Definition. Let \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that \( f(n) \) is of order \( g(n) \), written \( f(n) \in O(g(n)) \), if there exist \( N \in \mathbb{N} \) and \( C \in \mathbb{R} \) such that for all \( n \leq N \), \( f(n) \leq C \cdot g(n) \).

\[
\exists C \in \mathbb{R}, N \in \mathbb{N}: [\forall n \leq N: f(n) \leq C \cdot g(n)]
\]

To show that \( f(n) \in O(g(n)) \), one typically finds appropriate values (existential witnesses) for \( N \) and \( C \).

Example 1. \( 2n^2 + 5n + 3 \in O(n^2) \).

Find a dominating value in terms of \( n^2 \) for each term in \( 2n^2 + 5n + 3 \). 

\[
2n^2 \quad \& \quad 5n \quad \& \quad 3
\]

\[
0 \leq n \quad \& \quad 5 \leq n \quad \& \quad 2 \leq n
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
2n^2 \leq 2n^2 \quad 5n \leq n^2 \quad 3 \leq n^2
\]

(a) (b) (c)

To satisfy (a-c), take \( N = 5 \). To satisfy (a+b+c), take \( C = 4 \).

\[
\forall n \geq 5: 2n^2 + 5n + 3 \leq 4n^2
\]

Example 2. \( 2^{(n+10)} \in O(2^n) \).

Write down the inequality and “solve” it.

\[
\begin{align*}
2^{(n+10)} & \leq C \cdot 2^n & n \geq N, C > 0 \\
2^{10} \cdot 2^n & \leq C \cdot 2^n & \text{since } 2^{(n+10)} = (2^n)(2^{10}) \\
C \cdot 2^n & \leq C \cdot 2^n & \text{if let } C = 2^{10}
\end{align*}
\]

So this claim is true if \( C > 0 \). Since both sides of this inequality are now the same, the inequality is true for any value of \( n \). We can take \( N = 0 \), then. So:

\[
\forall n \geq 0: 2^{n+10} \leq 1024 \cdot 2^n
\]

That is,

\[
2^{(n+10)} \in O(2^n) \text{ for } C = 2^{10} \text{ and } N = 0
\]
Example 2. \( n^2 \notin O(n) \). Assume \( C \) and \( N \) exist such that, for all \( n \geq N \), \( n^2 \leq C \cdot n \). Since \( 0 \leq n \) (considered as a real number), we can divide both sides of the inequation by \( n \) to get \( n \leq C \). However, this inequality does not hold whenever \( n > C \), a contradiction to our assumption. In particular, no such \( C \) exists and, therefore, \( n^2 \notin O(n) \).

\[ \square \]

Example 3. \( n(\log_2 n) \notin O(n) \).

Assume \( n(\log_2 n) \in O(n) \) and deduce a contradiction.

Assume \( C \) and \( N \) exist for which

\[
\begin{align*}
n(\log_2 n) & \leq C \cdot n \quad \text{for all } n \geq N \\
\downarrow \\
\log n & \leq C \quad \text{(multiply by } \frac{1}{n}, \ n \geq N \geq 0) \\
n & \leq 2^C \quad \text{(raise both sides to a power of } 2, \ n \geq 0) \\
& \quad \text{for all } n \geq N \ ?!
\end{align*}
\]

The last inequality doesn't hold for arbitrarily large \( n \), so we have reached a contradiction. The original assumption that \( n(\log_2 n) \in O(n) \) must be false, and this is just what we set out to demonstrate.

\textit{Two explain the proof in a more “linear” way, one could write…}

Assume for the purpose of contradiction that \( N \) and \( C \) exist such that, for all \( n \geq N \), \( n(\log_2 n) \leq C \cdot n \). Now consider any \( n \) greater than the larger of \( N \) and \( 2^C \). Then since \( \log_2 \) is an increasing function,

\[
n(\log_2 n) > n(\log_2 2^C) = n \cdot C.
\]

This contradicts our assumption, so either \( N \) or \( C \) do not exist. Therefore \( n(\log_2 n) \notin O(n) \).

\[ \square \]
Example 4. \( n^2 + 7 \not\in O(3n + 5) \).

Again, prove this by contradiction. *Keep in mind that we are seeking a “big enough \( n \)” to refute the assumption, “For some \( N \) and \( C \), \( n^2 + 7 \leq C \cdot (3n + 5) \).”*

(a) Assume \( n^2 + 7 \in O(3n + 5) \) and let \( N \) and \( C \) be the existential witnesses.

(b) Then for \( n \geq N \), \( n^2 + 7 \leq C(3n + 5) = 3Cn + 5C \).

(c) Subtracting 7 from both sides, \( n^2 \leq 3Cn + 5C − 7 \).

(d) Divide both sides by \( n \) to get \( n \leq 3C + \frac{5C}{n} − \frac{7}{n} \).

(e) If \( n > 5C \) and \( n > 7 \) we can replace both fractions by 1 and still preserve the inequality, \( n \leq 3C + 2 \).

(f) Thus, if we take \( n \) to be the larger of \( N \) and \( 3C + 2 \), the inequality cannot hold.

This argument demonstrates by contradiction that \( C \) and \( N \) do not exist, proving the result.

\( \square \)
Proposition. If \( f(n) \in O(g(n)) \) and \( g(n) \in O(h(n)) \) then \( f(n) \in O(h(n)) \).

Proof: Assume \( N_1 \) and \( C_1 \) are witnesses to \( f(n) \in O(g(n)) \) and \( N_2 \) and \( C_2 \) are witnesses to \( g(n) \in O(h(n)) \). Let \( N_3 \) be the greater of \( N_1 \) and \( N_2 \). For all \( n \geq N_3 \)
\[
f(n) \leq C_1 \cdot (g(n)) \quad (n \geq N_1 \text{ and } f(n) \in O(g(n)))
\]
\[
\leq C_1 \cdot (C_2(n)) \quad (n \geq N_2 \text{ and } g(n) \in O(h(n)))
\]
Thus, for all \( n \geq N_3 \), and for \( C_3 = C_1 \cdot C_2 \), \( f(n) \leq C_3 \cdot h(h) \). That is, \( f(n) \in O(h(n)) \) with witnesses \( N_3 \) and \( C_3 \). □

Exponential Order

[Taken from lecture notes by Danial Leivant, 2005]

Proposition. For all \( n \in \mathbb{N} \), \( n < 2^n \).

Proof: by induction on \( n \in \mathbb{N} \). Base Case: \( 0 < 1 = 2^0 \).

Induction: Assume \( k < 2^k \). Then \( k+1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1} \). □

Proposition. For all \( n \in \mathbb{N} \), \( n \geq 4 \) implies \( n^2 < 2^n \).

Proof: We proceed by induction. Base Case: \( 4^2 \leq 16 = 2^4 \).

Induction: Assume that \( k^2 \leq 2^k \). Then
\[
(x + 1)^2 = x^2 + 2x + 1^H
\]
\[
< 2^k + 2x + 1 \quad \text{I.H.}
\]
\[
< 2^k + 2^k \quad \text{by the Lemma above}
\]
\[
= 2^{k+1}
\]
□

Proposition. For all \( n \geq 3 \), \( n^2 > 2n + 1 \).

Proof: by induction on \( n \geq 3 \). Base Case: \( 3^2 = 9 > 2 \cdot 3 + 1 \). Induction:
\[
(x + 1)^2 = x^2 + 2x + 1^H
\]
\[
\equiv (2x + 1) + 2x + 1 \quad \text{I.H., } n \geq 4
\]
\[
= 4x + 2
\]
\[
> 2x + 3 \quad \text{since } x > 1
\]
\[
= 2(x + 1) + 1
\]
□
Corollary. If $C \geq 4$ then for all $n \geq 4$, $C \cdot n < 2^n$.

Discussion. We are going to show that $n^k \in O[2^n]$ provided that $n$ is a power of 2, that is $m = 2^m$. Thus, means we are only “sampling” the functions for values of $n \in \{1, 2, 4, 8, 16, \ldots \}$ to see which dominates the other. That this is good enough is not obvious, but is a consequence of the fact that, considered as functions over $\mathbb{R}$, they increase “smoothly.”

Proposition. For all $k \geq 4$ there exists an $m \in \mathbb{N}$ such that $2^{2m} > (2^m)^k$.

Proof: Note that the right-hand side, $(2^m)^k = 2^{mk}$. By the Corollary above, since $k \geq 4$, we have $2^m > km$ for all $m \geq k$. Raising 2 to the power of both sides preserves the inequality, yielding $2^{2m} > 2^{km} = (2^m)^k$, as desired. □

Corollary. For every $k \geq 1$ and every $C \in \mathbb{R}$, there exists an $m \in \mathbb{N}$ for which $2^{2m} < C \cdot (2^m)^k$.

Proof: By the Proposition above there is a constant $D$ for which $(2^m)^k < 2^{2m}$ whenever $2^m > D$. Taking $m$ such that $2^m$ is the greater of $C$ and $D$ we have

$$C \cdot (2^m)^k \leq (2^m)(2^m)^k = (w^m)^{k+1} < 2^{2m}$$

□