INDUCTIVELY GENERATED DATA

Generative processes

- Virtually every infinite set considered in programming is ***generated by a process***.

  The fundamental example is the set $\mathbb{N}$ of natural numbers:

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- Implicit assumptions:

  The meanings of 0 and "next" are known and given.

Generating $\{0,1\}^*$

- **Base.** The ***empty string*** is in $\{0,1\}^*$.
- **Generative step.**
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• Note: We generate strings here "from the head";
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• But numerals are in fact generated from the tail:

$$[\mathbf{7654321}]_{10} = 1 + 10 \cdot [\mathbf{765432}]_{10}$$

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    Particular known objects are in S.
  - **Generative steps:**
    If certain objects are in S
    then so are vertain objects obtained from those.

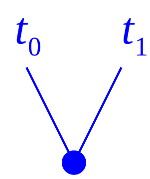
Another example: Binary trees

- ***Binary tree*** means here a *finite, ordered, unlabeled binary tree*

Base: The singleton tree • is in $\mathbf{BT}$.

Generative step:

If t_0, t_1 are binary trees then so is



Implicit assumptions:

We know what a singleton tree and juncture of trees mean.

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- Generate the set E of even natural numbers.

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  - Base: 0
  - Generative step: If $n \in E$ then $n{-}2 \in E$.

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- Main difference between IBT and PBT:
  No parentheses in PBT !

Lists of natural numbers

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  - $\square$ is a list of naturals.
  - If ℓ is a list and k a numeral then $k : \ell$ is a list.
- Examples: $1 : \square$, $0 : 101 : 10011 : 10 : \square$.

REASONING ABOUT INDUCTIVE DATA

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- ***So how can we hope to prove***

  that all natural numbers are such-and-such ?

Finitely generated infinities!

- The secret is that inductive data is generated by ***finite rules.***
- Therefore we have a finite tool for proving that all generated objects satisfy certain properties.

Following the process

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  and that each step maintains green-ness.
- Green-ness is here the process' ***invariant:***
  True at the outset, and preserved by the steps.

The principle of induction for $\mathbb{N}$

- Suppose $P(x)$ is a property of natural numbers x.
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- *A property of natural numbers that holds for zero*
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  is true of every natural number.
- The premise of the STEP is often called the “induction assumption”
  or the Induction Hypothesis (IH).

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  $2^{n+1} = 2^n + 2^n < 2^{n+1} + 2^{n+1} = 2^{n+2}$ nextt ($P(x)$ for $x = n{+}1$)
- By Induction, $2^x < 2^{x+1}$ for all $x \in \mathbb{N}$.

Try this...

- Prove by induction on $\mathbb{N}$ that $x \leqslant 2^x$ for all $x \in \mathbb{N}$.
  We are given that exponentiation is an increasing function.
- By Induction $x \leqslant 2^x$ for all $x \in \mathbb{N}$.

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  - Base: For $x = 0$ we have $x^2 = 0 < 1 = 2^x$.
  - Step: Assume $n \leqslant 2^n$. Then

$$
\begin{aligned}
n + 1 &\leqslant 2^n + 1 && \text{(IH)} \\
&= 2^n + 2^0 \\
&\leqslant 2^n + 2^n && \text{(exponentiation is increasing)} \\
&= 2^{n+1}
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0 + 1 + \cdots + x &= 0 + 1 + \cdots + n + (n+1) \\
&= \tfrac{n(n+1)}{2} + (n+1) \qquad \text{(IH)} \\
&= (n+1)(\tfrac{1}{2}n + 1) \\
&= \tfrac{1}{2}(n+1)(n+2) \\
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  That is, $(\star)$ for $x=n+1$.

- Conclude: $(\star)$ holds for every $x\in\mathbb{N}$.

Involving other data

- A property of natural numbers may refer to non-numeric data!

  ($\star$) *Every set with* x *elements has* 2^x *subsets*

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  Subsets of S: $A_1, \ldots, A_{2^n}$, $A_1 \cup \{a\}, \ \ldots, \ A_{2^n} \cup \{a\}$
  which are all different. So S has $2^n + 2^n = 2^{n+1}$ subsets.

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- By Induction ($\star$) for all $x \in \mathbb{N}$.

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- We refer to this as Shifted Induction:
  - **Base.** $2^2 = 4 > 2$
  - **Step.** $n^2 > n$ implies

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &> n + 2n + 1 \quad \text{(IH)} \\ &> n + 1 \quad \text{since } n > 0) \end{aligned}$$

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$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &> n + 2n + 1 \quad \text{(IH)} \\ &> n + 1 \quad \text{since } n > 0) \end{aligned}$$

- Conclusion: $x^2 > x$ for all integers $x > 1$.

Shifted Induction

- The template for such reasoning is ***Shifted Induction***
- Given a property $P(x)$ of natural numbers, and $b \in \mathbb{N}$,

- Assume:
  - **Shifted Base.** P true of b; and
  - **Shifted Step.** For all $n \geqslant b$, $P(n)$ implies $P(n+1)$
- Conclude: $P(x)$ for all $x \geqslant b$.
- Induction is a special case, with $b = 0$.

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  - Step. If $3^n > 5 \cdot 2^n$ then

$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n \\ &> 3 \cdot (5 \cdot 2^n) \quad \text{(IH)} \\ &> 2 \cdot 5 \cdot 2^n \\ &= 5 \cdot 2^{n+1} \end{aligned}$$

Inductive reasoning in general

- The principle of inductive reasoning applies
  to any inductively generated set S, not just $\mathbb{N}$.
- If $P(x)$ makes sense for $x \in S$,
  is true for every base element of S
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  and remains true under the generative steps for S,
  then $P(x)$ is true for all $x \in S$.
- The underlying reason is the same as for $\mathbb{N}$:
  as the elements of S are generated,
  the property P invariantly holds.

Involving non-numeric data

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  - **Step.** Assume $P(n)$.
    To prove $P(n+1)$ let S be a set with $n+1$ elements.
    Choose $a \in S$ and let $S^- = S - \{a\}$.
    By IH S^- has 2^n subsets $A_1, \ldots, A_{2^n}$.
    The subsets of S are $A_1, \ldots, A_{2^n}, A_1 \cup \{a\}, \ldots, A_{2^n} \cup \{a\}$, which are all different.
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  - By Induction on $\mathbb{N}$ $P(x)$ holds for all $x \in \mathbb{N}$.

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- By induction on $\{0,1\}^*$ $(\star)$ for all $w \in \{0,1\}^*$.

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- ***Problem: This is a property of a pair of strings!***

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- **Step:** Assume $(\star)$ for $x = w$.

  For $x = \sigma w$ we have for all $u \in \Sigma^*$

$$\begin{aligned} |\sigma w \cdot u| &= |\sigma(w \cdot u)| \\ &= 1 + |w \cdot u| \\ &= 1 + |w| + |u| \quad \text{(IH)} \\ &= (|\sigma w|) + |u| \end{aligned}$$

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- By induction on Σ^* conclude $(\star)$ for all $x \in \Sigma^*$.

Unambiguous PBT's

- A PBT t is **unambiguous** if it is exactly one of:
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- Theorem: *Every PBT is unambiguous*
- ***How to prove this?***
- Induction on terms does not work:
  If $t = \wedge\, t_0\, t_1$ what can we possibly conclude from assuming that t_0 and t_1 are unambiguous?

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  $w \in \Sigma^*$ is ***unambiguous*** if it can'e be read
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- If $w = t_1 \cdots t_k = t'_1 \cdots t'_m$
  then $m = k$ and $t_i = t'_i$ for $i \in [1..k]$.

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    So $w = q\, r\, t_2 \, \cdots \, t_k = q'\, r'\, t'_2 \, \cdots \, t'_m$. By IH $k{+}1 = m{+}1$, $q = q'$, $r = r'$ and $t_i = q_i$ for $i = 2..k$.

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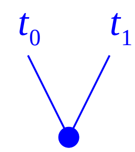
- By induction on Σ^* we conclude that every $w \in \Sigma^*$ is unambiguous.
- In particular, every PBT t is a concatenation of 1 string, and therefore must be unambiguous as a term.

Induction over binary trees

- Recall that the set of binary trees is generated

  from a base tree • by juncture:

  if t_0, t_1 are binary trees then so is



- Let $P(x)$ be a property that makes sense for any binary tree t.

- If we can show that

  - **Base:** $P(\bullet)$; and
  - **Step:** If both $P(t_0)$ and $P(t_1)$
    then $P(t)$ for the juncture t above of t_0 and t_1

  then $P(t)$ is true for all binary trees t.

Example: Odd size of binary trees

- ***Can a binary tree have an even number of nodes?***

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  Let t be obtained from t_0 and t_1. then the size of t is $n_0 + n_1 + 1$ which is again odd.
- By induction on binary tree we conclude that

  $P(t)$ for all binary trees t.

The mother of all inductions

- The Induction principle applies to all inductively generated sets.
- But induction on $\mathbb{N}$ is the one usually invoked. ***Why?***

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- Take generated set $\mathbb{G}$, $P(x)$ a property of $x \in \mathbb{G}$.
- Obtain induction over $x \in \mathbb{G}$ for property $P(x)$
  as induction over $n \in \mathbb{N}$ for the property:

  $P(n)$ *is true for all* $x \in \mathbb{G}$ *generated in* $\leqslant n$ *steps*

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- Note that $\mathbb{N}$ is the ***simplest*** infinite generated set:
  one initial object, ***one*** generative rule, involving ***one*** premise!