the expectation depends on the parity of $|w|$, the goals are therefore of the form

  Either $|w|$ is even and $[w] =_3 x$ or $|w|$ is odd and $[w] =_3 y$

  Let's abbreviate this as (x, y).

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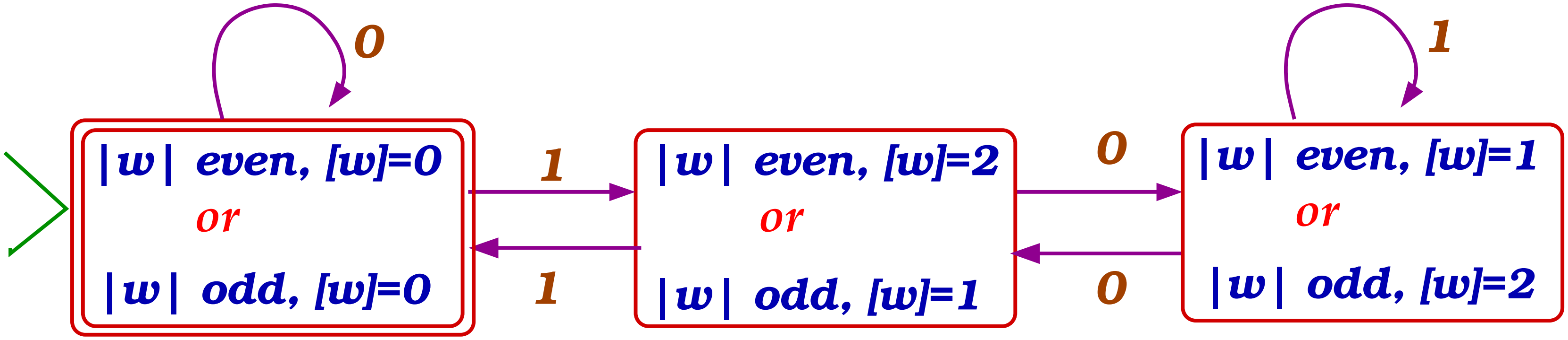
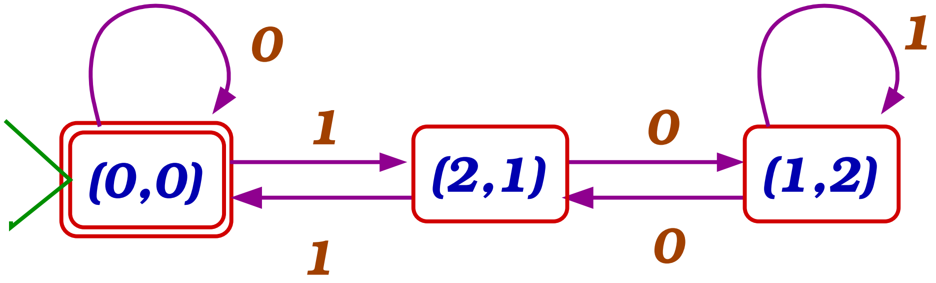
- From the observation above it follows that $(x,\ y) \xrightarrow{1} (y{+}2,\ x{+}1)$, and $(x,\ y) \xrightarrow{0} (y,\ x)$.

- This yields the following DFA:



Condensed:



RESIDUES AND THEIR APPLICATIONS

More examples of residues

- Take $L =$ English words.

  $L/\mathtt{invent}$ contains the strings $\mathtt{or}, \mathtt{ion}, \mathtt{ive}, \mathtt{ed}$ and $\mathtt{ing}$
  since $\mathtt{inventor}, \mathtt{invention}, \mathtt{inventive}$ and $\mathtt{invented}$ are words.

- ϵ is also in $L/\mathtt{invent}$ since invent is a word.

- The residue $L/\mathtt{ad}$ contains the strings $\mathtt{vance}, \mathtt{apt}, \mathtt{opt}, \mathtt{d},$ and ϵ.

- Take $L = \{\mathsf{ab}\}$, a singleton language.
  We have $L/\varepsilon = \{\mathsf{ab}\}, L/\mathsf{a} = \{\mathsf{b}\}$, and $L/\mathsf{ab} = \varepsilon$.

  For any other string w, $L/w = \emptyset$.

- For any language L we have $L/\varepsilon = L$:

  $w \in L$ iff $\varepsilon \in L/w$.

More examples yet

- $L = \{0,\ 00,\ 010\}$

$$\begin{aligned} L/\varepsilon &= L \\ L/0 &= \{\varepsilon,\ 0, 10\} \\ L/00 &= \{\varepsilon\} \\ L/01 &= \{0\} \\ L/010 &= \{\varepsilon\} \\ L/w &= \emptyset \text{for any other} w \end{aligned}$$

$L/00 = L/010$, so there are five (different) residues.

An example with language union

- $L = \{\mathtt{a}w \mid w \in \Sigma^*\} \cup \{\mathtt{baa}\}$.

$$\begin{array}{rcll} L/\varepsilon & = & L & \\ L/w & = & \Sigma^* & \text{if } w \text{ starts with } \mathtt{a} \\ L/\mathtt{b} & = & \{\mathtt{aa}\} & \\ L/\mathtt{ba} & = & \{\mathtt{a}\} & \\ L/\mathtt{baa} & = & \{\varepsilon\} & \\ L/w & = & \emptyset & \text{for any other } w \end{array}$$

There are 6 residues.

L and Σ^* are infinite languages, the others are finite.

A single-letter language

- $\Sigma = \{0, 1\}$, $L = \{0\}^*$.

- If $w \in \Sigma^*$ contains 1 then $L/w = \emptyset$.
  Otherwise $L/w = L$.
  There are two residues.

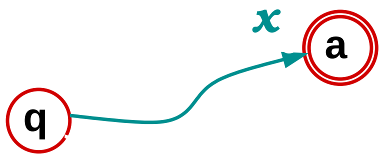
A language based on occurrence count

- $L = \{w \in \{0, 1\} \mid \#_0(w)$ is even $\}$.

  If $\#_0(w)$ is even then L/w is L,
  otherwise $L/w = \{w \mid \#_0(w)$ is odd $\}$

Each state determines a language

- Consider a DFA M recognizing L and a state q in it.
  Some string x may lead from q to acceptance.



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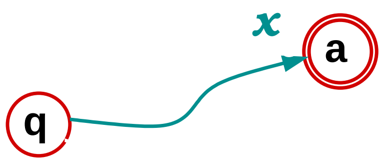
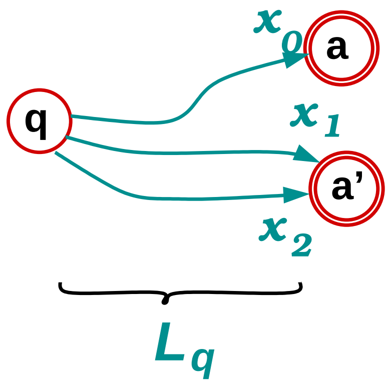


- Denote the set of all such x's by L_q.
  In particular, $L_s = L$.



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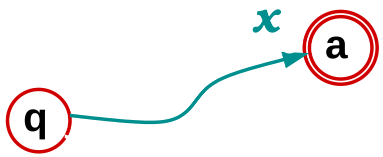

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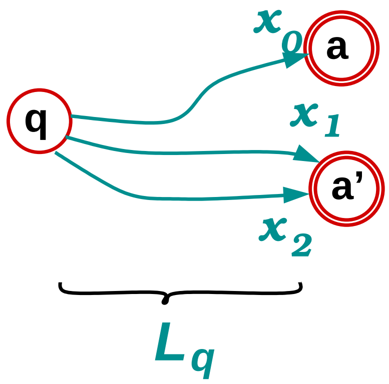

- Note: We focus on the future of q , not its past!
  (The past would be the set of strings leading to q)

States and residues

- Now suppose that $s \xrightarrow{w} q$.

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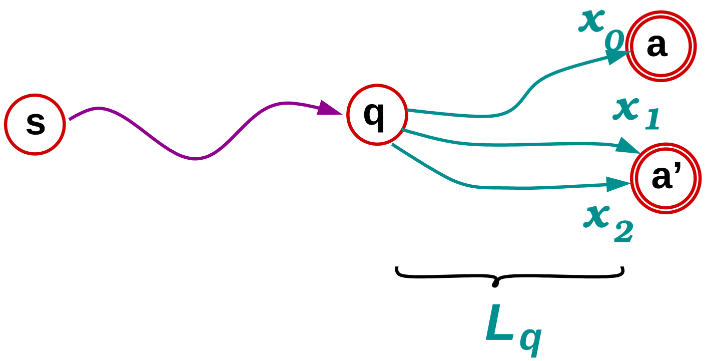

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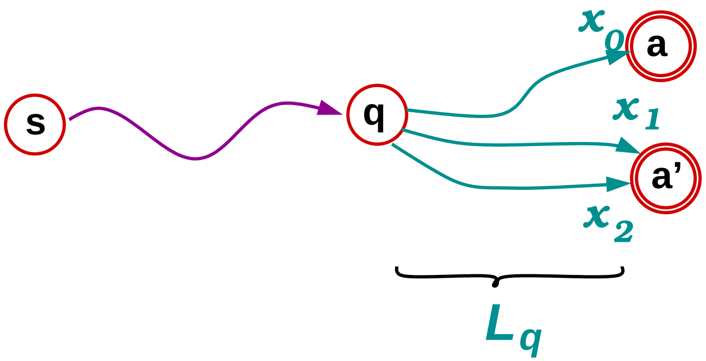
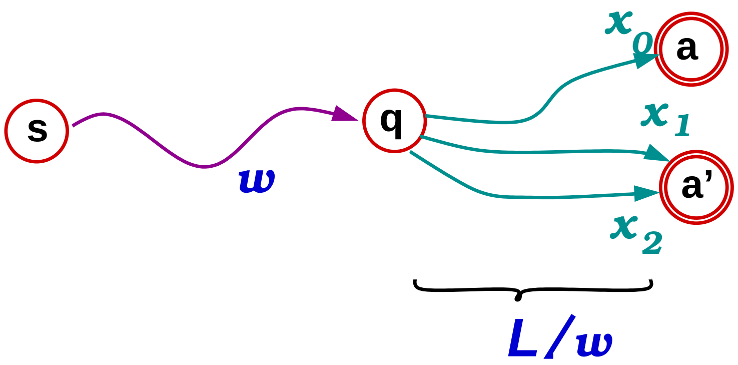


- L_q is $L/w =$ the residue of L over w:



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- **Theorem.** (Myhill-Nerode) ***A language recognized by a k-state DFA has $\leqslant k$ residues.***

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- **Theorem.** (Myhill-Nerode) ***A language recognized by a k-state DFA has $\leqslant k$ residues.***
- Proof. If $s \xrightarrow{u} q$ and $s \xrightarrow{v} q$ then $L/u = L/v$.
- Consequently:
  Theorem.
  A language with infinitely many residues is not recognized.

Languages with infinitely many residues

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$$L/1^n = \{x \mid \#_0(x) = \#_1(x) + n\} \ ,$$
since to compensate for an initial substring of n 1's the rest of the string should have n extra 0's.

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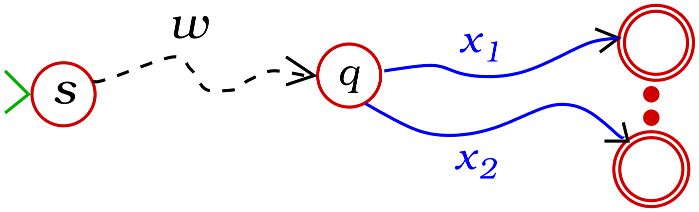
$\therefore$ L is not recognized, since it has infinitely many residues.

States and residues

- We developed automata by thinking of residues as states.
- Let M be an automaton over Σ.
  For a state q of M define
  $$L_q =_{\mathrm{df}} \{x \in \Sigma^* \mid q \xrightarrow{x} A\}$$
- In particular, for the start state $L_s = L$.
- If $s \xrightarrow{w} q$ then $L_q = L/w$.

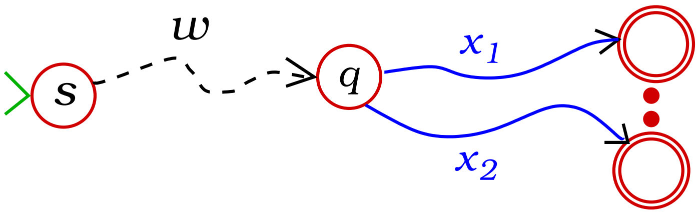


$L_q = L / w$

⋆ Each string leads from s to some state.

⋆ All strings leading from s to a state q have the same residue.

The Myhill-Nerode Theorem



$L_q = L / w$

- Every residue L/w is L_q for q as above.
- And two different residues $L/w \neq L/x$ must correspond to two different states.
- So we have an injection that maps residues to states,
  I.e. the number of residues is bounded by the number of states.
- **Theorem.** (John Myhill and Anil Nerode (1958)) (simplified and rephrased):
  $\mathcal{L}(M)$ cannot have more residues than M has states.
- Consequence: *A language with infinitely many residues cannot be recognized by any automaton!*

Showing that a language fails recognition

- We saw that $L = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$ has infinitely many residues.
- Consequence: It cannot be recognized by any automaton!!!
- General method: show that L is not recognized by showing that there are infinitely many residues.
- We do not need to consider all residues, ***only some infinite selection, defined by a template***
- We ***do not need to calculate*** the residues we choose, ***only show that each two of them are different***.
- We show them different by exhibiting a string which is in one but not in the other.

Example: Unary addition

- Representing unary addition, using unary numerals and the symbols for addition and equality:
- $L = \{\mathtt{1}^k + \mathtt{1}^m = \mathtt{1}^{k+m} \mid k, m \geqslant 1\}$
- What residues would you select?

- $L/\,\mathtt{1}^n + \mathtt{1} =$ for each $n \geqslant 1$.
- Suppose $i \neq j$.
  What string is in $L/\,\mathtt{1}^i + \mathtt{1} =$ but not in $L/\,\mathtt{1}^j + \mathtt{1} =$?

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- Since each two of these residues are different,
  L has infinitely many residues,
  and cannot be recognized by a DFA.

Example: Residues for perfect squares

- $L = \{\mathtt{1}^{n^2} \mid n \geqslant 0\}$.
- Consider the residues $L/\mathtt{1}^{n^2}$ for each $n > 0$.
- The first perfect square following n^2 is $(n{+}1)^2 = n^2 + 2n + 1$.
- So the shortest non-null string of $L/\mathtt{1}^{i^2}$ is $\mathtt{1}^{2i+1}$.
- It follows that $\mathtt{1}^{2i+1} \in L/\mathtt{1}^{i^2}$
  but $\mathtt{1}^{2i+1} \notin L/\mathtt{1}^{j^2}$ for any $j > i$.
- Since every two of these residues are different,
  L has infinitely many residues,
  and cannot be recognized by any automaton.

Building automata directly from residues

- We showed that every recognized language has finitely many residues.
- The converse is also true:
- If $L \subseteq \Sigma^*$ has finitely many residues, then $L = \mathcal{L}(M)$ where:
  - ⋆ The states of M are the residues.
  - ⋆ The initial state is $L/\varepsilon = L$.
  - ⋆ A state L/w is accepting iff it contains ε.
  - ⋆ The transitions are given by $L/w \quad \xrightarrow{\sigma} \quad L/w\sigma$.
- We used the same idea to construct automata, except that here we assume that the residues are given to us.
- We write *Res*(L) for the automaton constructed from residues.

Recognized = finitely many residues

- A language L is recognized iff it has finitely many residues.
- The DFA constructed from L's residues
  has the fewer states
- Given a DFA M recognizing L, and a state q,

Minimizing an automaton: Rationale

- Suppose M is a k-state DFA over Σ, recognizing L.

  For each accessible state q the language L_q is a residue of L. If M is the smallest automaton recognizing L

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- M might be constructed using residues as states
  and yet not be minimal, because the same residue might have been introduced twice for different property descriptions.

  But when M is not minimal we can still obtain
  a minimal automaton by identifying duplicates and unifying them.

Minimizing an automaton: Separating residues

- Say that a string x ***separates*** q from q'
  if x is in one of L_q and $L_{q'}$ but not in the other.
  That is, x is a witness for $L_q \neq L_{q'}$.
- Write $q \mathrm{D} q'$ if there is such an x ,
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  - Note: $\mathrm{D}_{n+1} \supseteq \mathrm{D}_n$.
  - Let's show that if $\mathrm{D}_{n+1} = \mathrm{D}_n$ then $\mathrm{D}_{n+2} = \mathrm{D}_{n+1}$

Minimizing an automaton: Bounding the separator

- Suppose $q \ \mathrm{D}_{n+2} q'$, i.e. some σx of length $n{+}2$ separates between q and q' .
  Let $q \xrightarrow{\sigma} p$ and $q' \xrightarrow{\sigma} p'$.
  Then x separates between p and p' , so $pdm_{n+1}p'$.
- But we assume $\mathrm{D}_{n+1} = \mathrm{D}_n$, so $p \ \mathrm{D}_n p'$,
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- Conclusion: For some n $\mathrm{D}_0 \subset \mathrm{D}_1 \subset \mathrm{D}_2 \subset \cdots \subset \mathrm{D}_n = \mathrm{D}_{n+1} = \mathrm{D}_n$
  where $n \leqslant$ the number of pairs of distinct states.
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- Conclusion: If $q\ \mathrm{D}q'$ then q, q' are separated
  by a string of length $\leqslant k(k-1)/2$.

Minimization algorithm for DFAs

Outline of a ***minimization algorithm***:
Given a k -state DFA M recognizing L :

1. For each pair q, q' determine if $L_q = L'_q$ by checking all strings of length $k(k-1)/2$.

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Given a k -state DFA M recognizing L :

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2. Obtain the minimal DFA recognizing L by unifying equivalent states.

MODIFYING & COMBINING AUTOMATA

Partial-automata

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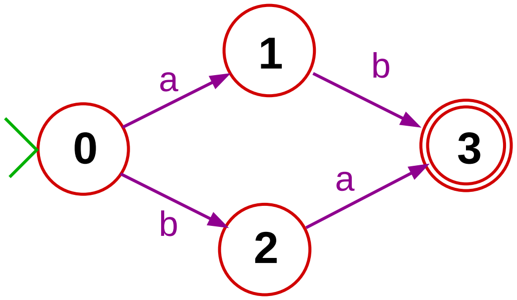

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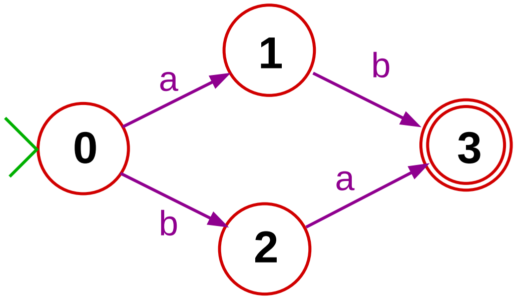

- Some people use “automaton” for our “partial-automaton” and “total-automaton” for our “automaton.”

From partial- to total-automaton

- **Theorem.** Every partial-automaton M can be converted into a total-automaton $\bar{M}$ equivalent to M, i.e. recognizing the same language.

  Do you seee how?

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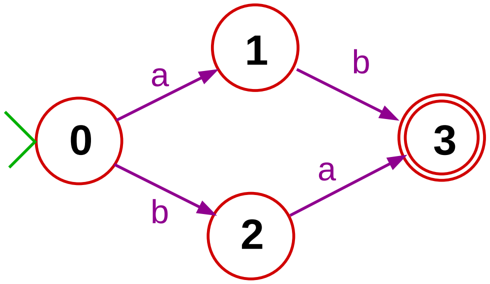
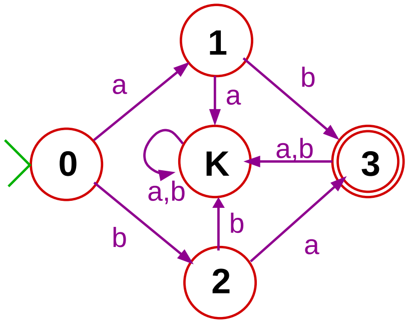


to



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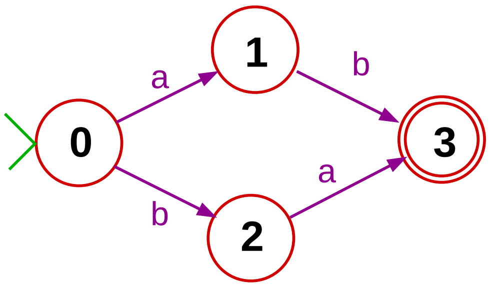

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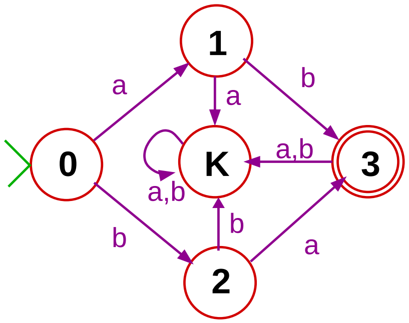

- That is, $\bar{M}$ is obtained by adding to M
  a sink state K , with all missing transitions of M
  as well as outgoing transition from K , pointing to K .

Example: Equiping strings with start signal

- $M = (\Sigma, Q, s, A, \delta)$ is a partial-automaton recognizing L .
  Convert M to M' recognizing $\mathtt{a} \cdot L$.
  ($\mathtt{a}$ can be construed as a start-signal.

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  Convert M to M' recognizing $\mathtt{a} \cdot L$.
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  Fix some $t \notin Q$ and let M' be
  M augmented with t as the new start state,
  and the transition $q \xrightarrow{a} s)$

Example: Equiping strings with end signal

- Let $\Box \notin \Sigma$.
  Convert M to M'' recognizing $L \cdot \Box$.

Example: Equiping strings with end signal

- Let $\square \notin \Sigma$.

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  This construction won’t work if $\square \in \Sigma$, why?

The complement of a recognized language

- **Theorem.** *If* $L \subseteq \Sigma^*$ *is recognized then so is* $\bar{L} = \Sigma^* - L$.

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  An automaton recognizing L is converted into one for $\bar{L}$.

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- Given DFA M , how do you get a DFA $\bar{M}$
  that accepts when M rejects, and rejects when M accepts?

The complement of a recognized language

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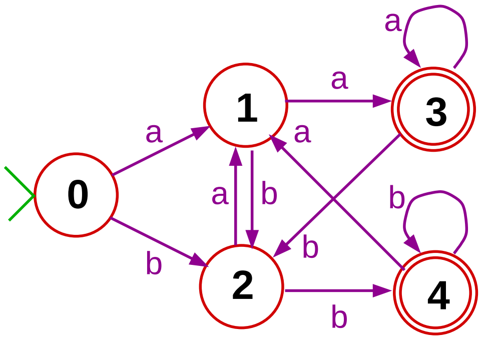
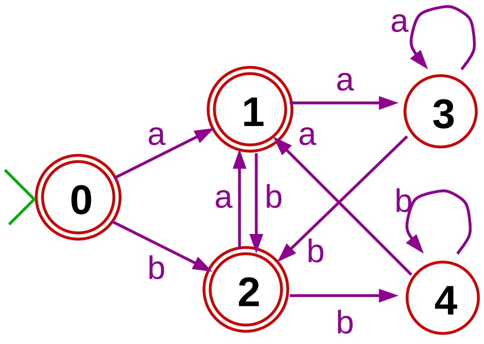
  For example, the automaton recognizing $\{w\sigma\sigma \mid w \in \Sigma^*, \sigma \in \Sigma\}$



is converted to



which accepts the strings of length < 2 and the strings ending with two different letters.

Application: Additional languages recognized

- Suppose M recognizes $\{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) = \#_b(w) \mod 2\}$.
- Then swapping states in M yields an automaton recognizing

$$\{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w) \mod 2\}$$

Application: Showing a language not-recognized

- Show that $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w)\}$ is not recognized.

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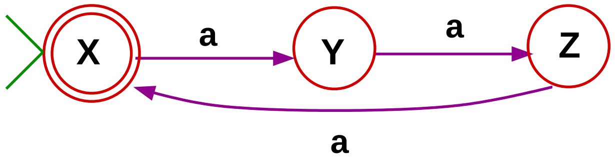
Application: Showing a language not-recognized

- Show that $L = \{w \in \{\mathtt{a},\mathtt{b}\}^* \mid \#_a(w) \neq \#_b(w)\}$ is not recognized.
- Clipping doesn't work!
- Use Clipping to show that the complement
  $\bar{L} = \{w \in \{\mathtt{a},\mathtt{b}\}^* \mid \#_a(w) = \#_b(w)\}$ is not recognized.
- Conclude: L is not recognized, or else $\bar{L}$ would be.

Combining two automata

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

- Suppose M_3 recognizes $L_3 = \{w \in \Sigma^* \mid \#_a(w) = 0 \mod (3)\ \}$



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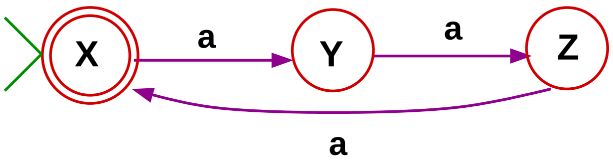
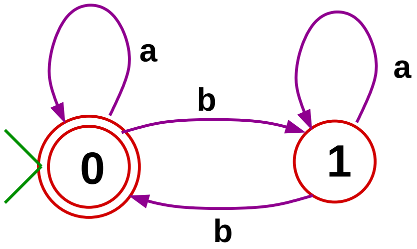


and

- M_2 recognizes $L_2 = \{w \in \Sigma^* \mid \#_b(w) = 0 \mod (2)\}$.



$\#_b w = 0 \mod 2$

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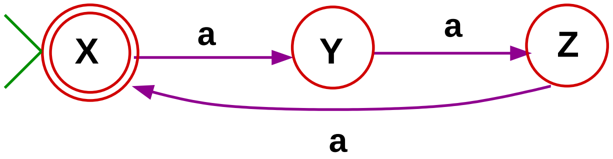

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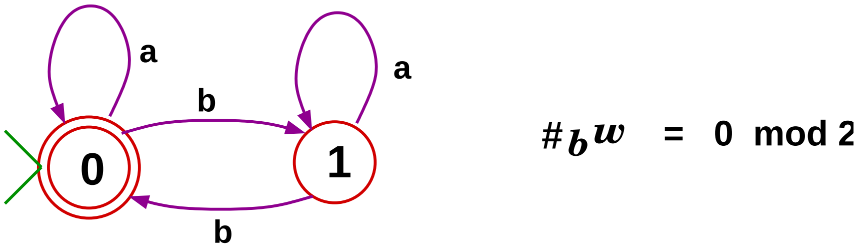
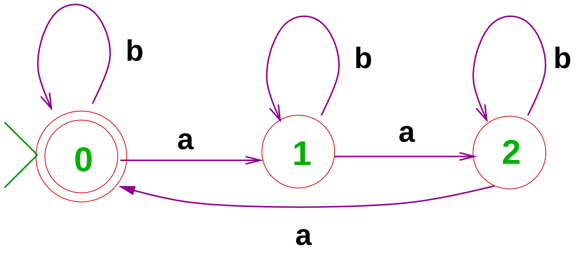
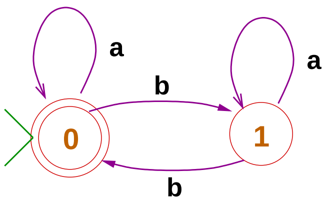
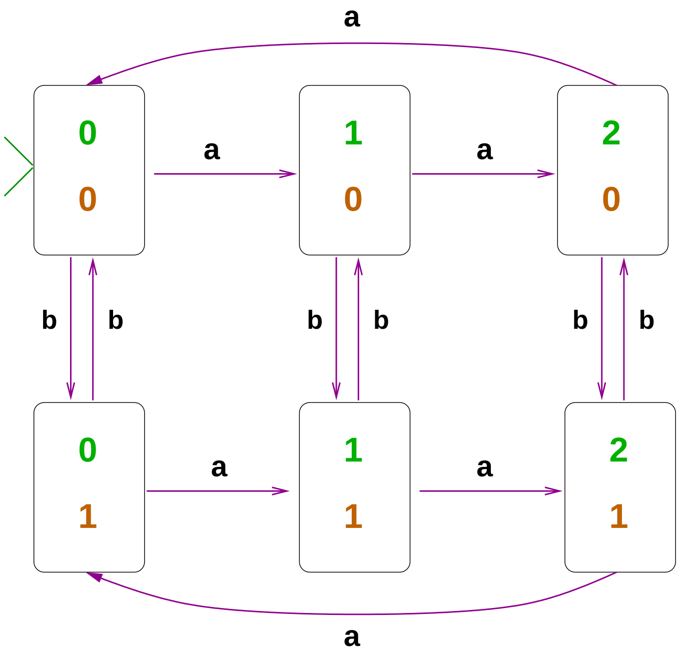


Parallel programming is tricky, but here we have a special form of parallelism: the two processors may work in tandem, because they read the same input one symbol at a time.

Two automata collaborating

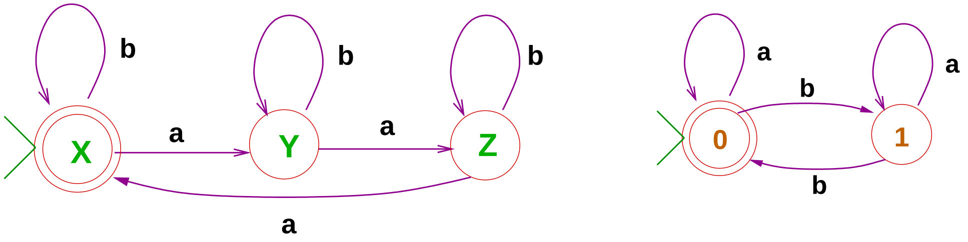


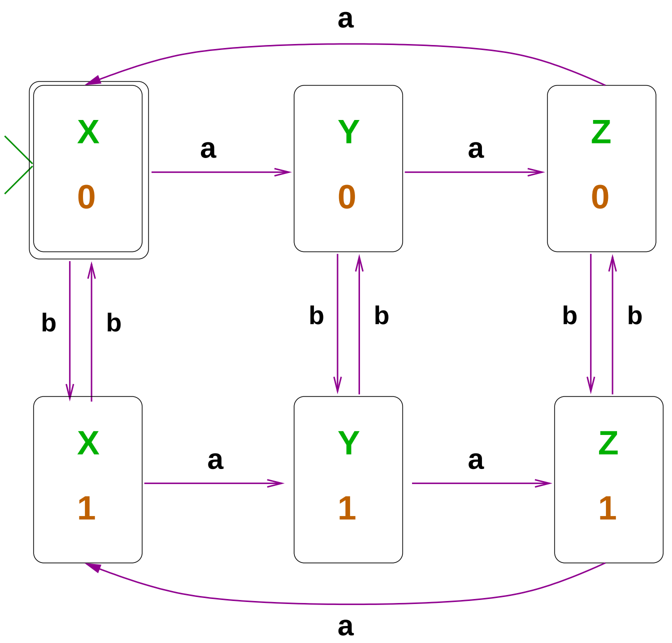




Conjuctive pairing

- Accepting when both accept:

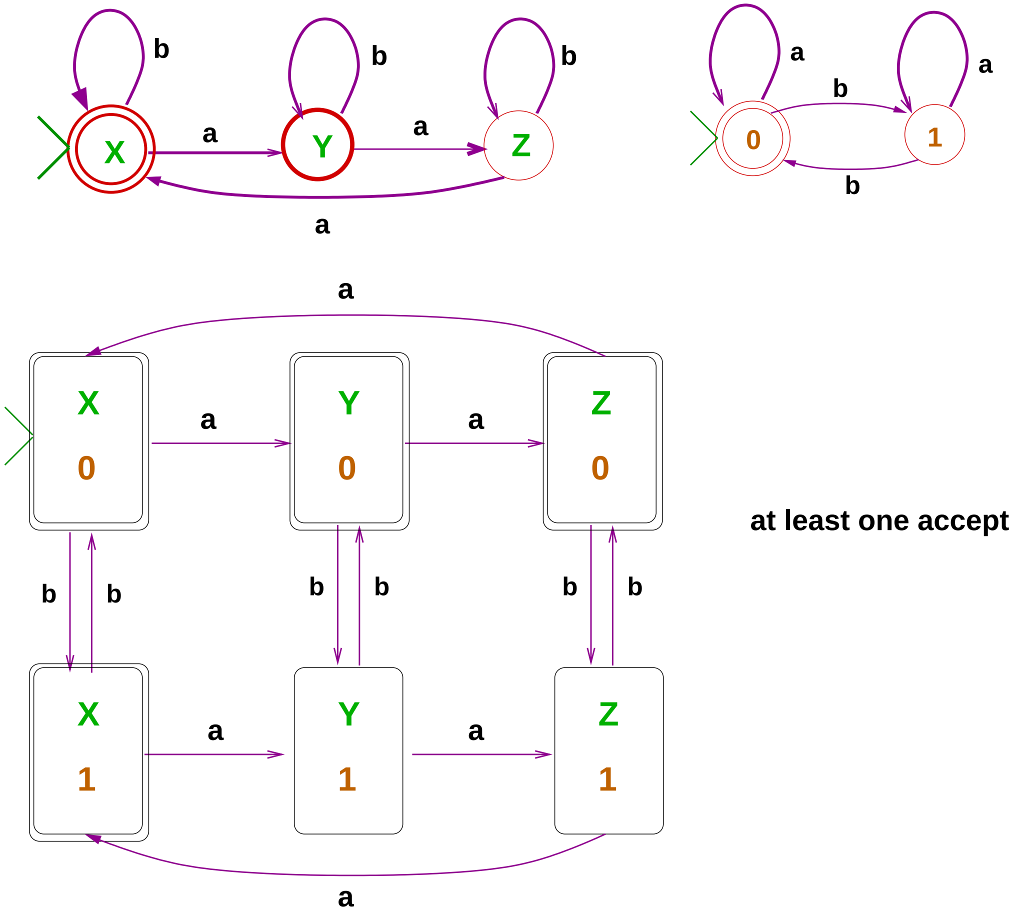




both accept

Disjunctive pairing

- Accepting when at least one automaton accepts:



at least one accepts

Formal definition of automata product

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Formal definition of automata product

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– In a ***conjunctive product*** the set of accepting states is $A \times A'$ (both automata accept).

– In a ***disjunctive product*** the set of accepting states is $(A \times Q') \cup (Q \times A')$ (at least one automaton accepts).

Some applications

- $L = \{\ \mathtt{a}\, w\, \mathtt{z}\ \mid\ w \in \Sigma^*\ \}$

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- An automaton over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ recognizingthe string that miss at least one letter.

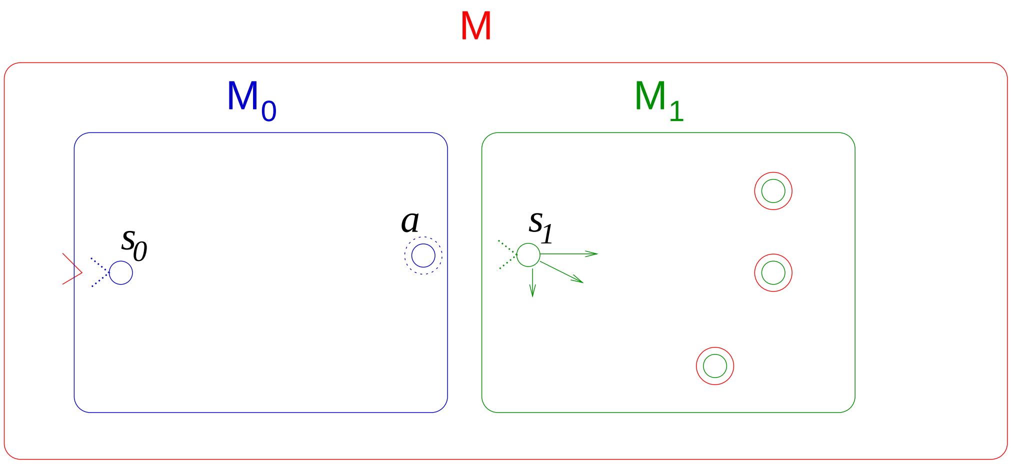
Nondeterministic Automata

Capturing operationally language concatenation

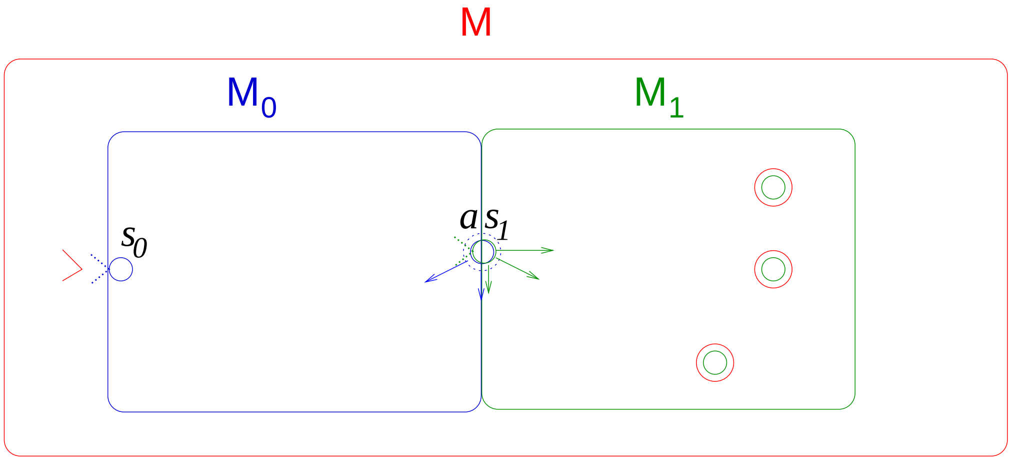
- We verified that combining recognized languages
  by union, intersection, and difference,
  yields recognized languages.

- What about concatenation?
  li Suppose we have two automata M_0 and M_1.
  Construct automaton M such that

$$\mathcal{L}(M) = \mathcal{L}(M_0) \cdot \mathcal{L}(M_1)$$

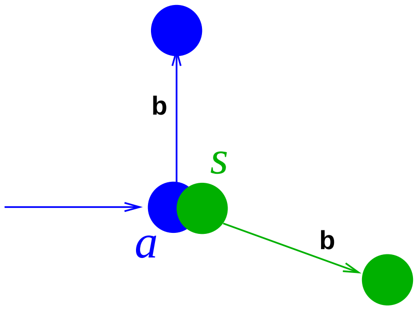


Trying to make this work



- Problem: Can't coalesce a and σ_1 :

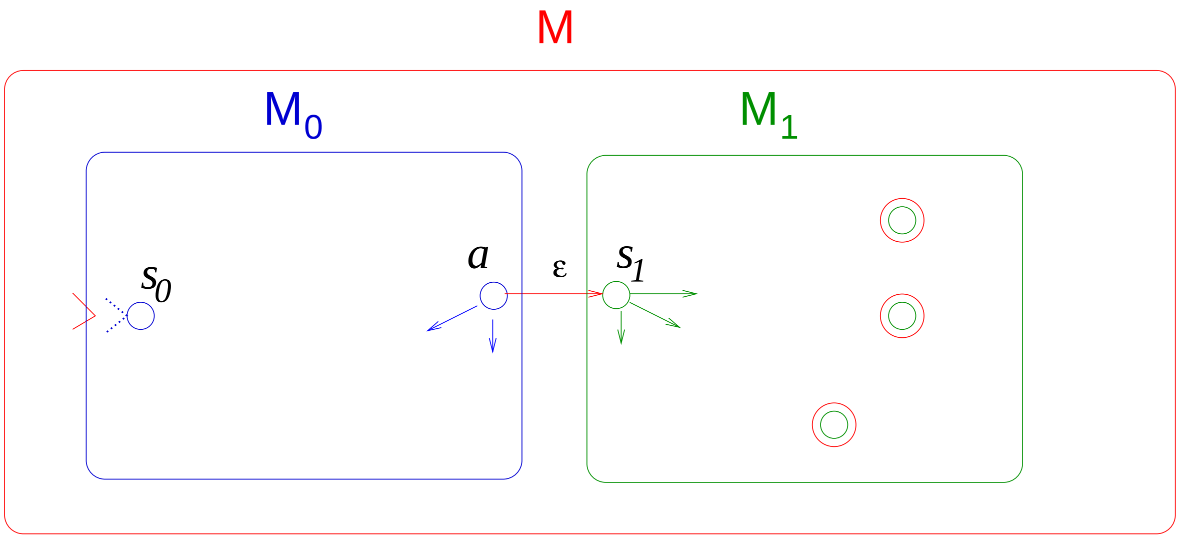
  They might have conflicting transitions rules:



And computation might proceed back and forth between M_0 and M_1 .

Spontaneous transitions

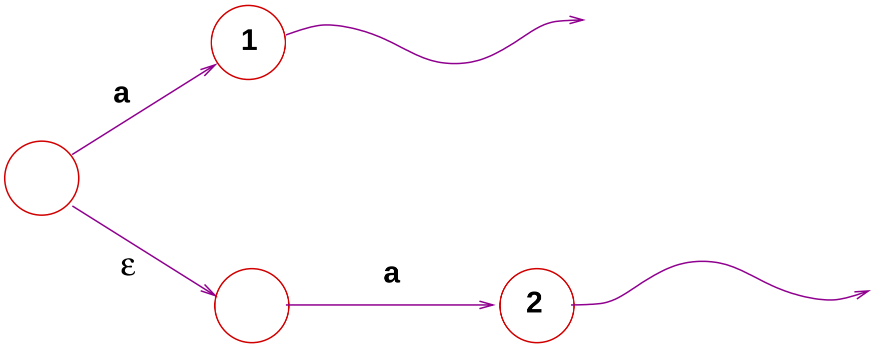
- How about allowing spontaneous transitions between states, $q \xrightarrow{p}$ without any symbol read.
- To streamline notation we can think of such transitions triggered by ε: $q \xrightarrow{\epsilon} p$.



- We call these $\boxed{\textit{epsilon-transitions}}$, in analogy to our previous notation: $q \xrightarrow{w} p$ for a combined transition from state q to p obtained by reading the string w.

Nondeterminism

- ε-transitions yield “ambiguous” computation:
  multiple transitions for a state+symbol may be created:



Admitting non-determinism

- We consider relaxing the requirements that each transition rule is a function (univalent and total) and triggered by reading a letter.
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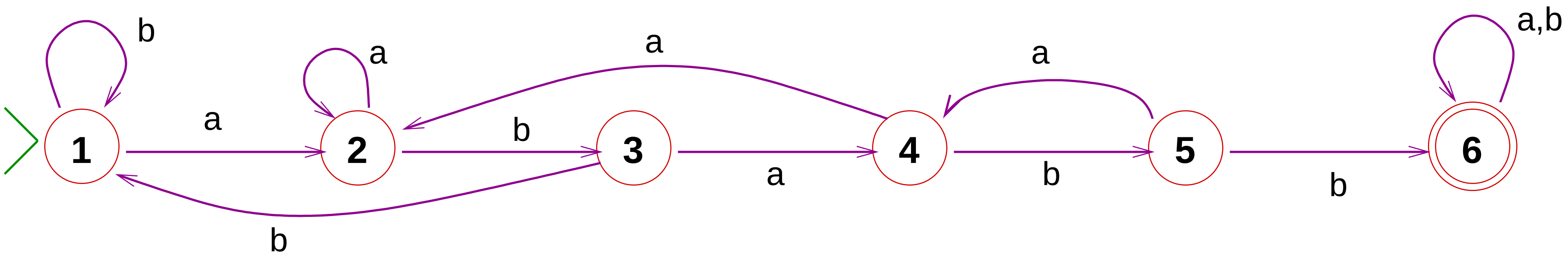
2. It can be simulated by ε-transitions, which do model natural phenomena; and
3. It is algorithmically natural, as we shall now see.

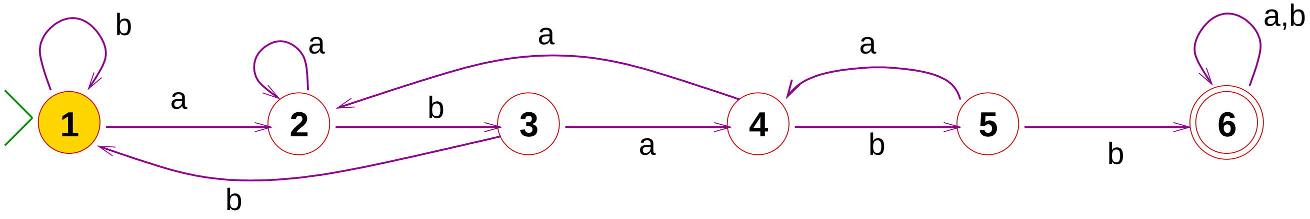
AUTOMATA AS ON-LINE ALGORITHMS

Automata as on-line algorithms

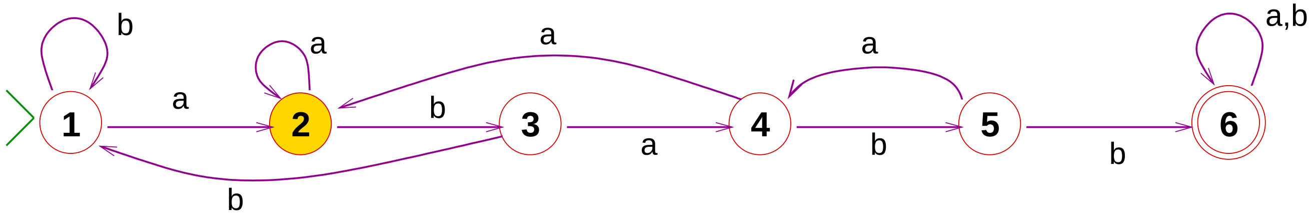
- Automata can be viewed as efficient real time algorithms, which move pointers (or “tokens”) around.
- An automaton to recognize the presence of `ababb`:



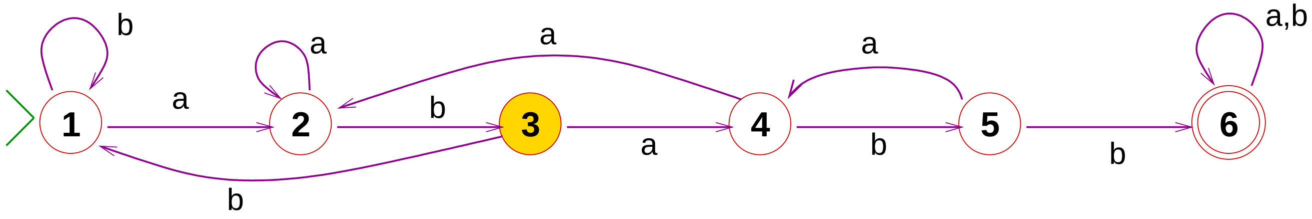
- It is visualized by moving a token for the state position.



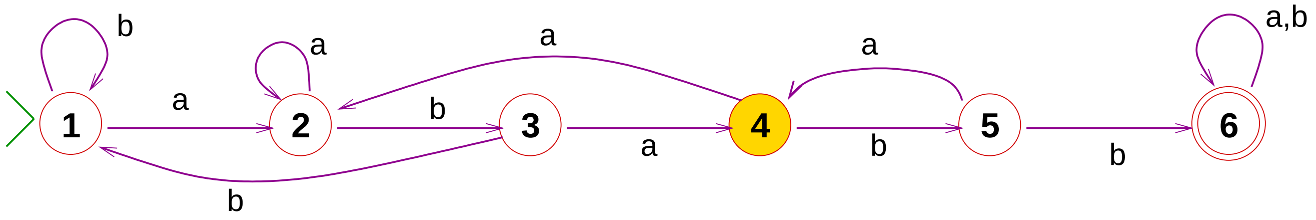
a b a b a b b a



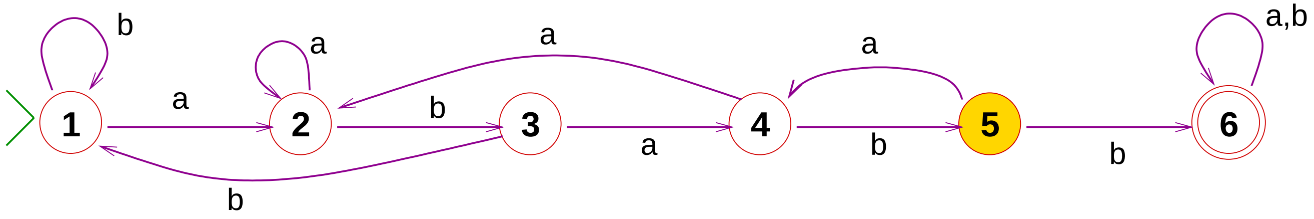
a b a b a b b a



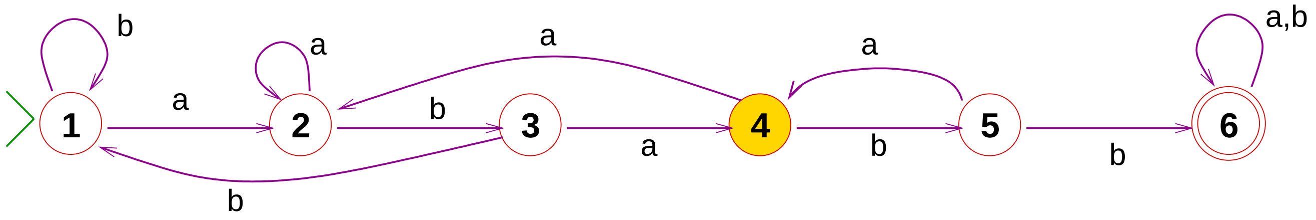
a b $\underline{\textbf{a}}$ b a b b a



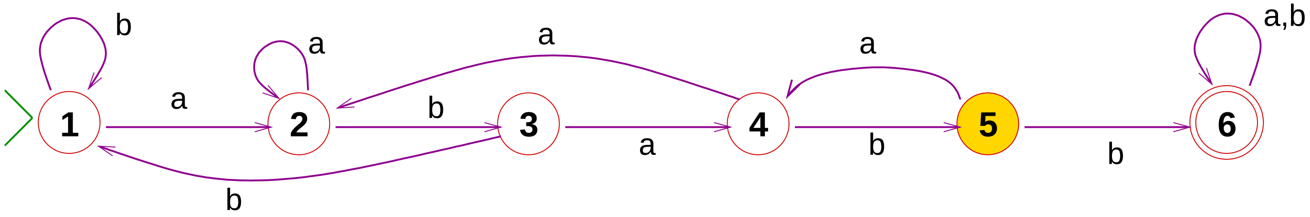
a b a b a b b a



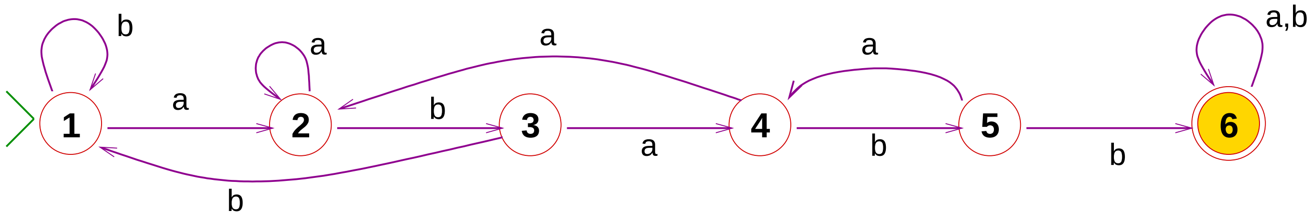
a b a b $\underline{\textbf{a}}$ **b b a**



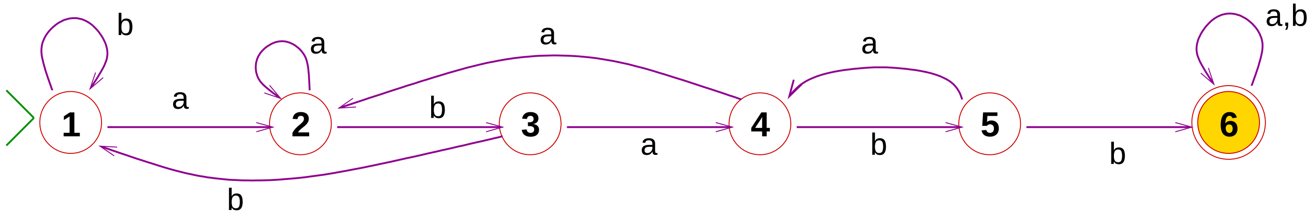
a b a b a $\underline{\mathbf{b}}$ **b a**



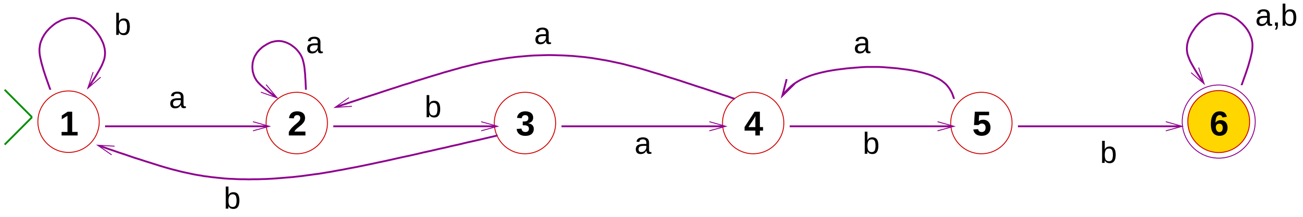
a b a b a b $\underline{\text{b}}$ a



a b a b a b $\underline{\textbf{b}}$ **a**

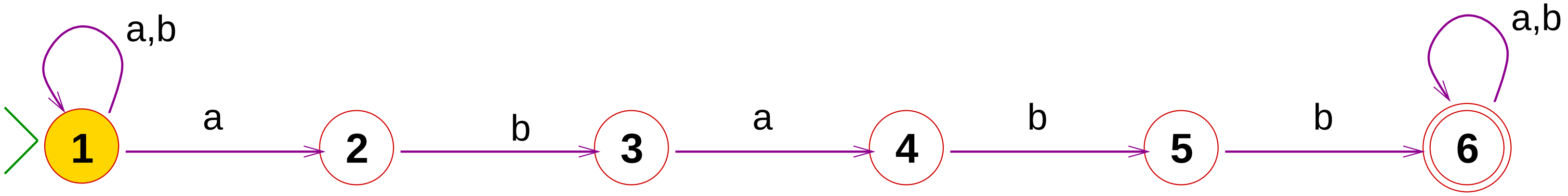


a b a b a b b a



a b a b a b b a _

An alternative, with token rules relaxed.



<u>a</u> b a b a b b a

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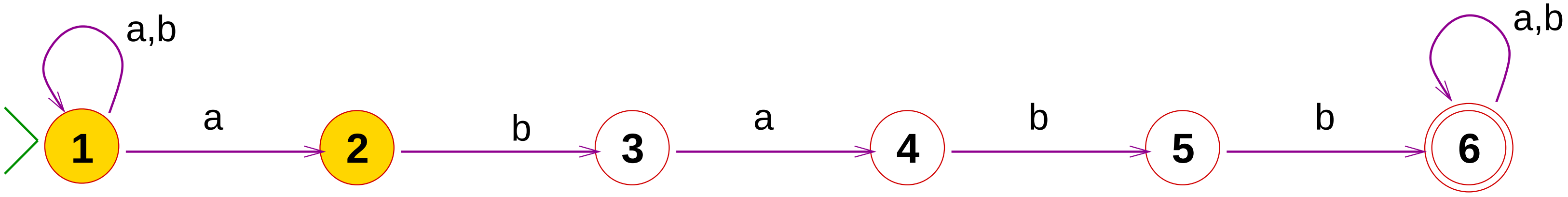
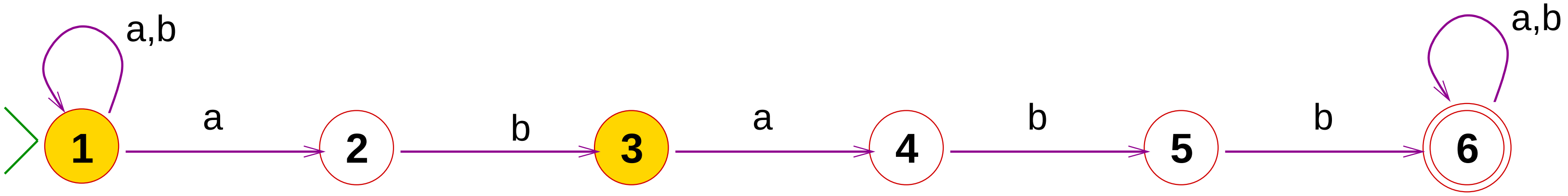


a b a b a b b a

An alternative, with token rules relaxed.



a b <u>a</u> b a b b a

- Next states marked are 1,2 and 4. Etc.

Non-deterministic automata

A non-deterministic automaton over Σ:

- Finite (non-empty) set Q of states
- Start state s and accepting states $A \subseteq Q$
- Transition mapping: $\delta : (Q \times \Sigma_\epsilon) \Rightarrow Q$
- Here $\Sigma_\epsilon = \Sigma \cup \{\varepsilon\}$
- Still using the notation $q \xrightarrow{\sigma} p$ for $\langle q, \sigma, p \rangle \in \delta$
- But $q \xrightarrow{\epsilon} p$ is also an option.

Computation state-traces

- If $w = \sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_n$ where $\sigma_i \in \Sigma_\varepsilon$,

  and $q \xrightarrow{\sigma_1} r_1 \xrightarrow{\sigma_2} r_2 \quad \cdots \quad r_{n-1} \xrightarrow{\sigma_n} p$

  then $q \overset{w}{\Longrightarrow} p$.

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  then $q \overset{w}{\Longrightarrow} p$.
- The sequence of states

$$q \;\; r_1 \;\; r_2 \quad \cdots \quad r_{n-1} \;\; p$$

as above is a $\boxed{\textit{state-trace}}$ of the NFA for input w.

Generative definition of $q \overset{w}{\Longrightarrow} p$

- **Base.** $q \xrightarrow{\epsilon} q$ for all $q \in Q$.
- **Step.** If $q \xrightarrow{\sigma} p$ by the NFA's transition,
  and $p \overset{w}{\Longrightarrow} r$ has been generated already (where $\sigma \in \Sigma_\epsilon$) then $q \overset{\sigma\cdot w}{\Longrightarrow} r$.

Acceptance by an NFA

- M ***accepts*** a string $w \in \Sigma^*$ if $s \overset{w}{\Longrightarrow} A$.

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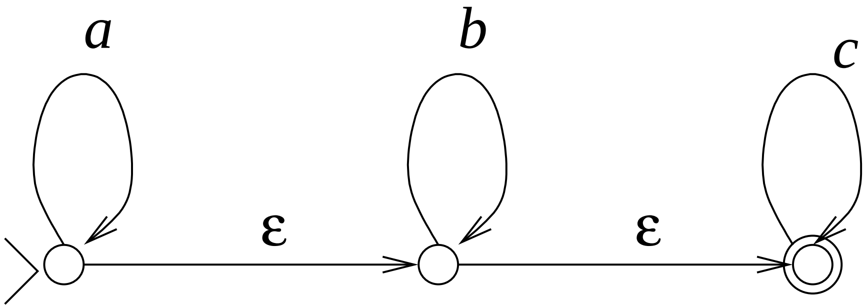
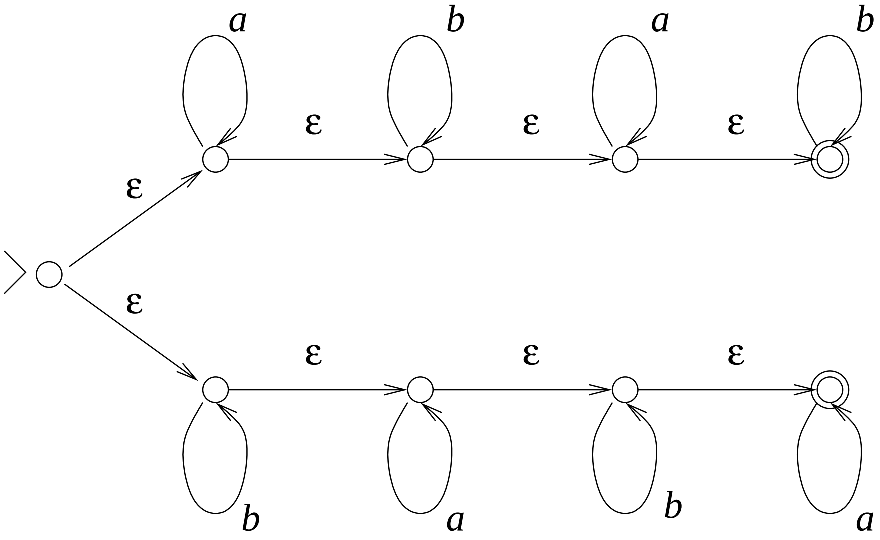
  2. A “lucky trace” may include ε-transitions.
- The $\boxed{\textit{language recognized}}$ by M
  is the set of accepted strings.

Example: $\mathcal{L}(\mathtt{a^*b^*c^*})$



Recognizing $\mathcal{L}(a^*b^*a^*b^* \cup b^*a^*b^*a^*)$



$a^*b^*a^*b^* \cup b^*a^*b^*a^*$

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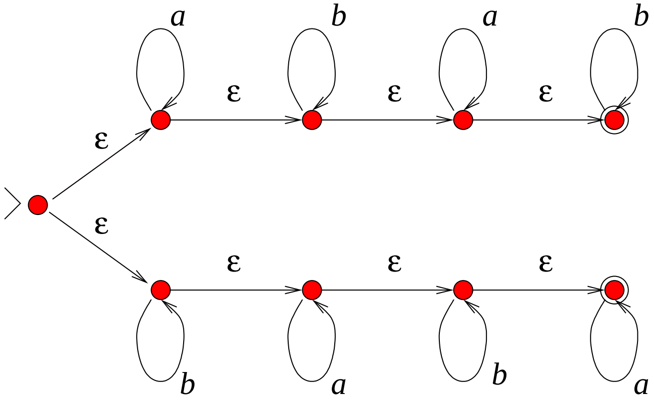
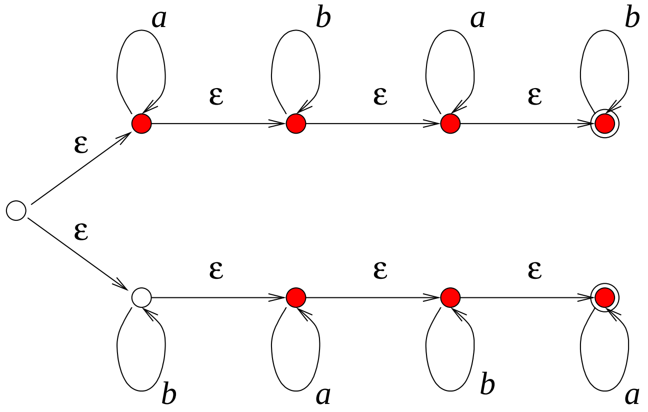


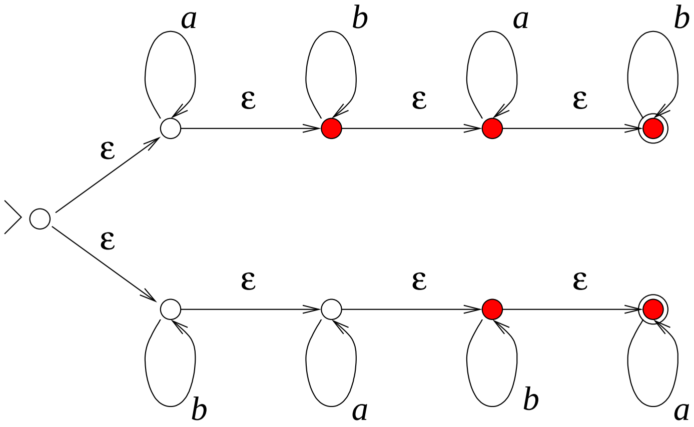
>abb

Recognizing $\mathcal{L}(\mathtt{a^*b^*a^*b^*} \cup \mathtt{b^*a^*b^*a^*})$



a>bb

Recognizing $\mathcal{L}(\mathtt{a^*b^*a^*b^*} \cup \mathtt{b^*a^*b^*a^*})$



ab>b

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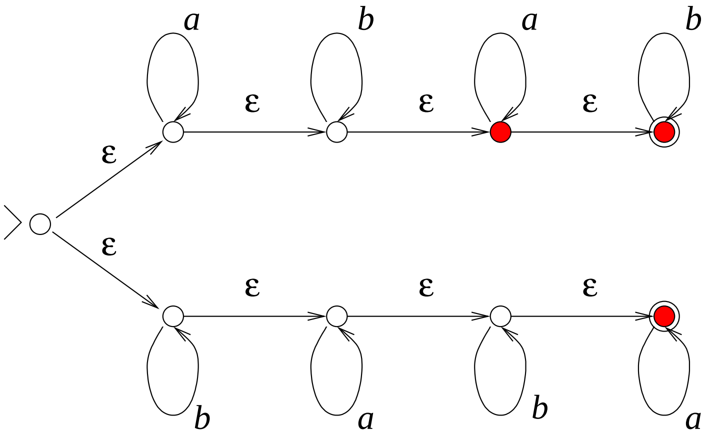


abb>

So the number of states is *reduced* with each step.

DFA-RECOGNIZED = NFA-RECOGNIZED

DFA-RECOGNIZED = NFA-RECOGNIZED

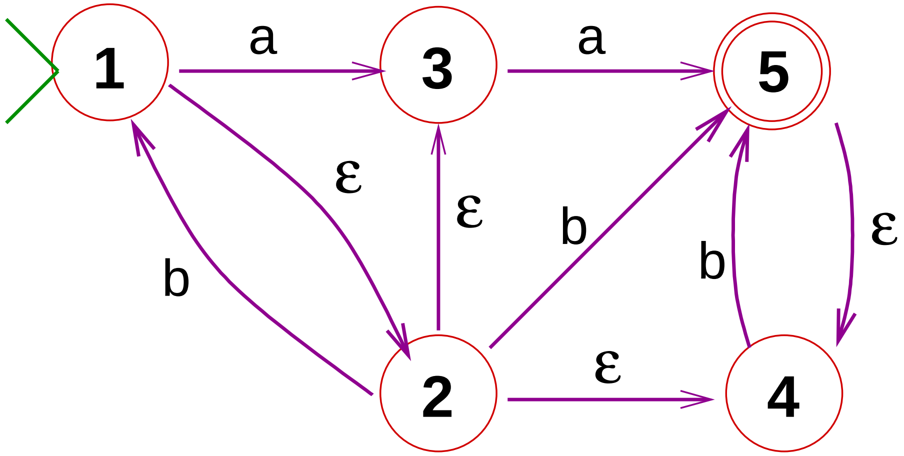
- DFA-RECOGNZD $\Longrightarrow$ NFA-RECOGNZD:

  TRIVIAL: Every DFA is an NFA

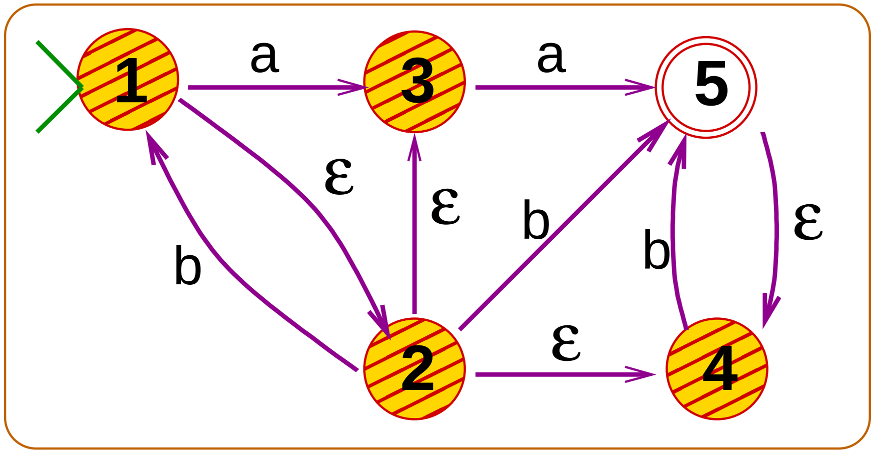
- NFA-RECOGNZD $\Longrightarrow$ DFA-RECOGNZD...

Converting NFAs to equivalent DFAs

- Given an NFA N :

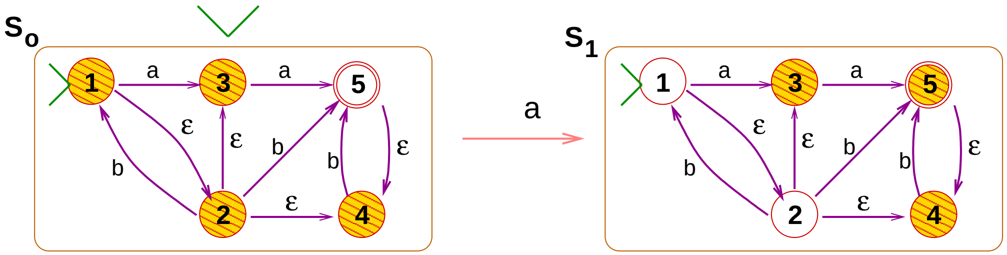


- Mark as “on” the states reachable before reading any input:

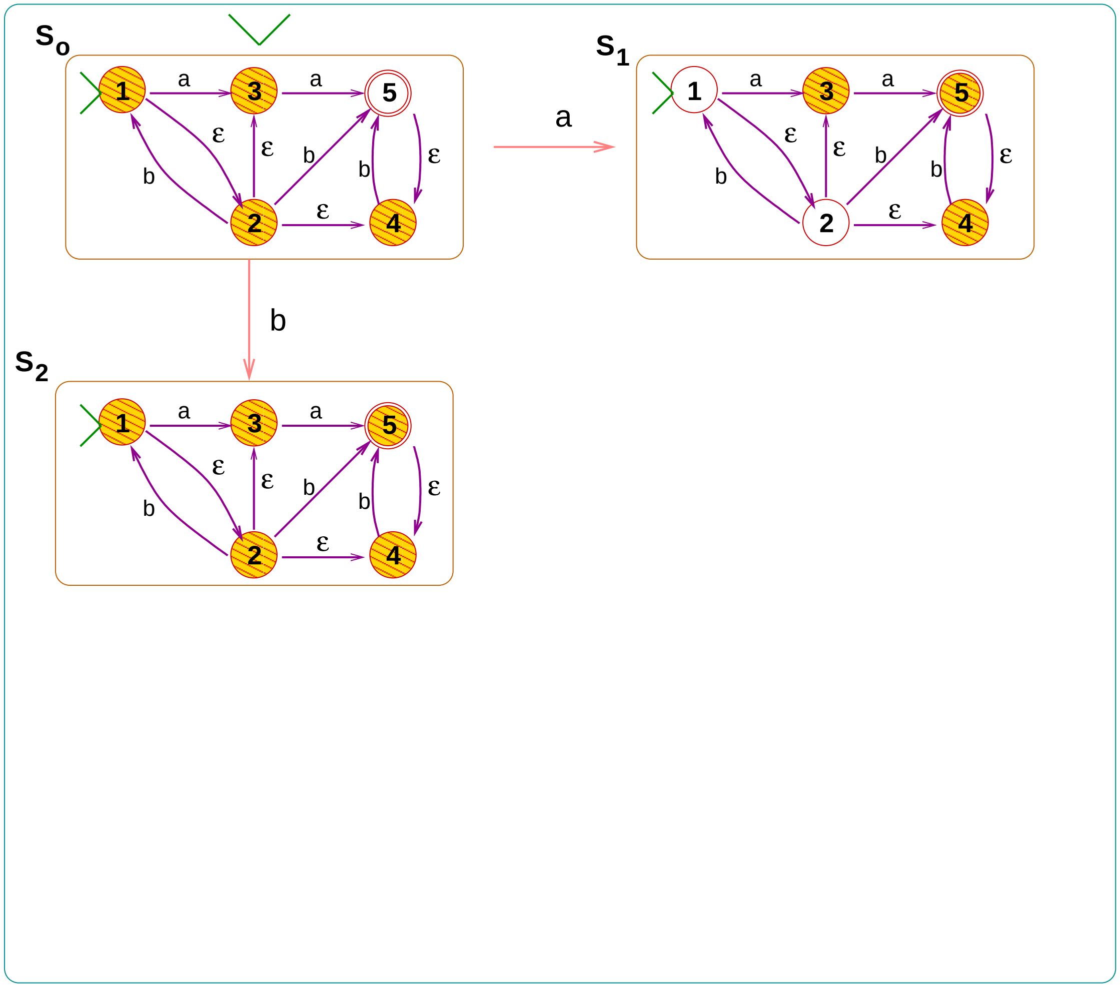


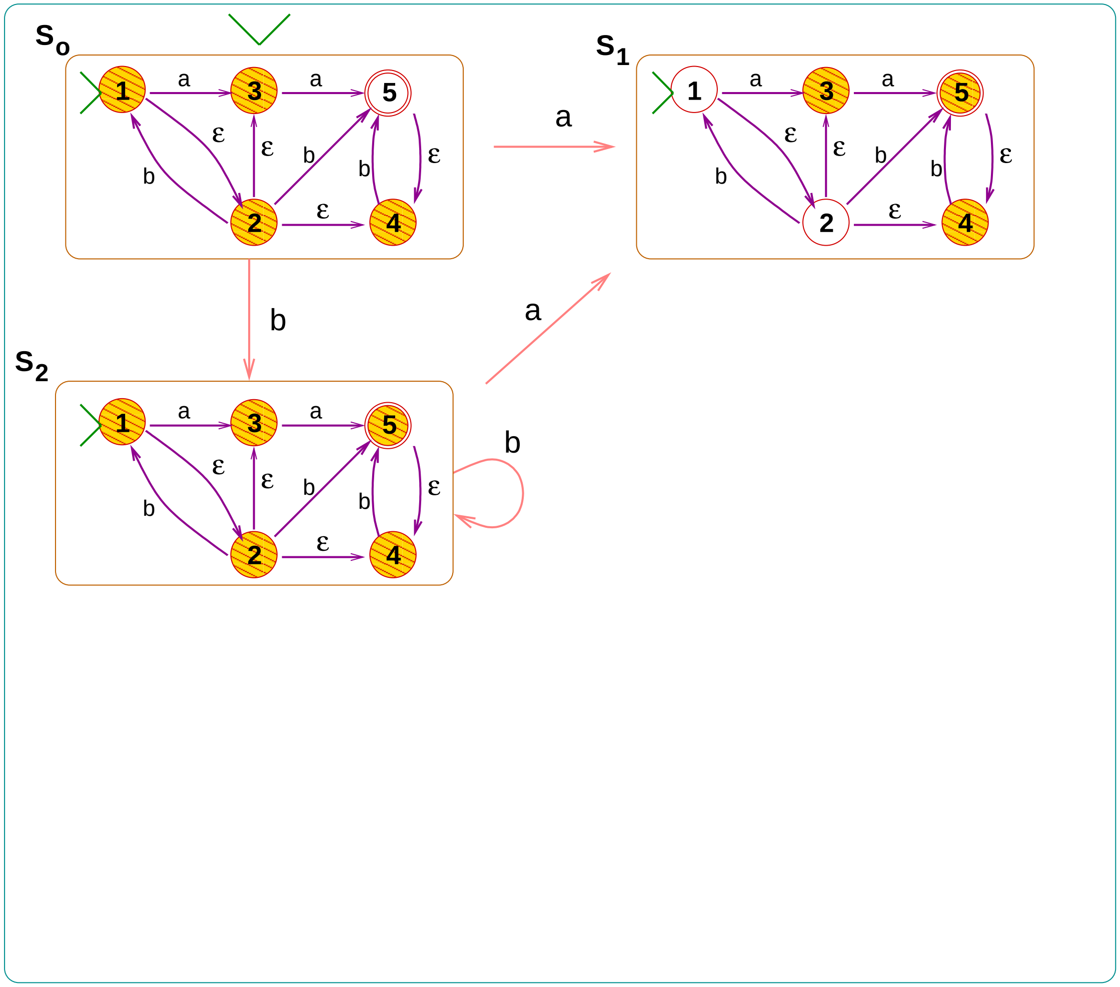
- This setup is the “start state” of our deterministic automaton.

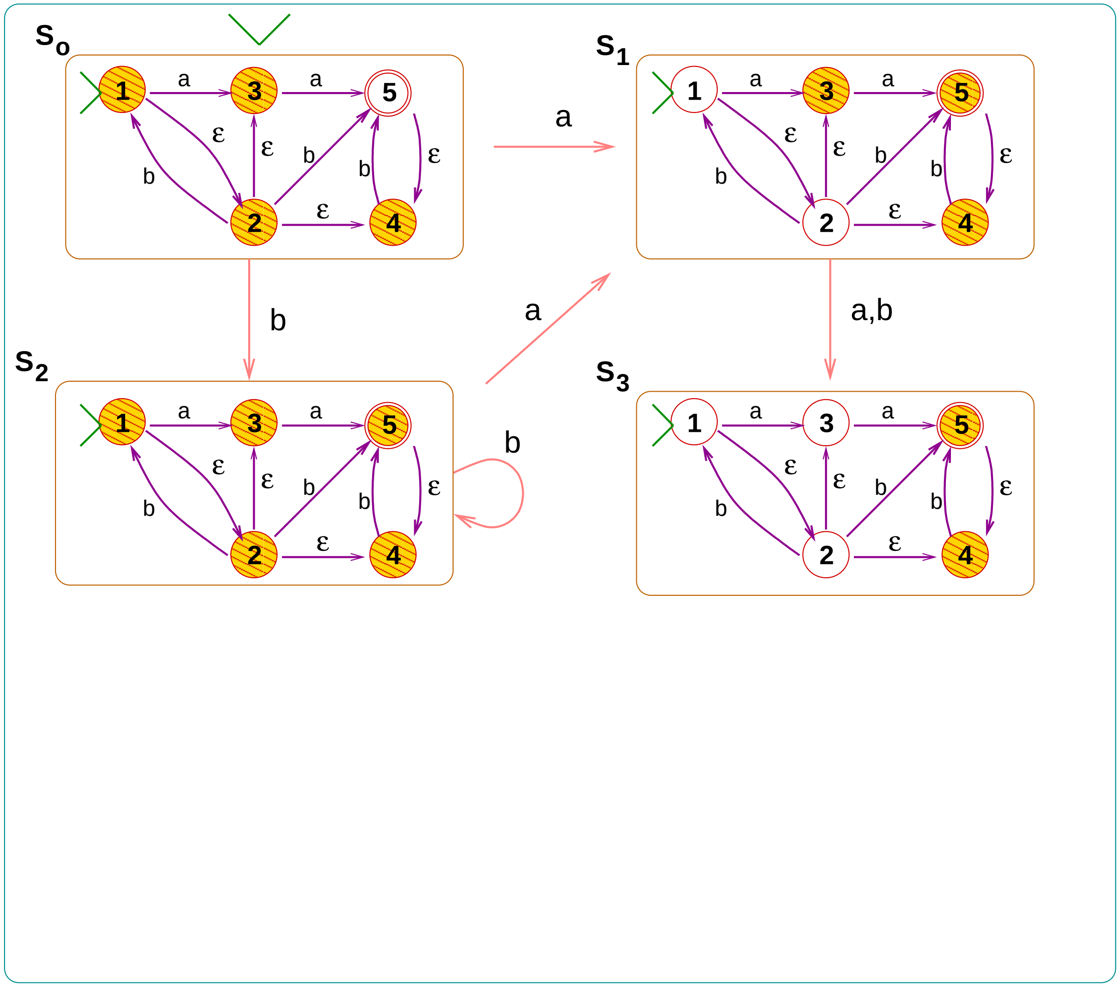
- On rreading a the NFA can be in one of possible states:

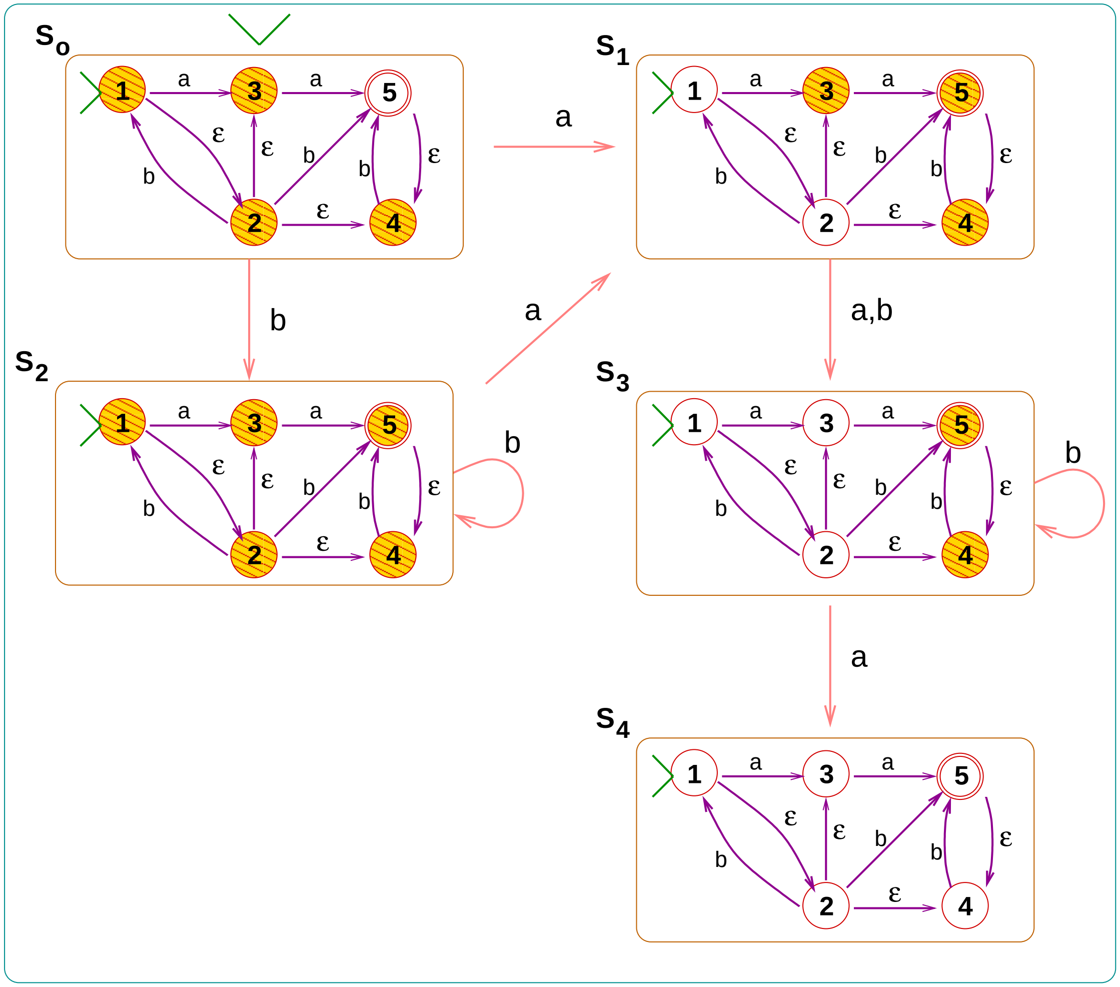


- Proceed to explore the set of reachable states of N :



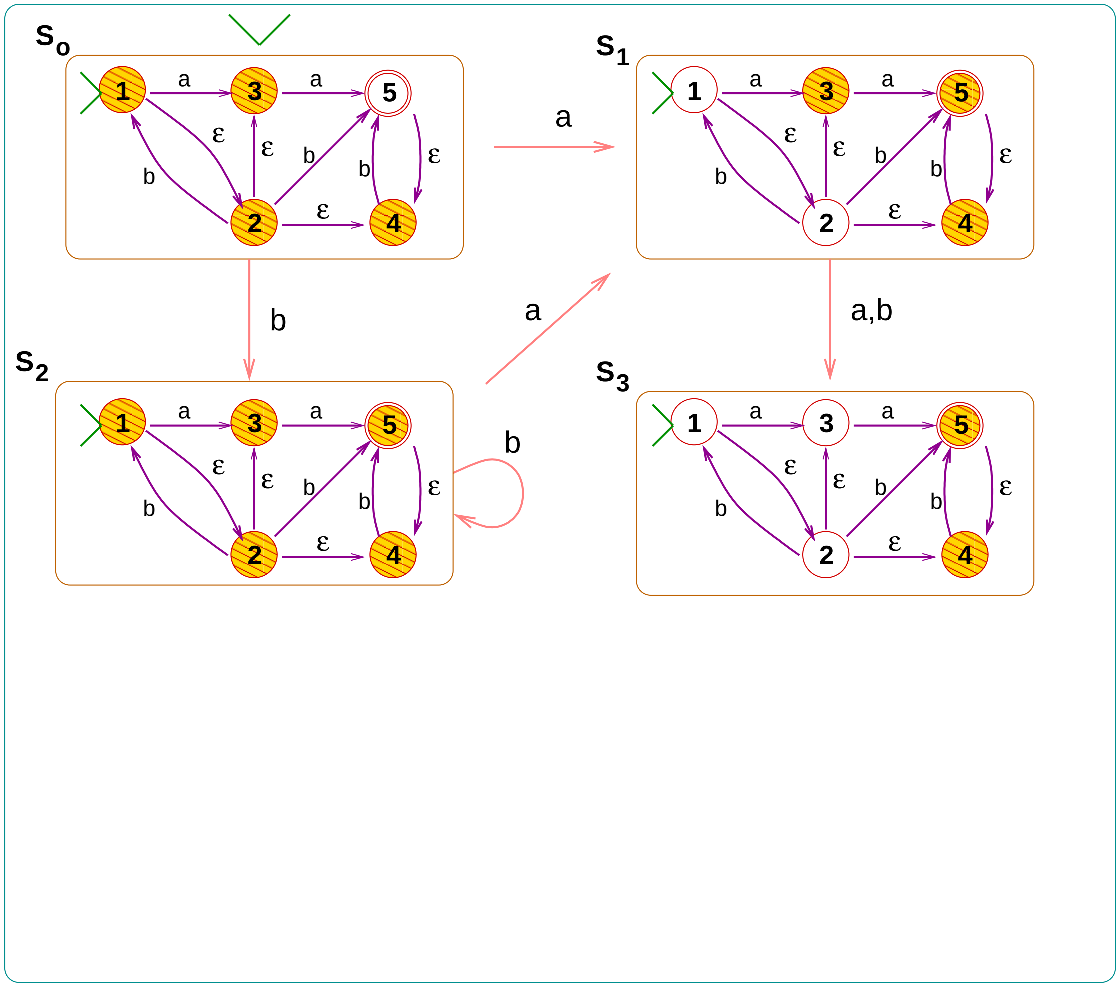

S o
1
3
5
2
4
a
a
ε
ε
b
b
b
ε
ε
a
S 1
1
3
5
2
4
a
a
ε
ε
b
b
b
ε
ε
b
a
S 2
1
3
5
2
4
a
a
ε
ε
b
b
b
ε
ε
b


S0
S1
S2
S3
a
b
a
a,b
b
1
2
3
4
5
ε


S0
1
3
5
2
4
a
a
ε
ε
b
b
b
ε
ε
a
S1
b
a
a,b
S2
b
S3
b
a
S4

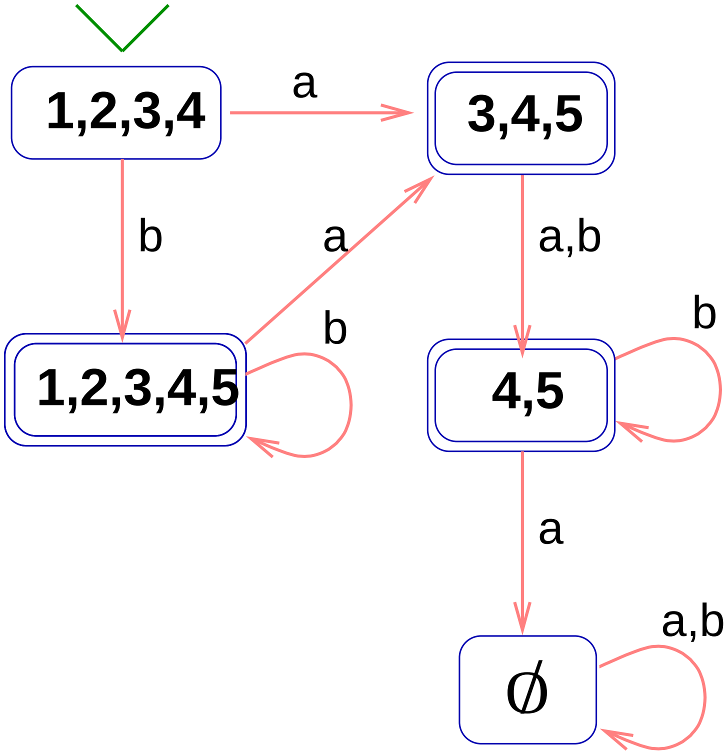
- Complete the transition for the final setup.

- The setups are the states of the new, deterministic, automaton.
- A setup is ***accepting*** if it contains an accepting state of N:


S0
1
a
3
a
5
ε
ε
b
b
b
ε
2
ε
4
a
S1
1
a
3
a
5
ε
ε
b
b
b
ε
2
ε
4
b
a
a,b
S2
1
a
3
a
5
ε
ε
b
b
b
ε
2
ε
4
b
S3
1
a
3
a
5
ε
ε
b
b
b
ε
2
ε
4

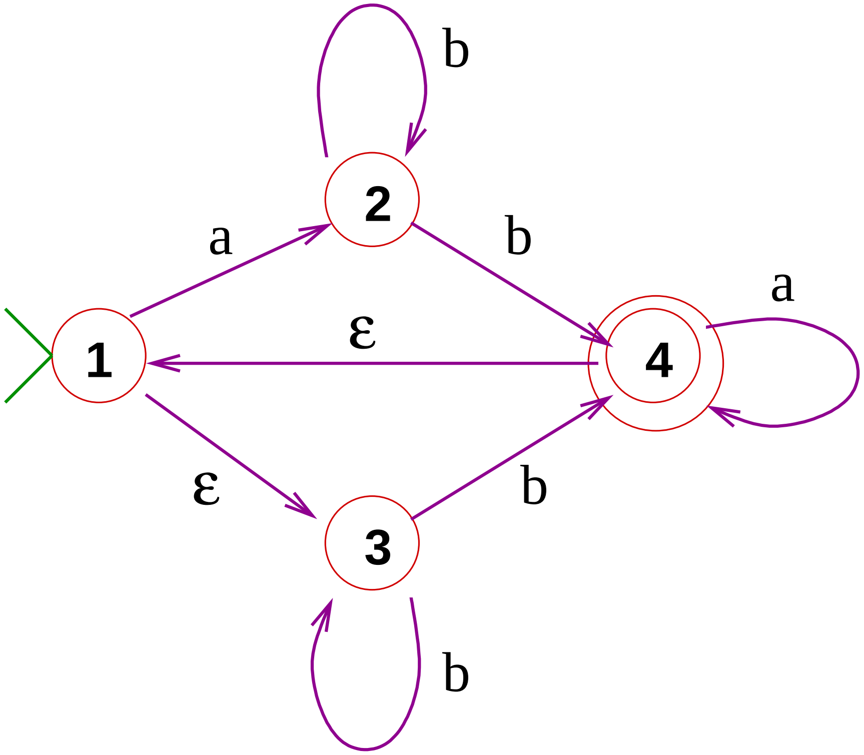
The resulting DFA

- Each state of the DFA obtained is a setup of N 's states:

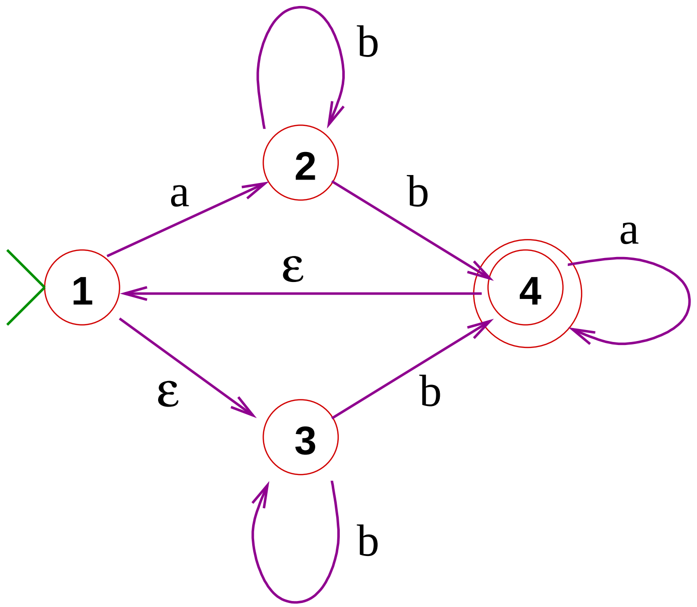


- We have constructed from an NFA N an equivalent DFA M.

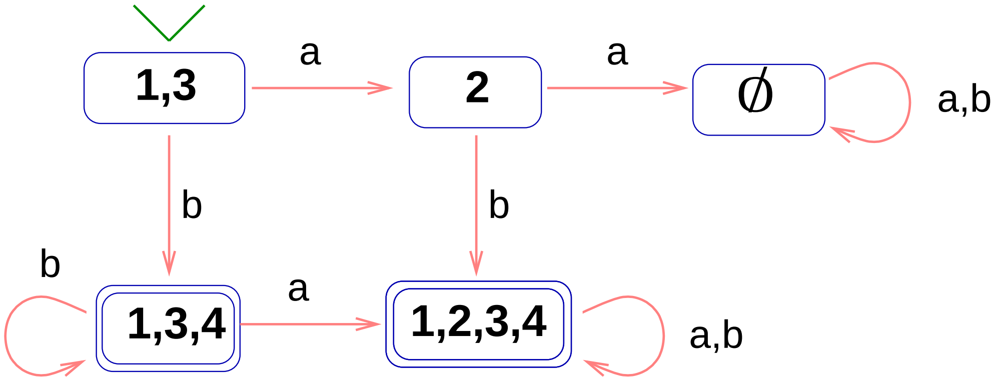
Another example



Another example

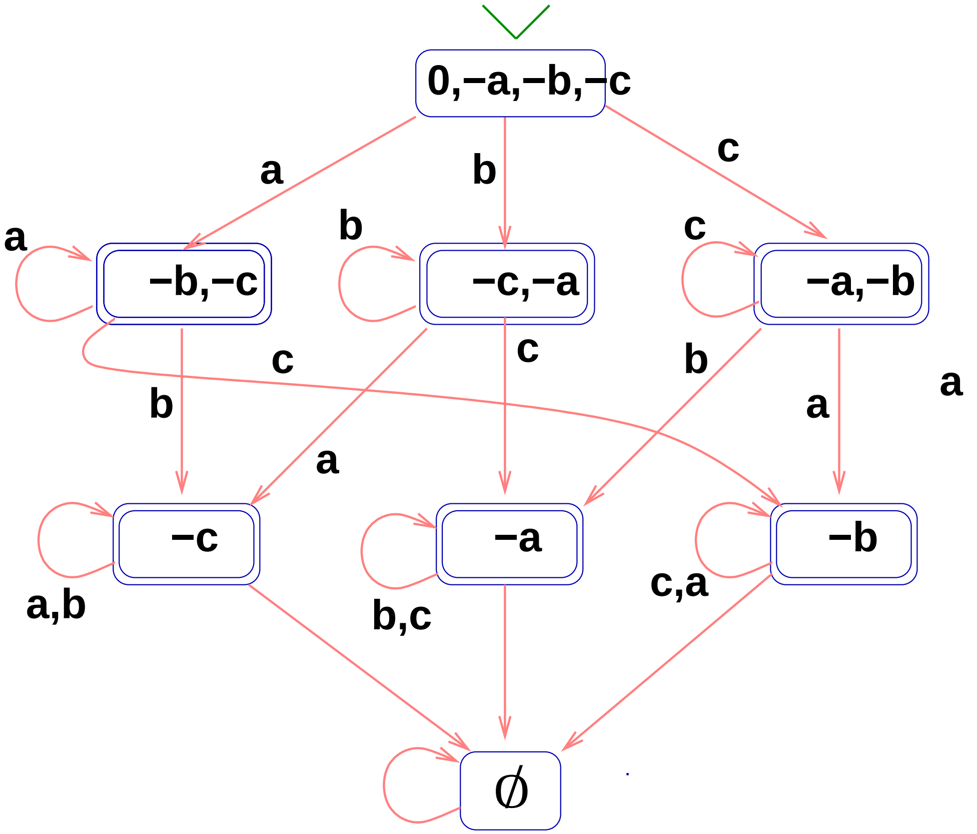


Another example

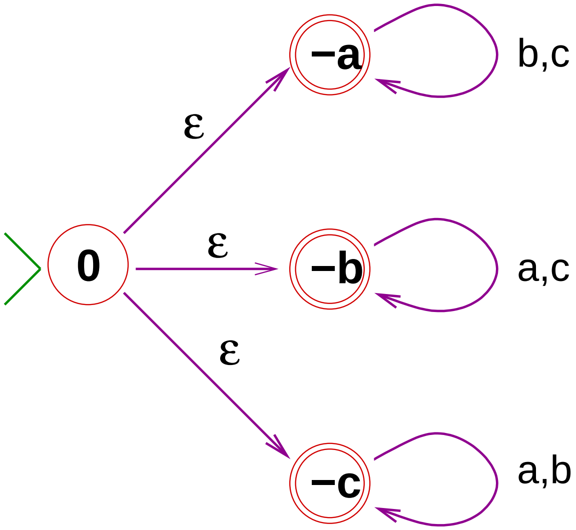


An exponential explosion

- If N has n states, then the DfA obtained
  may have up to 2^n states.
- Is that really necessary?
  Could we have a more efficient construction?
- No! Consider the language of strings over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ that miss at least one letter.
- The smallest DFA recognizing it is

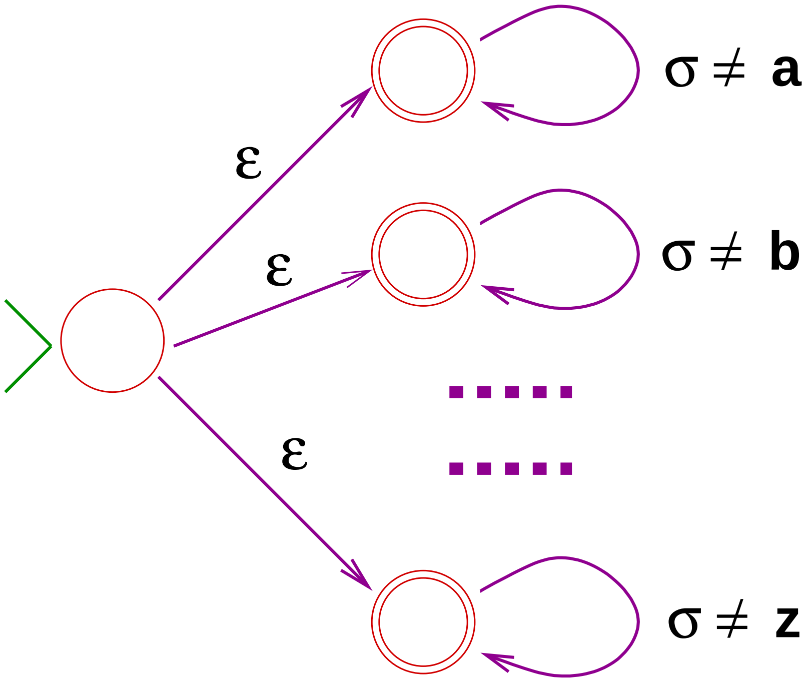

0,−a,−b,−c
a
b
c
a
−b,−c
b
−c,−a
c
−a,−b
c
b
c
b
a
a
a
−c
−a
−b
a,b
b,c
c,a
∅

- But here is a 4-state NFA recognizing it:



- For “missed-som” language over the Latin alphabet
  the smalles recognizing automaton has $2^{26} > 67$ million states!

- But here is a 27 state NFA recognizing it:


ε
ε
ε
σ ≠ **a**
σ ≠ **b**
σ ≠ **z**

Next ...

	Descriptive		Operational
Narrow	STRICT-REG		DFA
			=
Broad	REGULAR	$\Longrightarrow$	NFA

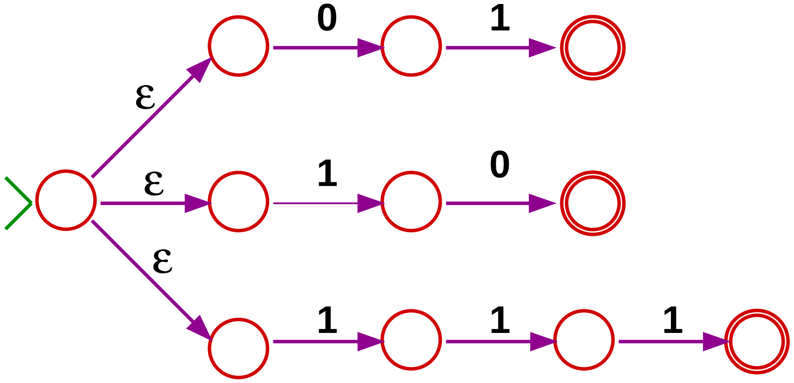
Reminder: Generating the regular languages

1. Every finite language is regular.
2. If L, K are regular, then so are their union, intersection, complement, concatenation, star, and plus.

- We show that all regular languages are recognized by NFAs (and therefore by DFAs).
- The proof is by induction on the generative dfn of the regular languages.

Finite languages are recognized

- For example $\{01, 10, 111\}$ is recognized by



- We know that it suffices to take the finite languages with 0 or 1 elements, each a string of size 0 or 1.

  By this construction, what would be the NFA recognizing

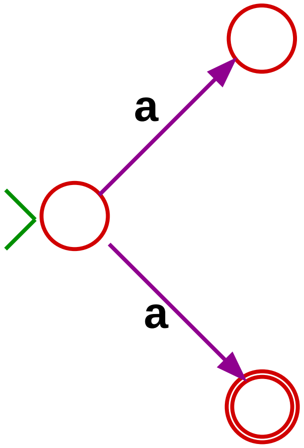
  $\{0\}$? $\{\varepsilon\}$? $\emptyset$?

Complement of recognized is recognized

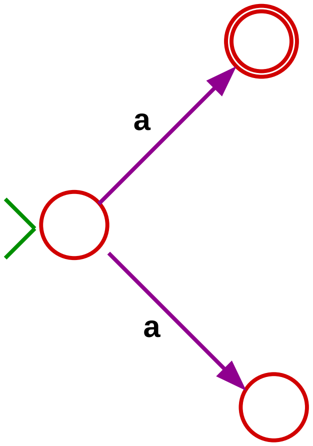
- We have seen:
  A language recognized by an NFA is recognized by a DFA M,
  so its complement is recognized by the DFA $\bar{M}$
  obtained by replacing in M acceptance and non-acceptance.

- Note: This idea doesn't work for NFAs:

N:



$\bar{\mathbf{N}}$**:**



NFA N accepts a and so does $\bar{N}$.

The $\cup$ and $\cap$ of recognized is recognized

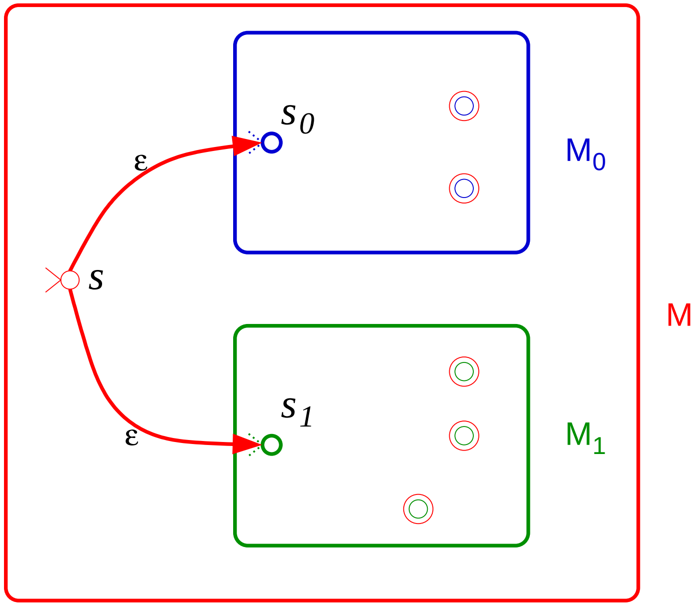
- We already showed this for DFAs.

The ∪ and ∩ of recognized is recognized

- We already showed this for DFAs.
- An alternative approach for union:

  Given $L_0 = \mathcal{L}(M_0)$ and $L_1 = \mathcal{L}(M_1)$,

  here's an NFA M that recognizes $L_0 \cup L_1$



- Once we have closure under union and complement, we obtain closure under intersection:
- 3-If L and K are both recognized, then so are $\bar{L}$ and $\bar{K}$, and therefore $\bar{L} \cup \bar{K}$, as well as its complement which is $= L \cap K$.

- Once we have closure under union and complement,
  we obtain closure under intersection:
- We have $\overline{L \cap K} = \bar{L} \cup \bar{K}$,
  so by complementing both sides we get $L \cap K = \overline{\bar{L} \cup \bar{K}}$
- 3-If L and K are both recognized,
  then so are $\bar{L}$ and $\bar{K}$,
  and therefore $\bar{L} \cup \bar{K}$, as well as its complement which is $= L \cap K$.

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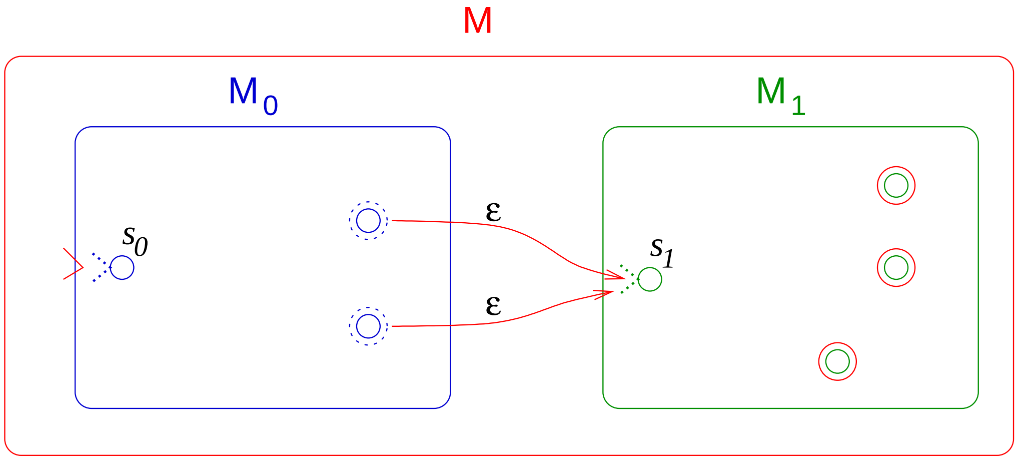
- 3-If L and K are both recognized, then so are $\bar{L}$ and $\bar{K}$, and therefore $\bar{L} \cup \bar{K}$, as well as its complement which is $= L \cap K$.

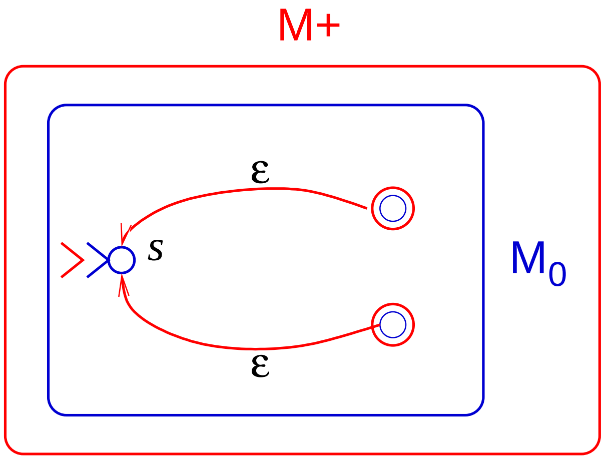
Concatenation of recognized is recognized

- Given $L_0 = \mathcal{L}(M_0)$ and $L_1 = \mathcal{L}(M_1)$,
  here’s an NFA M that recognizes their concatenagion:



Plus and star of recognized are recognized

- Given $L = \mathcal{L}(M)$ here's an NFA M^+ recognizing L^+:



- Since $L^* = L^+ \cup \{\varepsilon\}$ we conclude that L^* is also recognized.

Graphs with reg-exps as labels

- ⋆ Starting with the given NFA,

  Collapse labels: eg, replace $q \xrightarrow{a,b,\epsilon} p$ by $q \xrightarrow{a \cup b \cup \epsilon} p$

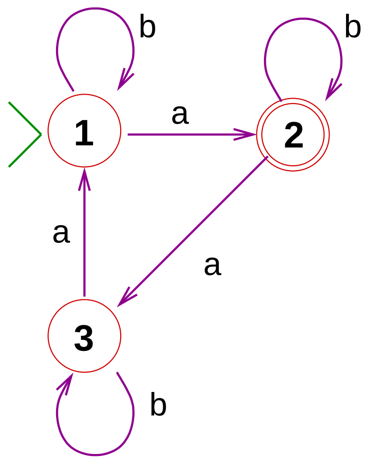
- ⋆ Create a new start state s_0

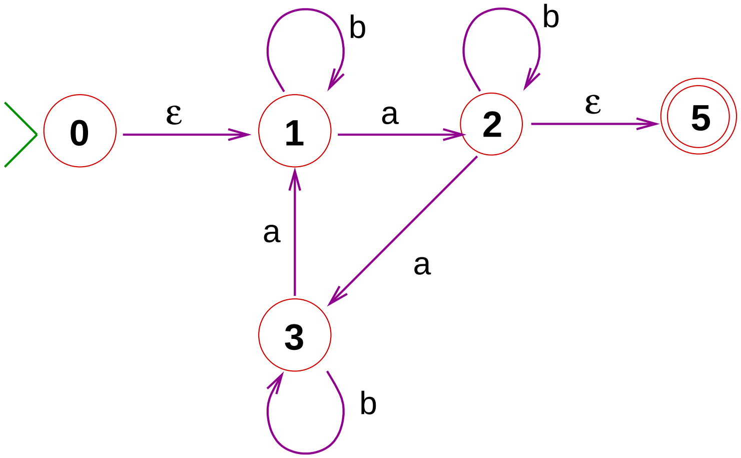
  with an ε-transition to the original start state of N.

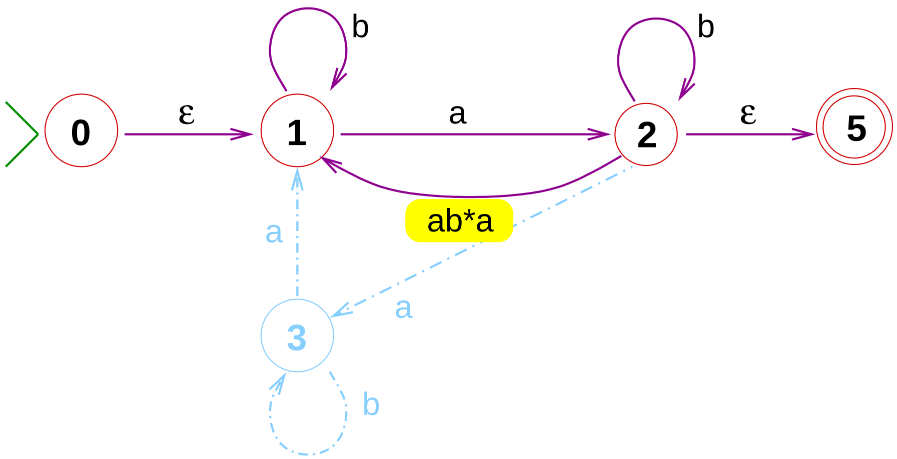
- ⋆ Create a new state a_0 as the only accepting state,

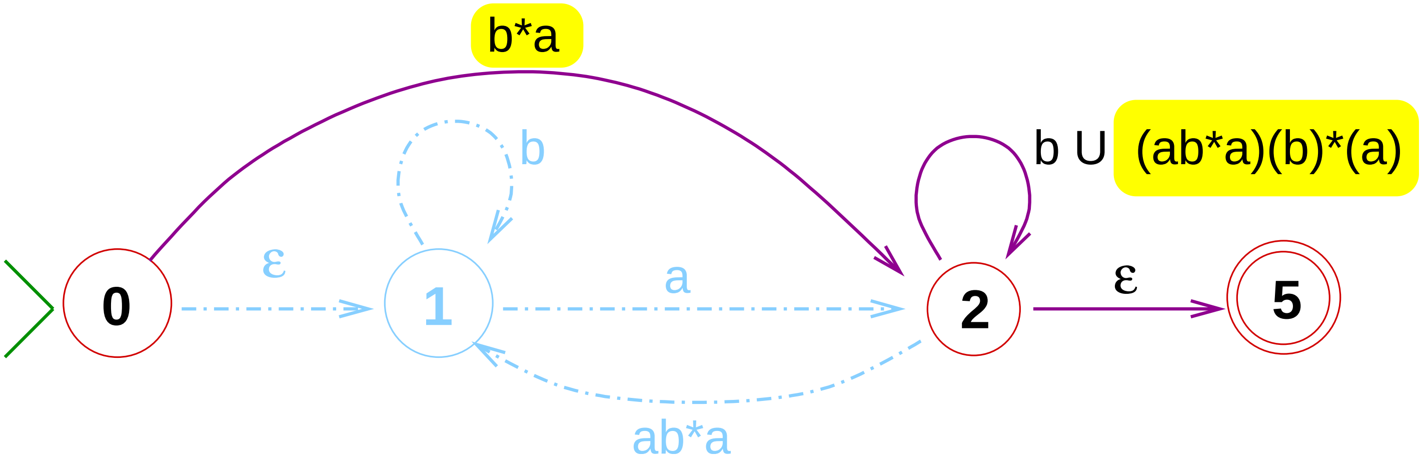
  and create an ε-transition from each accepting state of N to a_0.

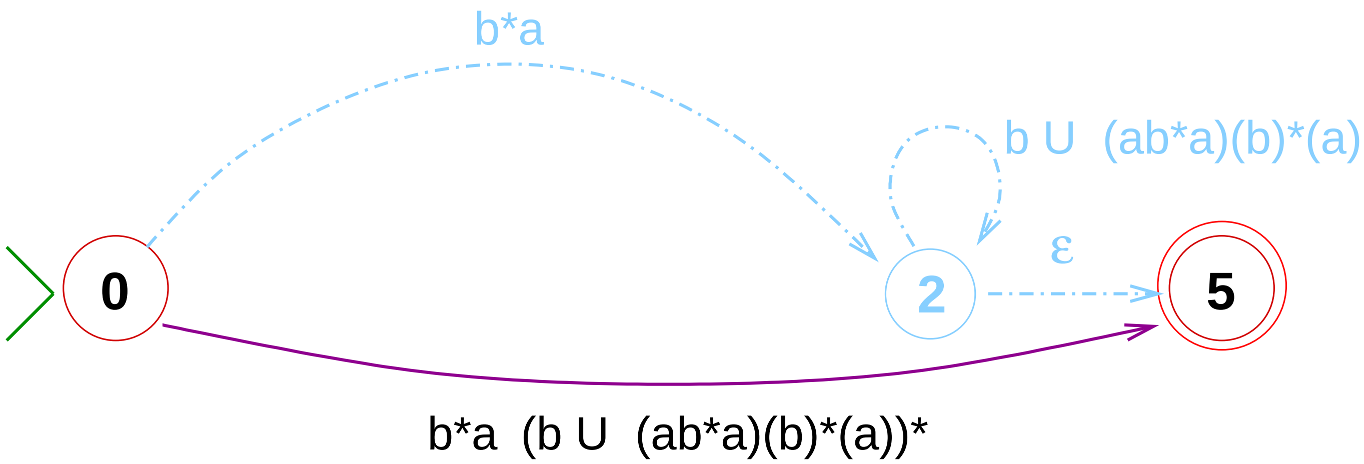
A working example




0
ε
1
b
a
2
b
ε
5
a
a
3
b

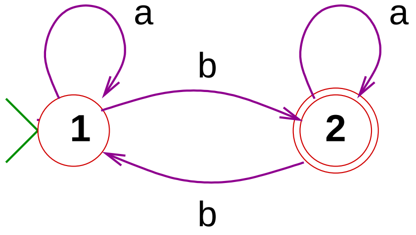

b
b
0
ε
1
a
2
ε
5
a
ab*a
a
3
b

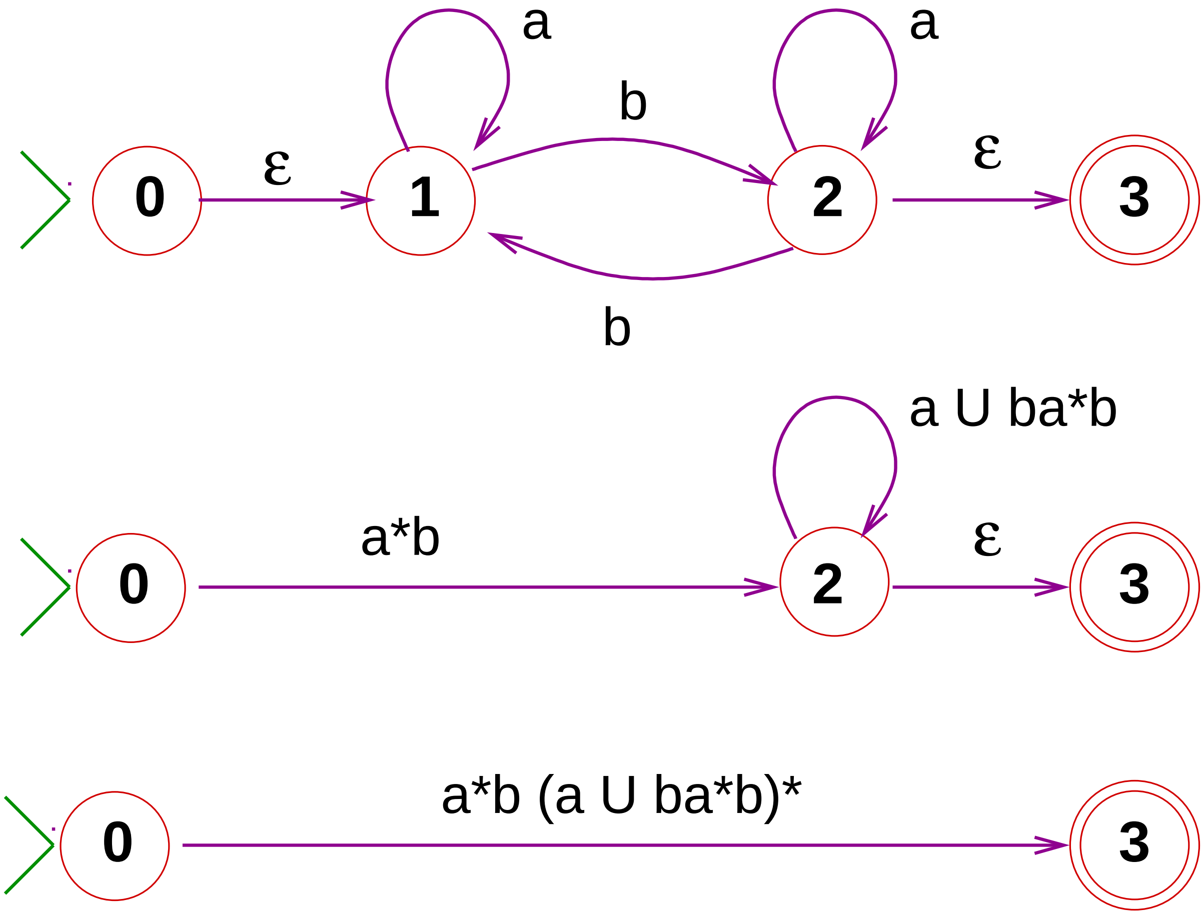

b*a
b
b U (ab*a)(b)*(a)
ε
a
ε
0
1
2
5
ab*a



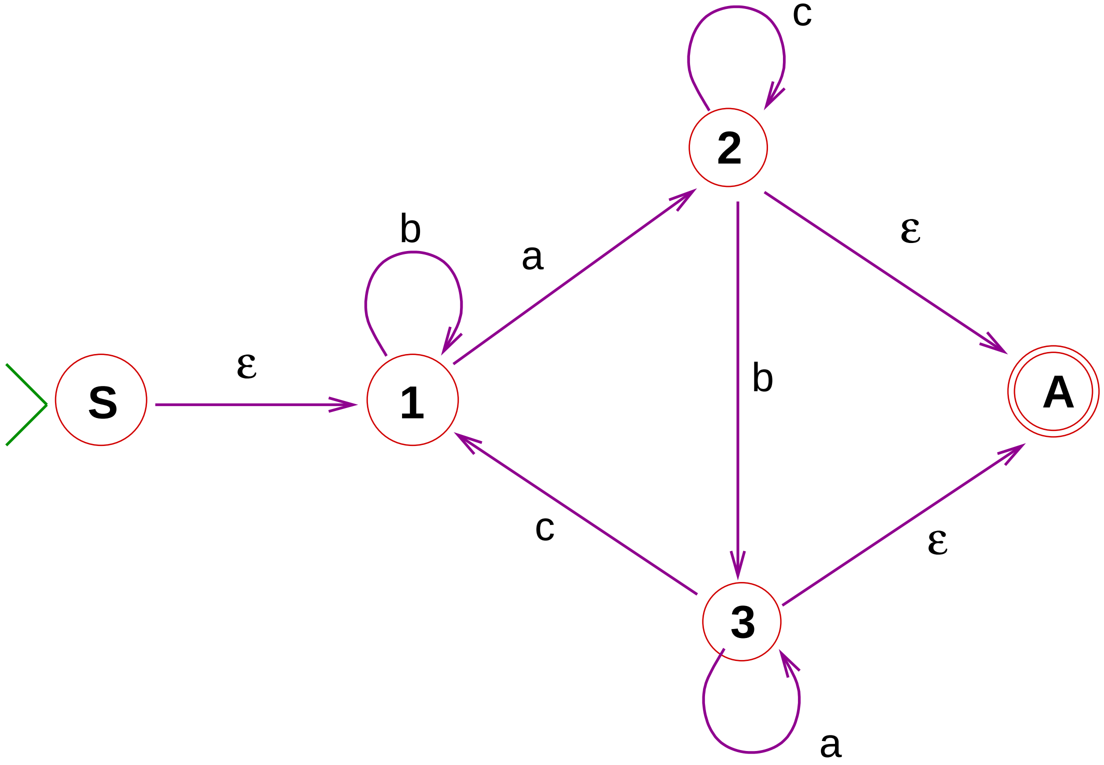
$$\mathcal{L}(N) = \mathcal{L}(\mathtt{b}^{*}\cdot\mathtt{a}\cdot(\mathtt{b}\cup(\mathtt{a}\cdot\mathtt{b}^{*}\cdot\mathtt{a})\cdot(\mathtt{b})^{*}\cdot(\mathtt{a}))^{*})$$

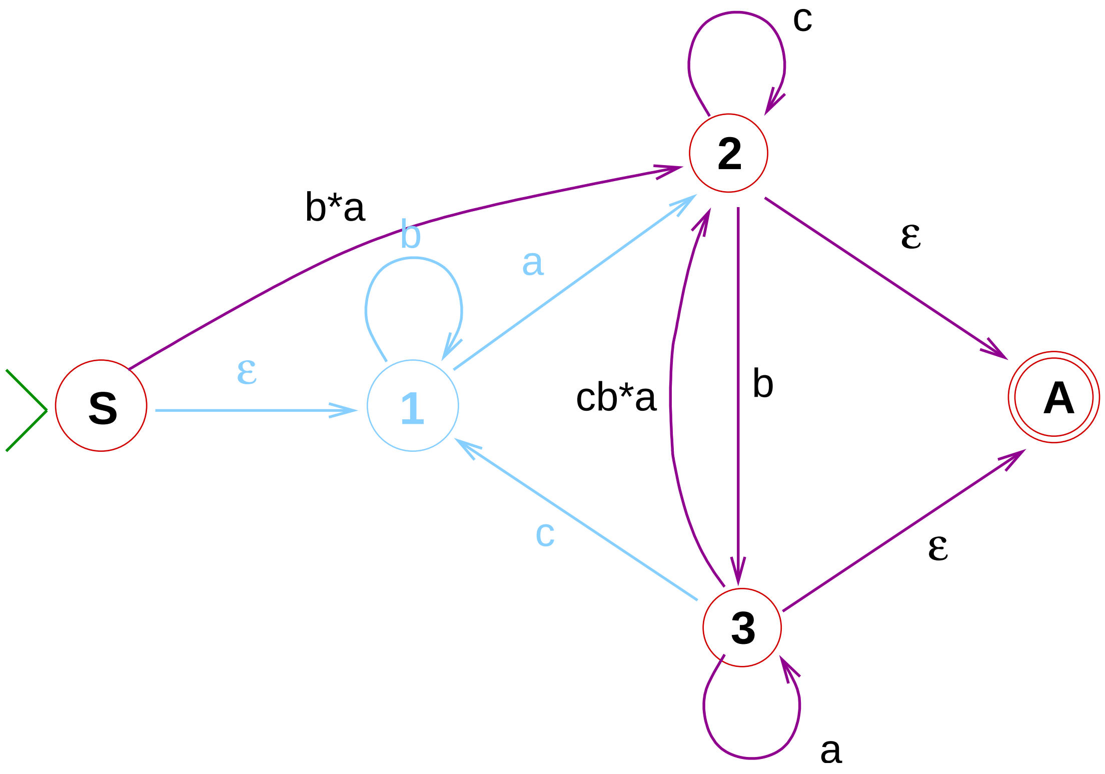
Another example

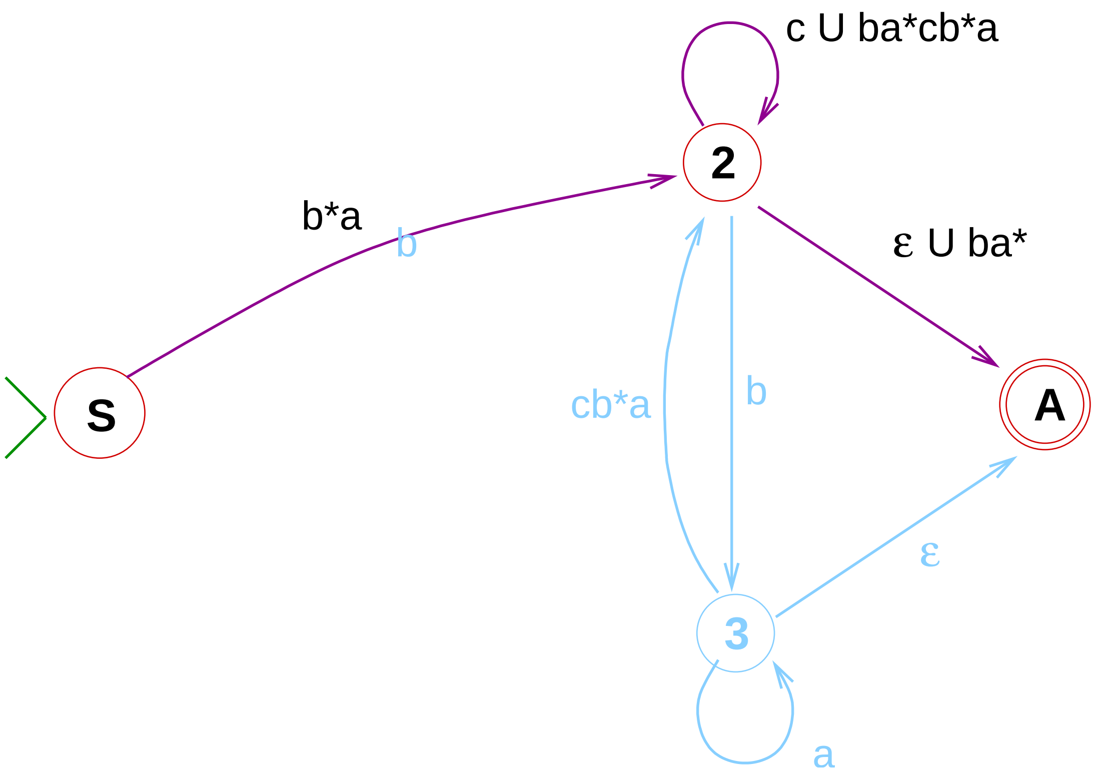


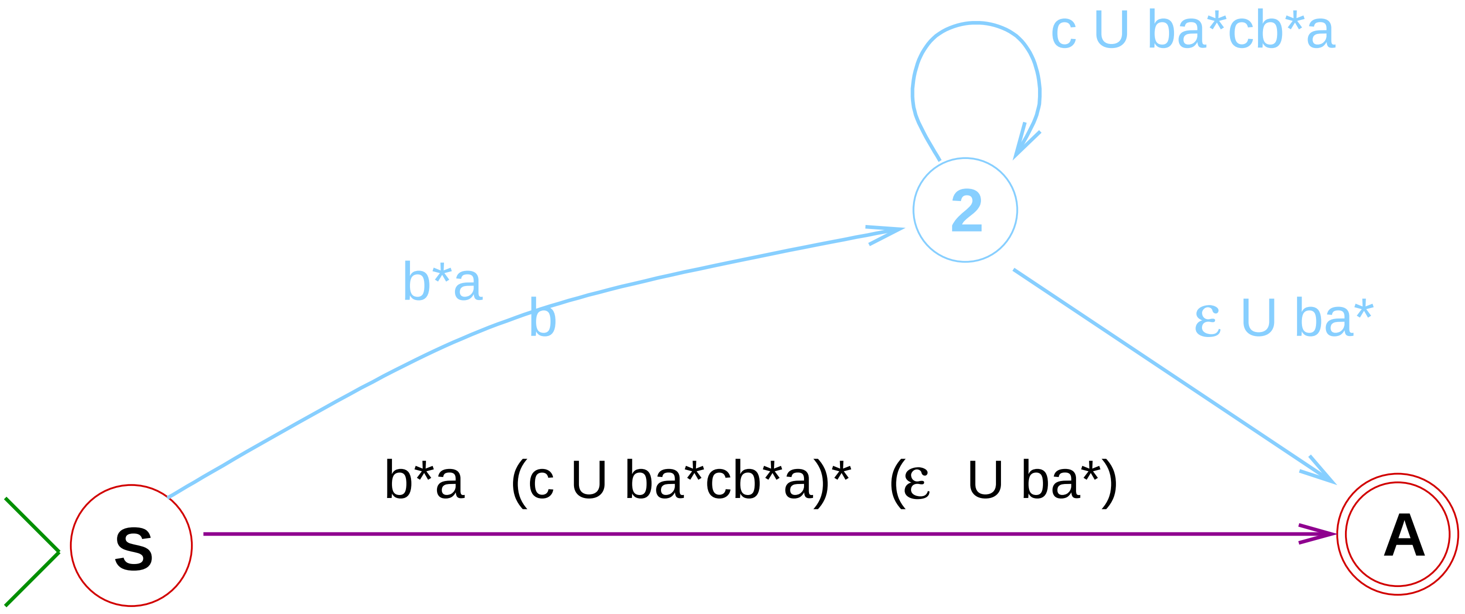

a
a
b
ε
ε
0
1
2
3
b
a U ba*b
a*b
ε
0
2
3
a*b (a U ba*b)*
0
3

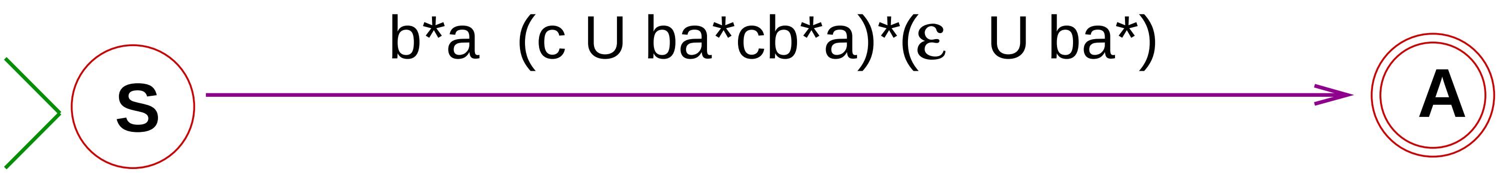
Yet another example




c
2
b*a
b
a
ε
S
ε
1
cb*a
b
A
c
ε
3
a


c U ba*cb*a
2
b*a
b
ε U ba*
S
cb*a
b
A
ε
3
a


c U ba*cb*a
2
b*a
b
ε U ba*
b*a (c U ba*cb*a)* (ε U ba*)
S
A


b*a (c U ba*cb*a)*(ε U ba*)
S
A

Summary

- The collection of DFA-recognized languages is closed under set operations (complement and product constructions)
- A language is NFA-recognized IFF it is DFA-recognized (Powerset construction)
- The collection of recognized languages is closed under all set/language operations.
- Therefore every regular language is recognized.
- Every recognized language is regular (state-elimination construction)

Two-way DFAs

Additional deterministic read-only algorithms

- Consider the language L over $[\mathtt{a}-\mathtt{z}]$ of words that include all letters. No English word is in L, but probably every book.
- L is a regular language: it is the intersection of the 26 languages $\{w \mid w$ has $\sigma\}$ for $\sigma = \mathtt{a}, \mathtt{b}\ldots$.
- The smallest DFA that recognizes L has $> 2^{26} > 67,000,000$ states.
- The smallest NFA recognizing L has 27 states.
- Is there a *deterministic* *algorithm* that does it with a manageable number of states?

A deterministic algorithm for the all-letters problem

- Algorithm: Scan for each digit separately, and repeat.

- This cannot be done if we only read forward!
  The cursor would have to be scrolled back (or repositioned).

- SO let’s imagine a device that behaves just like an automaton,
  but can move the cursor both ways.

Some challenges

- Symbol read determines not only next state,
  but also next move: forward or backward.
- To detect the ends of the input string it must have end-markers,
  say $>$ (the ***gate***) on the left,
  and $\sqcup$ (the ***blank***) on the right.
- Termination is not by reading through,
  but needs to be declared by a final accept state.
  (We need not guarantee termination.)

Two-way automata

A ***two-way automaton (2DFA)*** over an alphabet Σ:

- Finite set of states Q
- $s \in Q$, the *initial state*
- $a \in S$, the *accepting state*
- Transition partial-function: $\delta : Q \times \Gamma \rightharpoonup Q \times \mathbf{Act}$
  where $\Gamma = \Sigma \cup \{>, \sqcup\}$ and $\mathbf{Act} = \{+, -\}$.
- Write $q \xrightarrow{\sigma(\alpha)} p$ for $\delta(q, \sigma) = \langle p, \alpha \rangle$

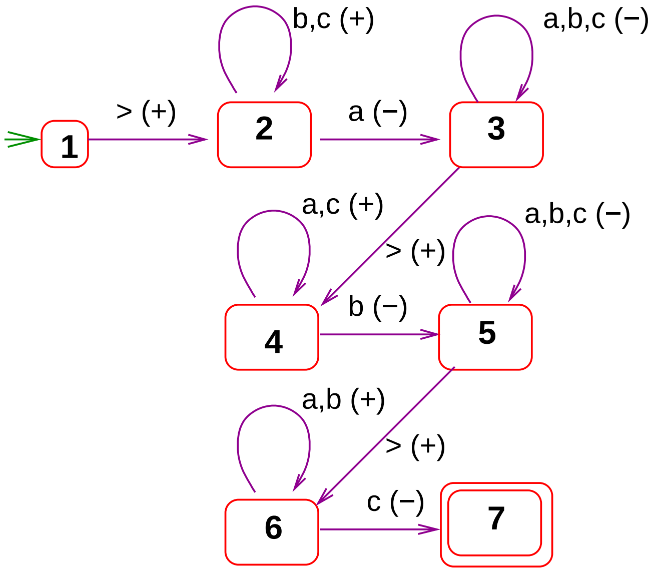
Two-way automata

- $\delta: \; Q \times \Gamma \;\rightharpoonup\; Q \times \mathbf{Act}$
  where $\Gamma = \Sigma \cup \{>, \sqcup\}$ and $\mathbf{Act} = \{+, -\}$.
- Write $q \overset{\sigma(\alpha)}{\rightarrow} p$ for $\delta(q, \sigma) = \langle p, \alpha \rangle$

The intent:

- Γ end-markers $\boxed{> \textit{ (gate)}}$ and $\boxed{\sqcup \textit{ (blank)}}$ added to Σ
- Example: Input 001201 appears as $>001201\sqcup$
- The $\boxed{\textit{actions}}$ $+$ and $-$ stand for “step forward” and “step back.”

Example: The strings using all of `a,b,c`



- With 26 in place of 3 we'd have 53 states,
  as opposed to $> 67,000,000$ states in the smallest DFA!

Operation of 2DFAs: configurations

- For DFAs we could generate the relation $p \xrightarrow{w} q$ inductively, as a function of w.
- This is no longer the case for 2DFAs: here we *must* account for the cursor position and keep record of the entire input for future use.
- A ***cursored-string*** over Σ is a $\Sigma-$string with one underlined symbol-position.
- A ***configuration (cfg)*** is a pair $(q, \check{w})$ where
  - ⋆ q is a state, and
  - ⋆ $\check{w}$ is a cursored-string,

    That is, $($ state, cursored-string $)$.
- Example: $(q, >0101\underline{1}00\sqcup)$
- The ***initial cfg for input w*** is the cfg $(s, \underline{>}w\sqcup)$.

The YIELD relation

- The ***Yield*** relation $\Rightarrow$
  (or $\Rightarrow_M$ when it matters which M) is obtained by:
-
  - ⋆ If $q \overset{\gamma(+)}{\longrightarrow} p$
    then $(q, u\underline{\gamma}\tau v) \;\Rightarrow\; (p, u\gamma\underline{\tau}v)$
  - ⋆ If $q \overset{\gamma(-)}{\longrightarrow} p$
    then $(q, u\tau\underline{\gamma}v) \;\Rightarrow\; (p, u\underline{\tau}\gamma v)$
  - ⋆ Nothing else
- If the given cfg is $(q, 01101\underline{0})$,
  and $q \overset{0(+)}{\longrightarrow} p$, then the transition above does not apply.

  The same holds when invoking a transition $q \overset{0(-)}{\longrightarrow} p$
  for a configuration with a cursor at the head of the string, such as $(q, \underline{0}11010)$.

Traces, acceptance, recognition

- A cfg $c = (q, u\gamma v)$ is ***terminal*** if no transition applies (no yield). It is a ***accepting*** its state is accepting state a.
- A ***trace*** of M for input w is a sequence of

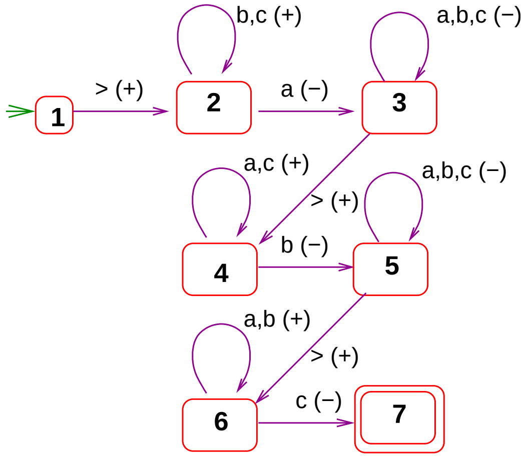
  $$c_0 \Rightarrow c_1 \Rightarrow \cdots$$

  where c_0 is initial for w, and either

  1. the sequence is infinite; or
  2. the sequence is finite, and its last cfg is terminal.
- The trace is ***accepting*** if it is finite and its last cfg is accepting.
- M ***accepts*** $w \in \Sigma^*$ if it its trace for input w is accepting.
- The language ***recognized*** by M is

  $$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$$

Example



Accepting trace for trace of M above for $w = \mathbf{bcab}$:

$$(1, \underline{>}\mathbf{bcab}\sqcup)$$

$\Rightarrow (2, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$

$\Rightarrow (2, >\mathtt{b}\underline{\mathtt{c}}\mathtt{ab}\sqcup)$

$\Rightarrow (2, >\mathtt{bc}\underline{\mathtt{a}}\mathtt{b}\sqcup)$

$\Rightarrow (3, >\mathtt{b}\underline{\mathtt{c}}\mathtt{ab}\sqcup)$

$\Rightarrow (3, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$

$\Rightarrow (3, \underline{>}\mathtt{bcab}\sqcup)$

$\Rightarrow (4, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$

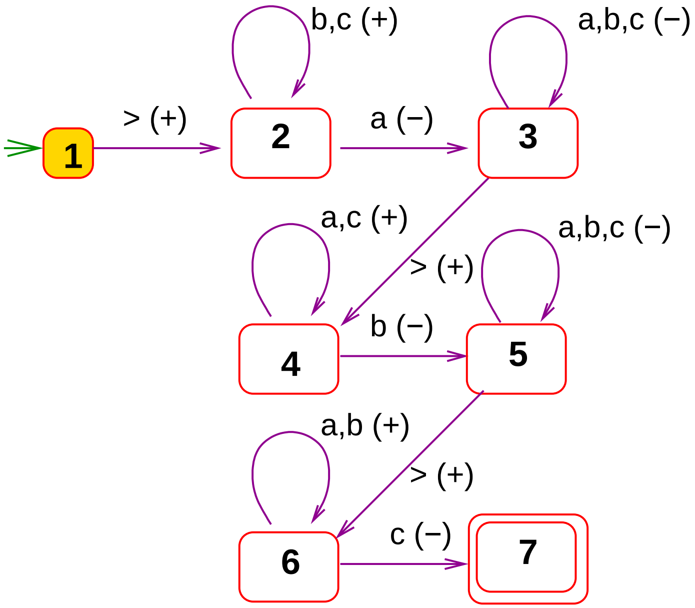
$\Rightarrow (5, \underline{>}\mathtt{bcab}\sqcup)$

$\Rightarrow (6, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$

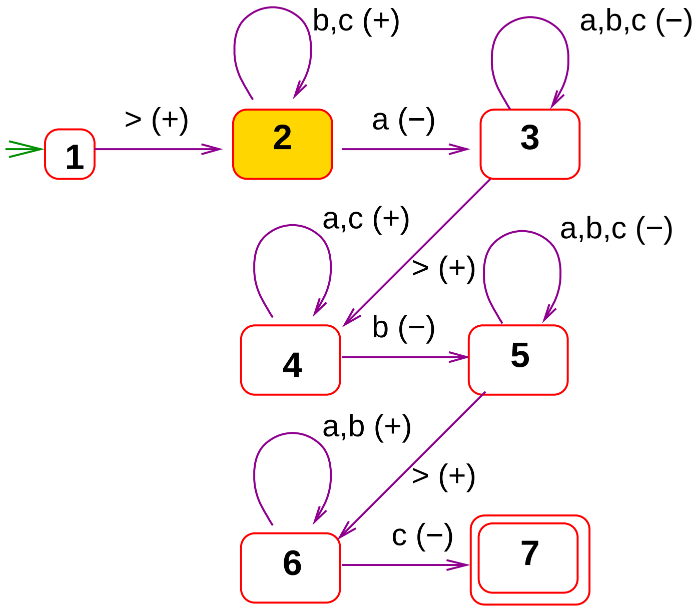
$\Rightarrow (6, >\mathtt{b}\underline{\mathtt{c}}\mathtt{ab}\sqcup)$

$\Rightarrow (7, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$

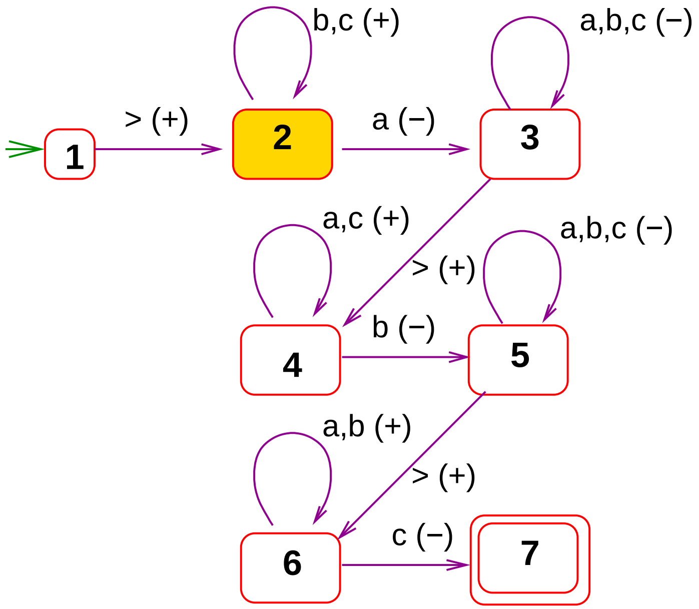
$(1, \underline{>}\mathbf{bcab}\sqcup)$



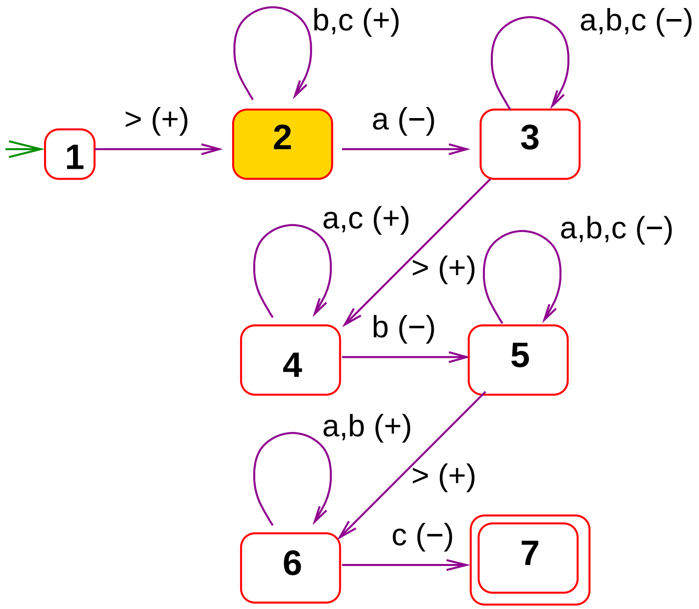
$(2,\ >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$



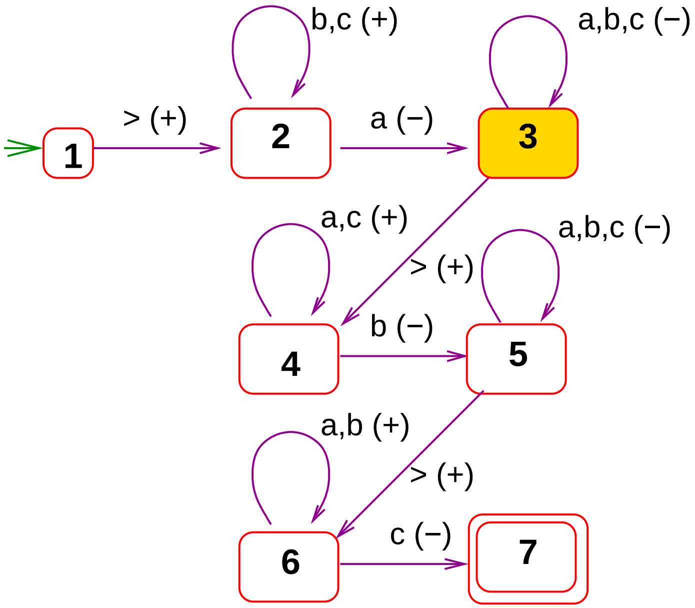
$(2, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup)$



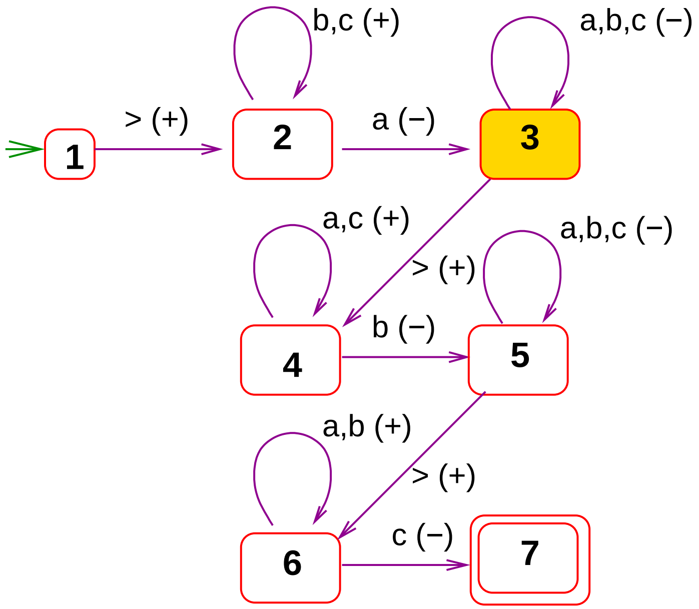
$(2, >\mathrm{bc}\underline{\mathrm{a}}\mathrm{b}\sqcup)$



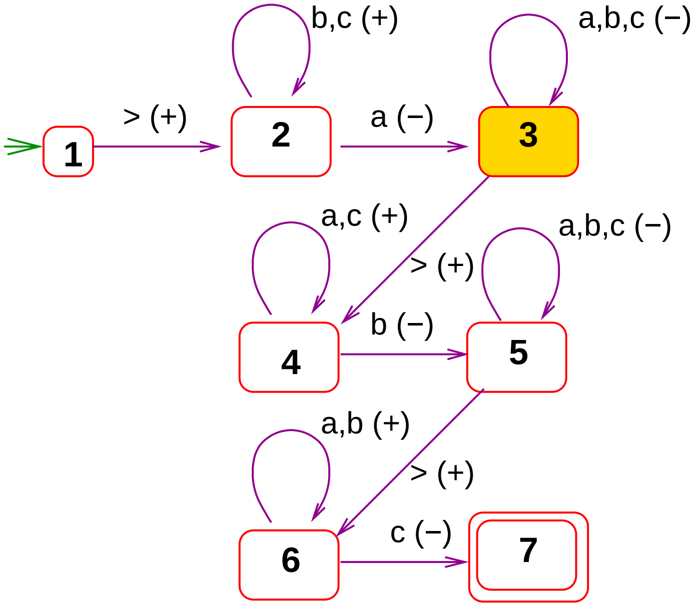
$(3,\ >b\underline{c}ab\sqcup)$



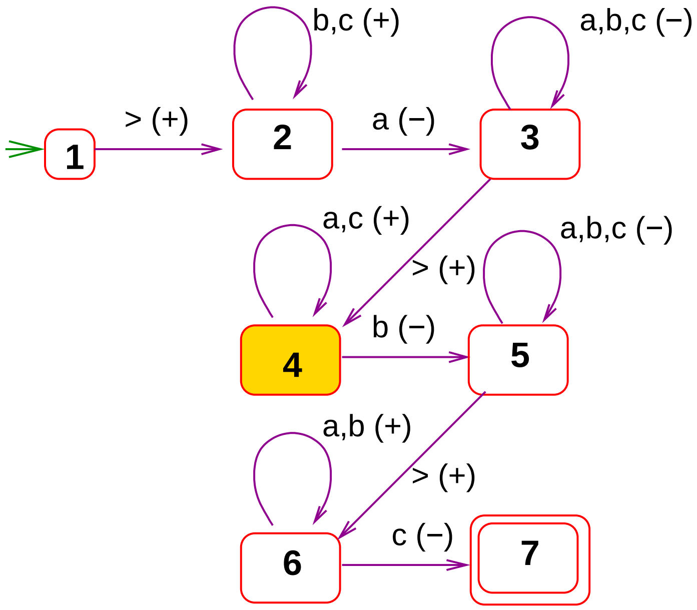
$(3, >\underline{\mathtt{b}}\mathtt{cab}\sqcup)$



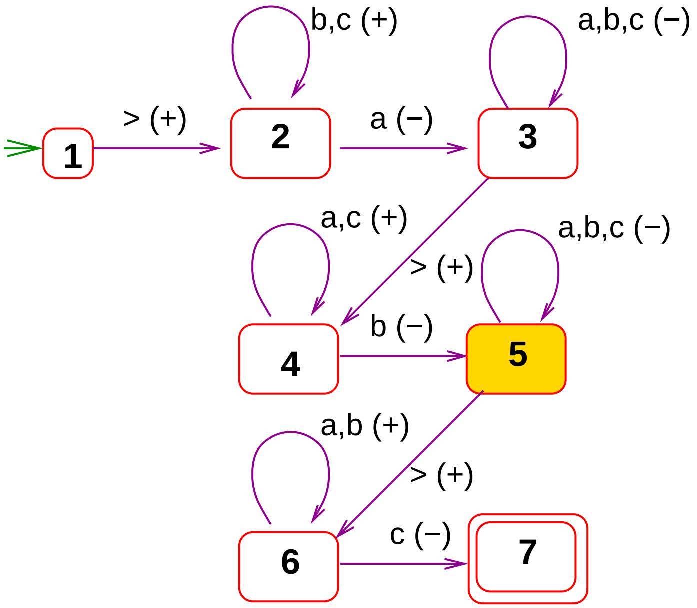
$(3, \underline{>}\mathtt{bcab}\sqcup)$



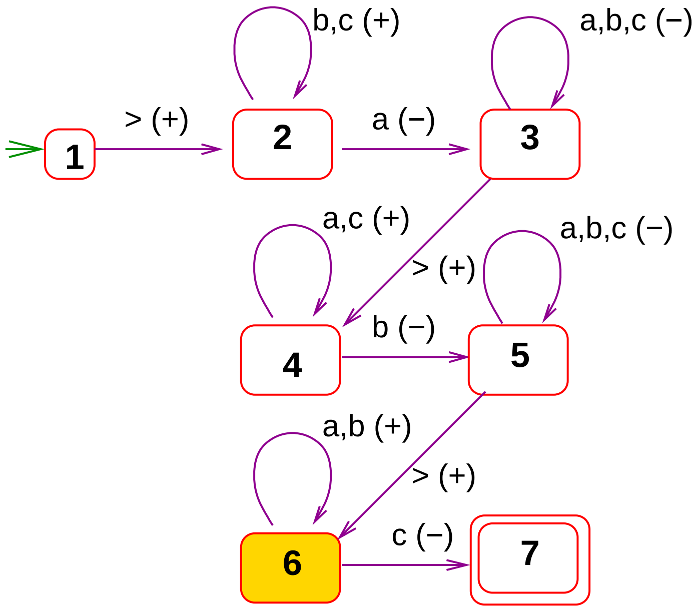
$(4,\ >\underline{b}cab\sqcup)$



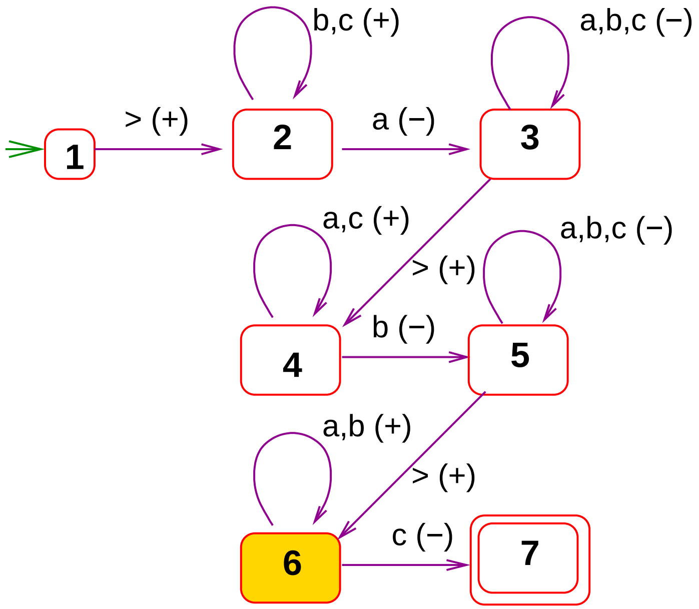
$(5, \underline{>}\text{bcab}\sqcup)$



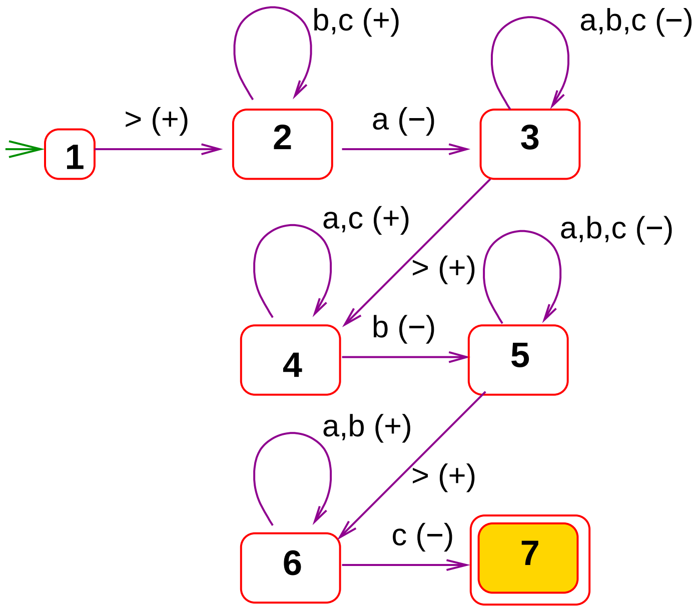
$(6, >\underline{b}cab\sqcup)$



$(6, >\mathrm{b}\underline{\mathrm{c}}\mathrm{ab}\sqcup)$



$(7, >\underline{b}cab\sqcup)$

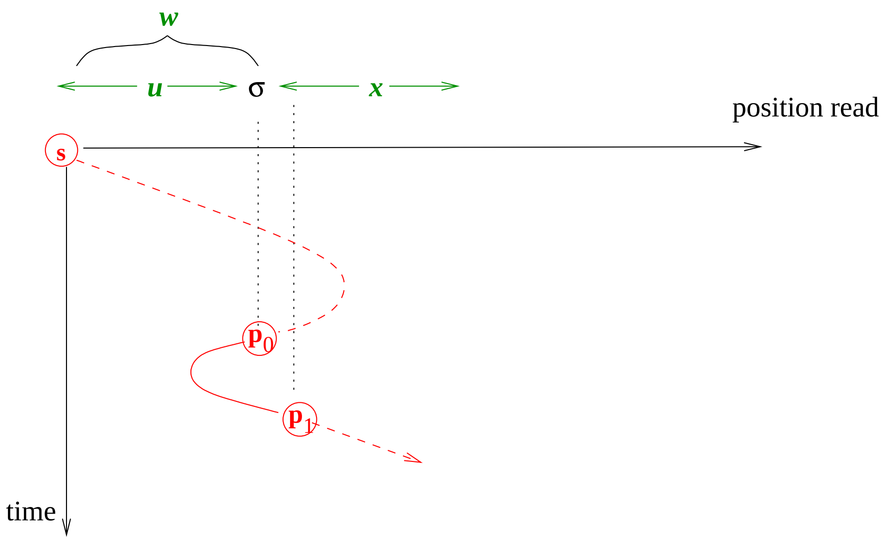


Two-way automata recognize just regular languages!

- Yet another characterization of regular languages!
- Adding nondeterminism to 2DFA still recognizes just regular languages!
- We still avoid extensible memory, so this is not a big surprise.

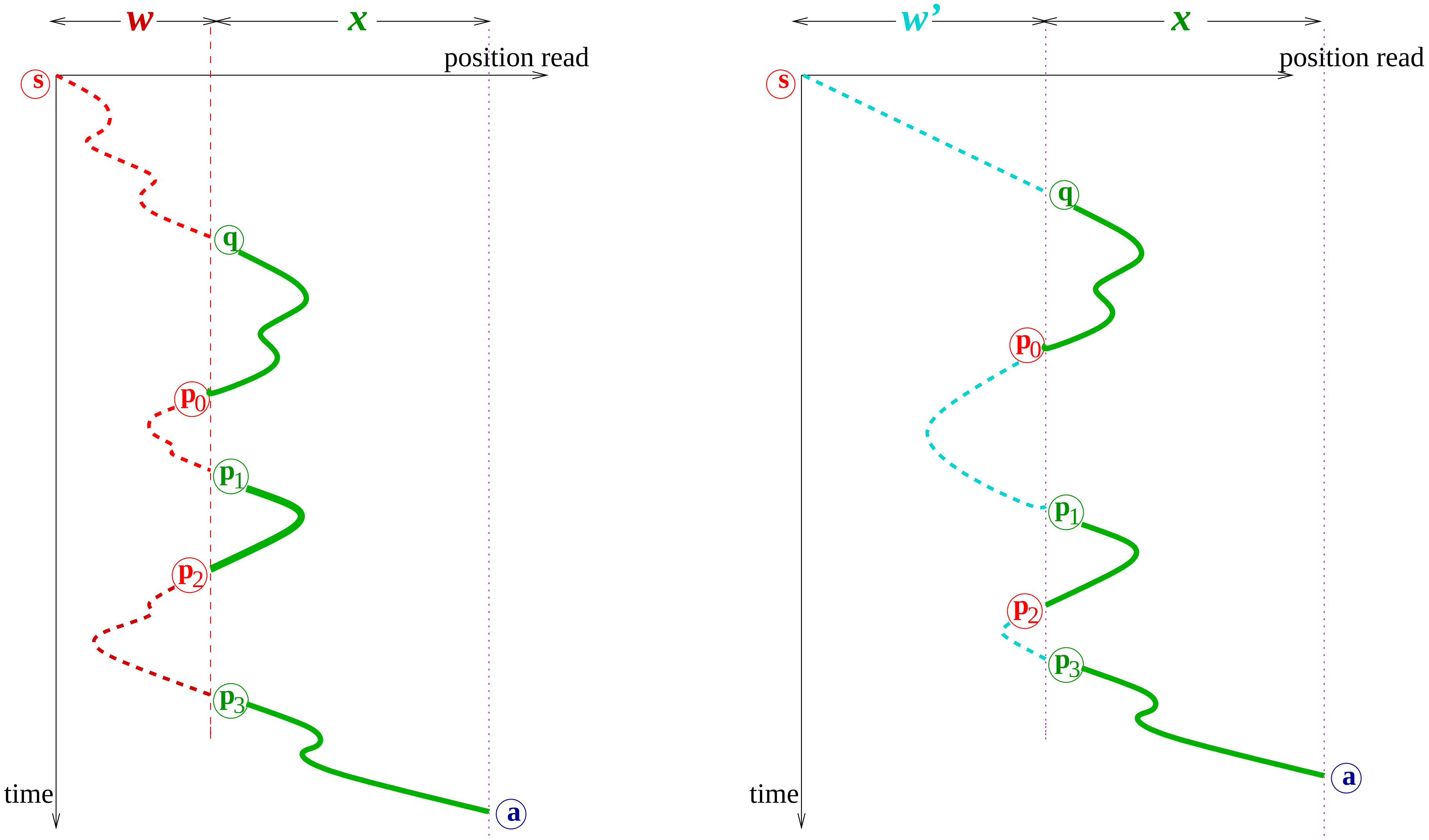
Proof outline

- DFA recognize languages with finitely many residues L/w.
- For each w a finite amount of info suffices to decide $x \in L/w$.
- For DFA the info is the state q reached: $s \xrightarrow{w} q$.
- For 2DFA the scan might cross out of w and into x .
  back in, and then out again into x.
- This is the info needed about w:
  If the reading cross back into w in a state
- The extra info:
  the pairs $(\textit{in}, \textit{out})$ of states
  s.t. crossing back into w in state *in*
  leads to crossing back out in state *out*.


w
u
σ
x
position read
s
p$_0$
p$_1$
time

Language recognized is regular!

- Say that $\langle p_0, p_1 \rangle$ is a *back-crossing pair.*
- L/w is determined by q reached by reading w,
  plus the set of back-crossing pairs for w:
  if w, w' reach the same state,
  and have the same crossing pairs, then $L/w = L/w'$.



x **in** L/w **IFF** x **in** L/w'

- For M with k states
  there are k^2 potential back-crossing pairs,
  and so 2^{k^2} possible descriptions of the situation at the border.
- Finitely many residues, albeit a lot, but still
  recognizing a regular language!

REGULARITY

The many facets of regularity

- Big equivalence of language properties, relating definitional to structural as well as computational properties.

The many facets of regularity

- Big equivalence of language properties, relating definitional to structural as well as computational properties.

$$\begin{aligned}\text{Regular} &\iff \text{Strictly-Regular}\\ &\iff \text{DFA-recognized}\\ &\iff \text{2DFA-recognized}\\ &\iff \text{NFA-recognized}\\ &\iff \text{has finitely many residues}\end{aligned}$$

- Another important characterization of regular languages is related to our automata-construction method.

The many facets of regularity

- Big equivalence of language properties, relating definitional to structural as well as computational properties.

  Regular $\iff$ Strictly-Regular
  $\iff$ DFA-recognized
  $\iff$ 2DFA-recognized
  $\iff$ NFA-recognized
  $\iff$ has finitely many residues

- Another important characterization of regular languages is related to our automata-construction method.
- One disappointment: It's all about languages and acceptors.
  What about functions and transducers?

The many facets of regularity

- Big equivalence of language properties, relating definitional to structural as well as computational properties.

Regular ⟺ Strictly-Regular
⟺ DFA-recognized
⟺ 2DFA-recognized
⟺ NFA-recognized
⟺ has finitely many residues

- Another important characterization of regular languages is related to our automata-construction method.
- One disappointment: It's all about languages and acceptors. What about functions and transducers?

FINITE STATE TRANSDUCERS

Finite-state transducers

- In a 2DFA the transition mapping indicates a choice of action: step forward or backward.

  In a deterministic ***finite-state transducer (DFT)*** the choice of action is an output string to be appended to an output device.

-

Finite-state transducers

- In a 2DFA the transition mapping indicates a choice of action: step forward or backward.

  In a deterministic **finite-state transducer (DFT)** the choice of action is an output string to be appended to an output device.

-

- Examples.

  - Double zeros: Input alphabet: 0,1.
    The DFS outputs 00 for 0 and 1 for 1.

Finite-state transducers

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  a choice of action: step forward or backward.

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    Output: phonetic text.
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  - ▸ Input alphabet: Latin.
    Output: Blanks replaced by ASCII $<$ *newline* $>$.

Formal definition of DFTs

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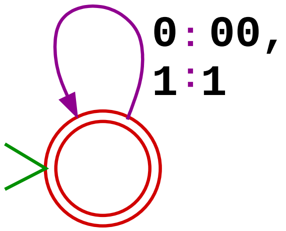
  - A partial-function $\delta : Q \times \Sigma \rightharpoonup \Gamma^* \times Q$.

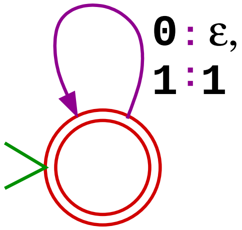
Examples

- Double zeros: The input is a binary string.
  Output: 00 for each 0 read and 1 for 1.



Examples

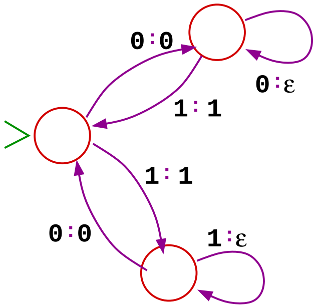
- Delete zeros: The input is a binary string.
  Output: ε for each 0 read and 1 for 1.



Examples

- Delete duplicate letters: The input is binary.

  Output: Remove duplicates, e.g. **001110** ↦ **010**.



Computing over streams

- A Given a set S a $\boxed{\textit{stream over } \Sigma}$ (or Σ***-stream***) is function $f : \mathbb{N} \to S$, i.e. an infinite sequence $a_0, a_1, \ldots$ where $a_i \in S$.
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- Example, every real number $a \in [0..1]$ has a decimal expansion as a stream $.a_0a_1a_2\ldots$ over the decimal digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. E.g. 1 is $.9999\ldots$, $\sqrt{2}/2$ is $.70710678118\ldots$ and $\pi/10$ is $.3141592653\cdots$.

Running DFA’s on streams: Büchi acceptors

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- Also bad:
  Acceptance is then determined by a prefix of the input.

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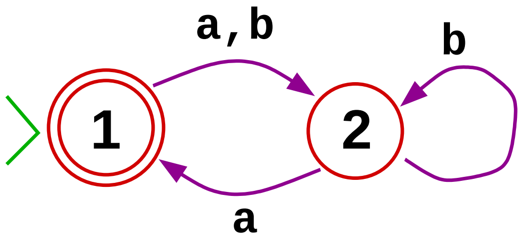
- The right idea (Büchi, 1962):
  Accept an input if its state-trace is in a “good” state
  infinitely many times.

Example 1

Here’s a DFA.



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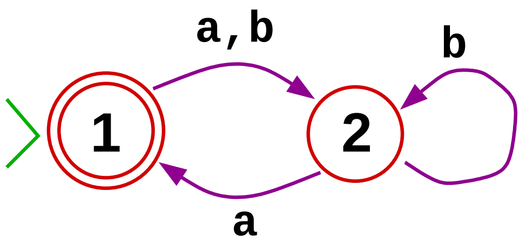

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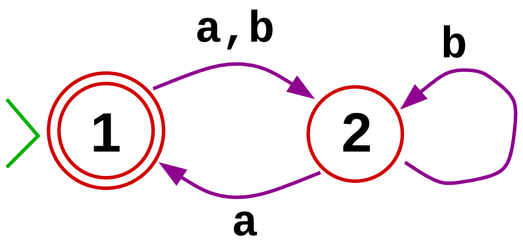

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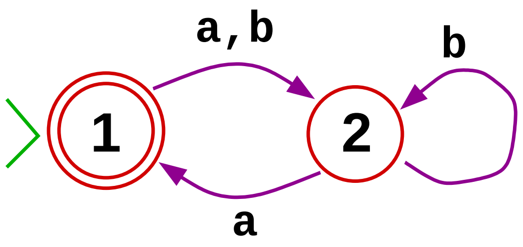

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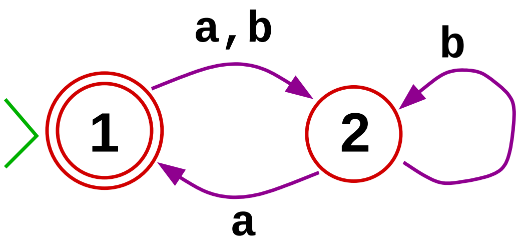

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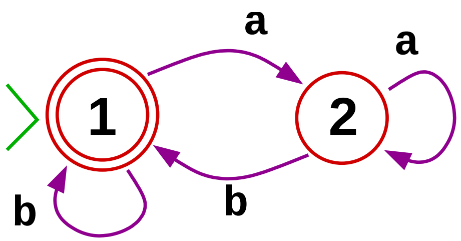
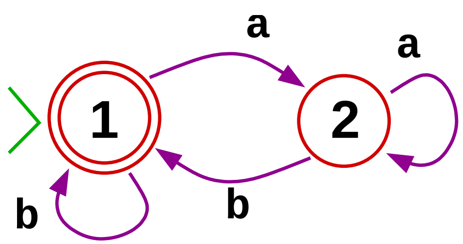
What streams are accepted?

With infinitely many a's.

Example 2



Example 2



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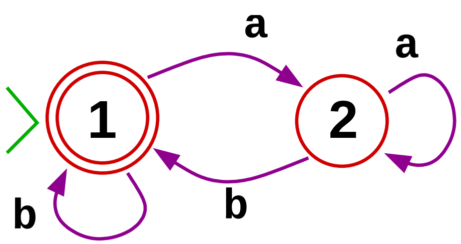
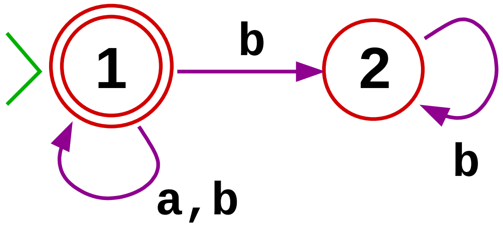


- What streams are accepted?
- Where every `a` is followed by some `b`.

Example 3



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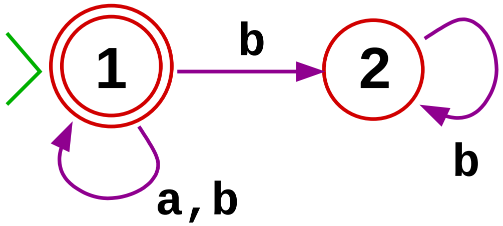

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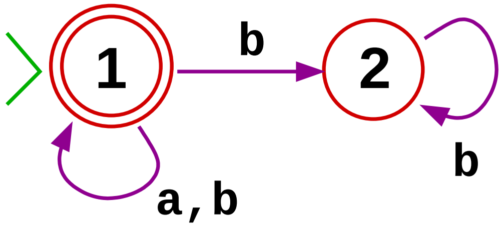

- What stream are accepted?
- With finitely many a’s.