C241 Test One

INSTRUCTIONS:

- Put your name and ID number in the upper-right of each page.
- Place your final answer on the test in the space provided. Scratch work is not graded, but neatness counts. Mark the parts of your answers clearly.
- Exam time is 75 minutes.
- There are eight questions, weighted as indicated.
- If you find an error or ambiguity in a problem, describe it and state how you would correct it. Then go on to answer the question using your interpretation.

1. (12 points) List the following sets:

   (a) \( \{2^i \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\} = \{1, 2, 4, 8, 16, 32, 64, 128, 256\} \)

   (b) \( \{i^2 \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\} = \{0, 1, 4, 9, 16, 25, 36, 49, 64\} \)

   (c) \( \{2k + 1 \mid k \in \mathbb{N}\} = \{1, 3, 5, 7, 9, \ldots\} \)

   (d) \( \{m \mid 23 < m < 29 \text{ and } m \text{ is a prime number}\} = \emptyset \)

2. (15 points) Let \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{1, 3, 5\} \). List

   (a) \( A \cup B = \{1, 2, 3, 4, 5\} \)

   (b) \( A \cap B = \{1, 3, 5\} \)

   (c) \( A \setminus B = \{2, 4\} \)

   (d) \( B \setminus A = \emptyset \)

   (e) \( (A \times \{1\}) \cup (B \times \{2\}) = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (1, 2), (3, 2), (5, 2)\} \)
3. (8 points) Recall (Definition 2.14) that \( G \subseteq A \times A \) is a rooted graph iff there is a node \( r \in A \) such that for every \( x \in A \) there is a path from \( r \) to \( x \) in \( G \). In the graph to the right, which nodes can serve as \( r \) in this definition?

*Only \( b \) can serve as the root of this graph. There is no path to \( b \) from any node—\( b \) has in-degree 0—neither \( a \) nor \( c \) nor \( d \) can be a root.*

4. (10 points) Let \( X = \{a, b\} \) and \( Y = \{ab\} \) be alphabets. Note that \( X \) has two elements and \( Y \) has one element. Define the language \( L \subseteq (X \cup Y)^+ \) to be

\[
L = \{u^*v^*u \mid u \in X^*, \ v \in Y^+\}
\]

List all the words in \( L \) that are less than seven letters long.

\[
\{ab, a^*ab^*a, b^*ab^*b, a^*a^*ab^*a^*a, a^*b^*ab^*a^*b, b^*a^*ab^*b^*a, b^*b^*ab^*b^*b, a^*ab^*ab^*a^*a, b^*ab^*ab^*ab^*b^*a, b^*b^*ab^*ab^*b^*b, a^*ab^*ab^*ab^*a^*a, b^*ab^*ab^*ab^*b^*b\}
\]

5. (15 points) Recall that a relation \( R \subseteq A \times A \) is:

- reflexive iff \( \forall x \in A: (x, x) \in R \).
- symmetric iff \( \forall (x, y) \in R: (y, x) \in R \).
- transitive iff \( \forall (x, y), (y, z) \in R: (x, z) \in R \).

(a) Add the minimal number of edges needed to make the graph \( R \subset A \times A \), below, symmetric and transitive.

(b) Is the new relation reflexive? Is it irreflexive?

*It is not reflexive because it does not contain \((a, a)\). It is not irreflexive because, for one instance, it contains \((b, b)\).*
6. (15 points) Two graphs are isomorphic if they “have the same structure.”

(a) Are the two graphs below isomorphic?

![Graph Diagram]

The function \( f = \{(a, N), (b, M), (c, P), (d, O)\} \) is an (in fact, the only) isomorphism between the two relations. The picture above suffices as a solution, but in more detail,

\[
\begin{array}{c|c|c}
(x, y) & (f(x), f(y)) & \checkmark \\
\hline
(a, a) & (N, N) & \checkmark \\
(a, b) & (N, M) & \checkmark \\
(a, c) & (N, P) & \checkmark \\
(b, b) & (M, M) & \checkmark \\
(b, c) & (M, P) & \checkmark \\
(d, b) & (O, M) & \checkmark \\
(d, c) & (O, P) & \checkmark \\
\end{array}
\]

(b) Write a formal definition of isomorphism.

**Definition.** Two directed graphs \( R \subseteq A \times A \) and \( S \subseteq B \times B \) are isomorphic iff there exists a bijection \( f: A \rightarrow B \) such that \((x, x') \in R\) iff \((f(x), f(x')) \in S\). The \( f \) is called an isomorphism between \( R \) and \( S \).

7. (10 points) A binary tree is a tree in which the out-degree of every node is either 0 or 2. Draw all the distinct (non-isomorphic) binary trees containing seven nodes.

![Binary Tree Diagrams]
8. (10 points) A partial description of the Stmt Programming Language is shown below. Write a program in the Stmt programming language divides $A$ by $B$ using only addition and subtraction, leaving the quotient in program variable $q$ and the remainder in $r$. It matters only that the program is correct; don’t worry about efficiency. Your solution should satisfying the assertions

$$\{ x = A \in W \text{ and } y = B \in W \}$$

begin
  $q := 0$;
  $r := x$;
  while $r \geq y$ do
    begin
      $q := q + 1$;
      $r := r - y$
    end
  end

$\{ qy + r = A \text{ and } r < B \}$

Stmt language:

\[
\langle \text{stmt} \rangle ::= \langle \text{identifier} \rangle := \langle \text{term} \rangle \quad \text{(assignment)}
\]

\[
| \quad \text{if} \langle \text{test} \rangle \, \text{then} \, \langle \text{stmt} \rangle \, \text{else} \, \langle \text{stmt} \rangle \quad \text{(conditional)}
\]

\[
| \quad \text{while} \langle \text{test} \rangle \, \text{do} \, \langle \text{stmt} \rangle \quad \text{(repetition)}
\]

\[
| \quad \text{begin} \langle \text{stmt} \rangle ; \ldots ; \langle \text{stmt} \rangle \, \text{end} \quad \text{(compound)}
\]
9. (10 points)

- Definition 2.5 states that the composition of two relations, \( R \subseteq X \times Y \) and \( S \subseteq Y \times X \) is
  \[
  S \circ R = \{(x, z) \mid \exists y \in Y: (x, y) \in R \text{ and } (y, z) \in S\}
  \]

- Proposition 2.1 states that the composition of two functions is a function.

- Definition 2.6(b) says a function \( f: X \to Y \) is injective iff
  \[
  \forall x, x' \in X: f(x) = f(x') \text{ implies } x = x'
  \]

**Prove:** The composition of two injections is an injection.

**Proof:** Suppose \( f: X \to Y \) and \( g: Y \to Z \) are injections. Suppose further that \( (g \circ f)(x) = (g \circ f)(x') \). This means \( g(f(x)) = g(f(x')) \). Since \( g \) is an injection it must be that \( f(x) = f(x') \); and since \( f \) is an injection \( f(x) = f(x') \) implies \( x = x' \). We have shown that \( (g \circ f)(x) = (g \circ f)(x') \) implies \( x = x' \) satisfying Definition 2.6(b). Therefore, \( g \circ f \) is an injection. □