C241 Homework Assignment 8

1. Estimate the performance of the *Bubble Sort* program

```
begin
for i from 1 to N - 1 by 1 do
  begin
    for j from 1 to i - 1 by 1 do
      else
        begin
          t := A[j];
          A[j] := A[j + 1];
          A[j + 1] := t
        end
  end
end
```
2. Define two functions, $f$ and $g$, for which $f \notin O(g)$ and $g \notin O(f)$. 
3. Prove: If $f_1 \in O(g_1)$ and $f_2 \in O(g_2)$ then $f_1(n) \times f_2(n) \in O(g_1(n) \times g_2(n))$
4. Is $2^n \in \Theta(n^2)$ or is $n! \in \Theta(2^n)$?
5. Prove: If $f(n) \in O(h(n))$ and $g(n) \in O(h(n))$ then $f(n) + g(n) \in O(h(n))$. 
6. Consider the program to the right, which computes over values in \( \mathbb{N} \). Define the outer loop’s invariant assertion \( I_1 \) to be:

\[
I_1 \equiv z \cdot x^y = A^B
\]

and the inner loop’s invariant assertion \( I_2 \) to be:

\[
I_2 \equiv z \cdot x^y = A^B \land y \neq 0
\]

Since \( z \) is initialized to 1, \( I_1 \) is true when the program first reaches the outer loop; and since \( I_2 \equiv I_1 \land y \neq 0 \), \( I_2 \) holds whenever the inner loop is reached. Assuming that \( I_1 \) and \( I_2 \) are loop invariants, by the Theorem on Loop Invariants we will have \( I_1 \land y = 0 \) when the program terminates and hence

\[
z \cdot x^y = z \cdot x^0 = z \cdot 1 = A^B
\]

as desired. So it remains to be proved that \( I_1 \) and \( I_2 \) are invariants for their respective loops. Prove this.

**HINT.** Start with the inner loop.
Supplemental Problem. (Hotel Paradox)

[This problem appears in many forms in many places.]

Three grad students go to a conference and decide to save money by sharing a hotel room. On checking in, the desk clerk says the charge for a room containing two beds and a cot is $30. After the students have paid, the desk clerk realizes that the charge should have been only $25. She gives the doorkeeper a $5 bill and asks that it be refunded to the students.

On the way to the room the doorkeeper realizes that he cannot divide $5 by 3 evenly, so he decides to refund the students $1 each and keeps $2 for himself.

The students are happy that they paid only $9 each for the room. The doorkeeper is happy that he gets a $2 “tip.” But $3 \times $9 + $2 = $29. Where did the other $1 go?