The implication problem for measure-based constraints

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Abstract

We study the implication problem of measure-based constraints. These constraints are formulated in a framework for measures generalizing that for mathematical measures. Measures arise naturally in a wide variety of domains. We show that measure constraints, for particular measures, correspond to constraints that occur in relational databases, data mining applications, cooperative game theory, and in the Dempster–Shafer and possibility theories of reasoning about uncertainty. We prove that the implication problem for measure constraints is in general decidable. We introduce inference systems for particular classes of measure constraints and show that some of these are complete, yielding tractability for the corresponding implication problem.

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1. Introduction

In this paper, we study the implication problem of measure-based constraints. These constraints are formulated in a framework for measures generalizing that for mathematical measures. Our framework is sufficiently rich to include measures occurring in a variety of domains. In the domain of databases, queries expressed in query languages such as SQL frequently use measures such as max, min, count, and sum. Also in this domain, we study measures for the degree of data uniformity in relational databases. In the domain of data mining, applications such as the frequent item sets problem utilize the frequency measure. This measure counts for a set of items the number of times this set is contained in a collection of baskets [1]. In cooperative game theory [2,3], the worth function\textsuperscript{3} of a game played by a set of players is a function that assigns to each coalition (group) of players a real number representing the combined worth or power of that coalition in the game. In the Dempster–Shafer theory of reasoning about uncertainty [4,5], belief, plausibility, doubt, nonbelief, and commonality functions are considered. These functions are used to measure, for an event, the aggregated evidence

\textsuperscript{3}In game theory, worth functions are also known as characteristic functions, or as power functions.
furnished by its sub-events, or that furnished by compatible events. (A good exposition of this theory can be found in [6].) Another approach to reasoning about uncertainty is possibility theory [7], in which possibility and necessity functions are introduced to measure the degree to which a proposition is true.

Our framework for measures generalizes that for finite mathematical measures. In that framework, given a finite set $S$, a nonnegative function $m : 2^S \rightarrow \mathbb{R}^+_{\geq 0}$ is called a mathematical measure if $m(\emptyset) = 0$, and for each pair $Y$ and $Z$ of disjoint subsets of $S$ (i.e., $Y \cap Z = \emptyset$) we have that $m(Y \cup Z) = m(Y) + m(Z)$. The latter property is called additivity. Furthermore, mathematical measures also have the isotonicity and modularity properties:

$$m(X) \leq m(X \cup Y) \quad \text{isotonicity;}$$

$$m(Y \cup Z) + m(Y \cap Z) = m(Y) + m(Z) \quad \text{modularity.}$$

Whereas mathematical measures satisfy isotonicity and modularity, many (non-mathematical) measures, such as those mentioned earlier, have weaker properties. Consider, for example, the max function defined over finite sets of nonnegative numbers. This function clearly has the isotonicity property since $\max(X) \leq \max(X \cup Y)$, but it does not have the modularity property. However, $\max$ has the weaker property called submodularity:

$$\max(Y \cup Z) + \max(Y \cap Z) \leq \max(Y) + \max(Z).$$

Given the insights above, a more general framework for measures could be based on the isotonicity and submodularity properties. Such a framework would be too narrow, however. To see this, consider the $\min$ function over finite sets, which we want to consider as a measure. However, $\min$ is anti-isotone, since $\min(X) \geq \min(X \cup Y)$, and $\min$ is "anti"-submodular (henceforth called supermodular) since $\min(Y \cup Z) + \min(Y \cap Z) \geq \min(Y) + \min(Z)$.

A framework in which both combinations above of isotonicity/anti-isotonicity and submodularity/supermodularity would be allowed, would still be inadequate, however. In the Dempster–Shafer theory of uncertainty, belief functions are isotone and supermodular and nonbelief functions are anti-isotone and submodular.

In our framework, we will therefore adopt all measures whose properties can be stated in terms of any of the four combinations between isotonicity or anti-isotonicity, on the one hand, and submodularity or supermodularity, on the other hand. We will describe the framework in Section 2.

A natural issue that arises when considering measures is their rate of change. For example, for an isotone, submodular measure $M$, we consider the rate of change at a set $X$, relative to $Y$, to be the quantity $M(X \cup Y) - M(X)$. (Notice the analogy with first finite differences [8].) It turns out that studying the rate of change of the rate of change of $M$ (i.e., the second finite difference) is also significant. The topic of first and second finite differences will be covered in Section 3.

Of special concern for this paper are the situations wherein these finite differences of a measure acquire the value 0. In such a situation, we will say of a measure that it satisfies a measure constraint. (Where there is no ambiguity, we just use the term constraint.) For some measures, the measure constraints have natural interpretations in the domain wherein these measures occur. We will study measure constraints in Section 4.

In Section 5, we will deal with the main topic of the paper, i.e., the implication problem for measure constraints. We will show that the implication problem for measure constraints is in general decidable. We will introduce sound inference systems for reasoning about several particular classes of measure constraints, and we will show completeness for some of these systems, yielding tractable decidability of implication. For some classes of measure constraints, the implication problem is strongly linked to the implication problem for well-known classes of database constraints, while, for other classes, this is not the case.

In Section 6, we discuss some directions of future research. These can roughly be divided into two streams: finding new applications for this theory and generalizing the theory to higher-order constraints.

2. Measures

In this section, we first introduce a framework to define various kinds of measures and give examples occurring in various computing domains. This framework generalizes that for mathematical measures. We then validate this framework by showing that, indeed, a wide variety of measures considered in the literature fit into this framework.

In the rest of the paper, $S$ denotes a finite set, $M$ denotes a nonnegative real function over $2^S$, and the variables $U$, $V$, $X$, $Y$, and $Z$ range over $2^S$.
Definition 1. The function $M$ is

isotone if $M(X) \leq M(X \cup Y)$;

anti-isotone if $M(X) \geq M(X \cup Y)$;

submodular if $M(Y \cup Z) + M(Y \cap Z) \leq M(Y) + M(Z)$;

supermodular if $M(Y \cup Z) + M(Y \cap Z) \geq M(Y) + M(Z)$.

If $M$ is isotone and submodular, isotone and supermodular, anti-isotone and submodular, or anti-isotone and supermodular, then $M$ is called a measure. \footnote{Notice that $M$ is a mathematical measure if and only if $M(\emptyset) = 0$, and $M$ is isotone, submodular, and supermodular.}

In [9], Choquet introduced the so-called capacities. A function $M$ is a $k$-alternating capacity if, for each $X \subseteq S$ and for each set $\mathcal{Y}$ of at most $k$ subsets of $S$, $\sum_{\mathcal{Y}} (-1)^{|\mathcal{Y}|} M(X \cup \bigcup_{Z \in \mathcal{Y}} Z) \leq 0$. A function $M$ is a $k$-monotone capacity if, for each $X \subseteq S$, and for each set $\mathcal{Y}$ of at most $k$ subsets of $S$, $\sum_{\mathcal{Y}} (-1)^{|\mathcal{Y}|+1} M(X \cap \bigcap_{Z \in \mathcal{Y}} Z) \geq 0$. It is easy to show that an isotone and submodular (respectively isotone and supermodular) measure is a 2-alternating capacity (respectively 2-monotone capacity), and vice versa.

In our setting, however, all four combinations of the properties isotonicity and anti-isotonicity, respectively submodularity and supermodularity are allowed and can actually occur. Moreover, isotonicity or anti-isotonicity can also occur without submodularity or supermodularity, and vice versa. $\min_2$, defined by $\min_2(X) = \min(X - \{\min(X)\})$, is isotone, but not supermodular, even though $\min_2$ is. Obviously, $\min_2$ is not submodular either. The other implication does not hold either. Submodularity alone, for instance, does imply neither isotonicity nor anti-isotonicity. Arguably, the best known examples of functions that are submodular and neither isotone nor anti-isotone are cut functions of graphs \cite{10}. Let $G = (S, E)$ be a finite directed graph. For $V \subseteq S$, define the cut of $V$, denoted by $\delta(V)$, as follows:

$\delta(V) = \{(v, w) \in E \mid v \in V \text{ and } w \notin V\}$.

The cut function $f : 2^S \rightarrow \mathbb{R}^{\geq 0}$ of the graph $G$ is defined as follows:

$f(V) = |\delta(V)|$.

It can be shown that $f$ is a submodular function, i.e., $f(V \cup W) + f(V \cap W) \leq f(V) + f(W)$ \cite{11}.

However, $f$ is only isotone or anti-isotone in trivial cases. Indeed, it is always true that $f(\emptyset) = f(S) = 0$. It suffices that $f(X) \neq 0$ for one non-trivial subset of $S$ for $f$ to be neither isotone nor anti-isotone. This is, for example, the case for $S = \{a, b\}$ and $E = \{(a, b)\}$. Then $f(\emptyset) = 0, f(\{a\}) = 1, f(\{b\}) = 0$, and $f(\{a, b\}) = 0$.

The four types of measures distinguished in Definition 1 are related. Let $M$ be a nonnegative function from $2^S$ to $\mathbb{R}$. Define functions $M^{\neg}$ and $M^{co}$ by $M^{\neg}(X) = M(S) + M(\emptyset) - M(X)$, and $M^{co}(X) = M(\overline{X})$. Clearly, $(M^{co})^{\neg} = (M^{\neg})^{co} = M(\emptyset) + M(S) - M(\overline{X})$. Therefore, we denote this last function as $M^{co}$ neg. The following connections are now straightforward:

**Proposition 2.** The following four statements are equivalent:

(1) $M$ is isotone and submodular;

(2) $M^{\neg}$ is anti-isotone and supermodular;

(3) $M^{co}$ is anti-isotone and submodular; and

(4) $M^{co}$ neg is isotone and supermodular.

In Proposition 2, the roles of isotone and anti-isotone, respectively submodular and supermodular, are interchangeable. In summary, one can relate each measure of a certain type to a measure of any of the other types.

Before we proceed to examples, we observe the following interesting connection between measures and concave and convex functions. This connection may help the reader in visualizing the type of functions that measures are.

**Proposition 3.** Let $M : 2^S \rightarrow \mathbb{R}^{\geq 0}$ be a function such that $|X| = |Y|$ implies $M(X) = M(Y)$. Let $F : \{0, \ldots, |S|\} \rightarrow \mathbb{R}^{\geq 0}$ be the function defined such that, for $k = 1, \ldots, |S|$, $F(k) = M(X)$ for each subset $X$ of $S$ with $|X| = k$. Then, $M$ is an isotone (respectively anti-isotone) submodular (respectively supermodular) measure if and only if $F$ is an isotone (respectively anti-isotone) concave (respectively convex) function.

**Proof.** We only prove that $M$ is an isotone, submodular measure if and only if $F$ is an isotone, concave function, as the other statements can be derived from this one using Proposition 2.

Clearly, $M$ is isotone if and only if $F$ is isotone. Let $M$ be submodular. To show that $F$ is concave, we have to demonstrate that, for each $k$ with $0 < k < |S|$, \[ F(k - 1) + F(k + 1) \leq F(k), \]
or, equivalently, that \( F(k+1) + F(k-1) \leq F(k) + F(k) \). Let \( X \) be a subset of \( S \) of size \( k-1 \), and let \( y \) and \( z \) be two different elements in \( S - X \). (Such elements exist since \( |S - X| \geq 2 \).) Then, by the submodularity of \( M \), \( M(X \cup \{y,z\}) + M(X) \leq M(X \cup \{y\}) + M(X \cup \{z\}) \). Hence, \( F(k+1) + F(k-1) \leq F(k) + F(k) \).

Now, assume that \( F \) is a concave function, and let \( X \) and \( Y \) be subsets of \( S \). We have to show that \( M(X \cup Y) + M(X \cap Y) \leq M(X) + M(Y) \), or, equivalently, \( F(|X \cup Y|) + F(|X \cap Y|) \leq F(|X|) + F(|Y|) \). Letting \( l = |X - Y| \), \( m = |X \cap Y| \), and \( n = |Y - X| \), this is equivalent to showing that \( F(l + m + n) - F(l + m) \leq F(m + n) - F(m) \), but this is a consequence of the concavity of \( F \).

In the remainder of this section, we will give examples of measures in the areas of aggregate functions and data uniformity in databases, data mining, cooperative game theory, and the theory of reasoning about uncertainty.

### 2.1. Databases—aggregation functions

Computations requiring aggregation functions occur frequently in database applications such as query processing, data cubes [12], and spreadsheets. The most often used such functions are derived from the basic aggregate functions, \( \text{count} \), \( \text{sum} \), \( \text{max} \), and \( \text{min} \). Except in the case of \( \text{count} \), we assume that the sets on which they work consist of nonnegative integers. Clearly, the functions \( \text{max} \), \( \text{count} \), and \( \text{sum} \) are isotone, submodular measures, and the function \( \text{min} \) is an anti-isotone, supermodular measure.

### 2.2. Databases—data uniformity

Measuring the degree of uniformity of categorical data can influence how such data are stored or processed, or both. To measure such uniformity, or lack thereof, it is useful to consider probability-based functions such as the Gini index (introduced in economics to study the distribution of incomes [13]), the Simpson measure (introduced in ecology to study the diversity of species [14]), and the Shannon entropy measure (introduced in information theory to study the distribution of messages during communications [15]). The Gini index is a special case of a class of entropy measures introduced by Tsallis [16]. Furthermore, the Shannon entropy is a limit case in this class.

More formally, let \( S \) be a relation schema (i.e., a finite set of attributes), let \( q \) be a finite nonempty relation over \( S \) (i.e., a finite nonempty set of tuples over \( S \), a tuple over \( S \) being a function associating to each attribute of \( S \) a value from some given domain), let \( p \) be a probability distribution over \( q \) satisfying \( p(t) \neq 0 \) for each tuple \( t \) in \( q \), and let \( q > 1 \) be a real number.

The \( q \)-Tsallis entropy measure \( H_q \) is defined by

\[
H_q(X) = \frac{1}{q-1} \sum_{x \in \pi_X(q)} p_X(x)(1 - p_X^{q-1}(x)),
\]

where \( p_X(x) = \sum_{t \in q, t_X = x} p(t) \). The Gini index (denoted by \( G \)) is the 2-Tsallis entropy measure, and the Shannon entropy measure (denoted by \( H \)) is defined by

\[
H(X) = \lim_{q \rightarrow 1} H_q(X).
\]

Thus,

\[
G(X) = \sum_{x \in \pi_X(q)} p_X(x)(1 - p_X(x)) = 1 - \sum_{x \in \pi_X(q)} p_X^2(x);
\]

\[
H(X) = -\sum_{x \in \pi_X(q)} p_X(x) \ln p_X(x).
\]

The Simpson measure \( S \) is defined by

\[
S(X) = \sum_{x \in \pi_X(q)} p_X^2(x),
\]

and is therefore equal to \( 1 - G(X) \). Hence, \( S(X) = G^{\text{neg}}(X) + (1 - \sum_{t \in q} p^2(t)) \), the latter term being a nonnegative constant.

Clearly, \( H_q \) is a nonnegative function. Also, it is well known (e.g., [17]) that \( H_q \) is isotone. Furthermore, Raggio [18] proved the following inequality:

\[
H_q(X \cup Y) + H_q(X \cap Y) \leq H_q(X) + H_q(Y).
\]

Consequently, for each \( q > 1 \), the \( q \)-Tsallis entropy measure is an isotone, submodular measure, and therefore so are the Gini index (\( H_2 \)) and the Shannon entropy measure (\( \lim_{q \rightarrow 1} H_q \)). Consequently, the Simpson measure is an anti-isotone, supermodular measure.

**Example 4.** Consider the relation \( q \) over the schema \( S = \{ A, B, C \} \) shown in Table 1. Let \( p \) be the uniform distribution on the tuples of \( q \), i.e., for all

\[\text{For } X \subseteq S, \pi_X(q) \text{ denotes the projection of the relation } q \text{ onto } X, \text{ which is a relation over } X; a \text{ tuple } x \text{ over } X \text{ belongs to } \pi_X(q) \text{ if and only if there is a tuple } t \in q \text{ such that } x = t_X, \text{ the restriction of } t \text{ to the attributes of } X.\]
The main objective of cooperative game theory is to provide a formal framework for reasoning about multi-player games wherein players can form coalitions for joint cooperations is such games [2,3]. Formally, a cooperative game is a pair \((N, v)\) where \(N\) is a non-empty set of players and \(v\) the so-called worth function, which is a real function over \(2^N\). Intuitively, for each coalition of players \(X \subseteq N\), \(v(X)\) represents the combined worth of \(X\) in the game. It is commonly assumed that \(v(\emptyset) = 0\).

Various classes of cooperative games have been considered on the basis of certain properties of their worth functions. We mention the following. A game \((N, v)\) is called

1. monotone if \(v\) is isotone, i.e., \(v(X \cup Y) \geq v(X)\);
2. convex if \(v\) is supermodular, i.e., \(v(X \cup Y) + v(X \cap Y) \geq v(X) + v(Y)\); and
3. concave if \(v\) is submodular, i.e., \(v(X \cup Y) + v(X \cap Y) \leq v(X) + v(Y)\).

**Example 7.** Consider the following game between three players: player \(S\) is a seller wanting to sell a commodity at price \(s\), player \(B_1\) is a potential buyer

**Table 1**

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(X) (\emptyset)</th>
<th>([A])</th>
<th>([B])</th>
<th>([C])</th>
<th>([A, B])</th>
<th>([B, C])</th>
<th>([A, C])</th>
<th>([A, B, C])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G)</td>
<td>0.000</td>
<td>0.500</td>
<td>0.375</td>
<td>0.625</td>
<td>0.625</td>
<td>0.750</td>
<td>0.625</td>
</tr>
<tr>
<td>(S)</td>
<td>1.000</td>
<td>0.500</td>
<td>0.625</td>
<td>0.375</td>
<td>0.375</td>
<td>0.250</td>
<td>0.375</td>
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**Table 2**

<table>
<thead>
<tr>
<th>(A)</th>
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<th>(C)</th>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>1</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>(X) (\emptyset)</th>
<th>([A])</th>
<th>([B])</th>
<th>([C])</th>
<th>([A, B])</th>
<th>([B, C])</th>
<th>([A, C])</th>
<th>([A, B, C])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H)</td>
<td>0.000</td>
<td>0.500</td>
<td>0.673</td>
<td>0.673</td>
<td>1.055</td>
<td>1.332</td>
<td>1.055</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.609</td>
</tr>
</tbody>
</table>

t \in \(q\), \(p(t) = 0.25\). Table 1 lists the values of the Gini-index and the Simpson measure for future reference.

**Example 5.** Consider the relation \(q\) over the schema \(S = \{A, B, C\}\) shown in Table 2. Let \(p\) be the uniform distribution on the tuples of \(q\), i.e., for all \(t \in q\), \(p(t) = 0.2\). Table 2 lists the values of the Shannon entropy measure (three significant places) for future reference.

2.3. **Data mining—frequent item sets**

The frequent item sets problem is arguably one of the best-known problems in data mining. The problem is, given a list of baskets, to find those item sets that are contained frequently inside the baskets [1]. The frequency functions arising in this problem are anti-isotone, supermodular measures.

More formally, let \(S\) be a set of items and let \(\mathcal{B}\) be a finite list of baskets, each containing a set of items (i.e., subsets of \(S\)). Let \(\mathcal{B}(X)\) denote the sublist of baskets in \(\mathcal{B}\) that contain \(X\). Define \(freq(X) = |\mathcal{B}(X)|/|\mathcal{B}|\). The frequency function \(freq\) is anti-isotone, since \(freq(X) \geq freq(X \cup Y)\), and supermodular, since \(|\mathcal{B}(Y \cap Z)| \geq |\mathcal{B}(Y) \cup \mathcal{B}(Z)| = |\mathcal{B}(Y)| + |\mathcal{B}(Z)| - |\mathcal{B}(Y) \cap \mathcal{B}(Z)| = |\mathcal{B}(Y)| + |\mathcal{B}(Z)| - |\mathcal{B}(Y \cup Z)|\). We note that Calders and Paredaens [19] consider rarity functions. By definition, \(rare = freq^{1/8}\). Hence, rarity is an isotone, submodular measure.

**Example 6.** Consider the list of baskets \(\mathcal{B}\) over the set of items \(S = \{A, B, C\}\) shown in Table 3. Table 3 also lists the values of the frequency and rarity functions for future reference.

<table>
<thead>
<tr>
<th>(X) (\emptyset)</th>
<th>([A])</th>
<th>([B])</th>
<th>([C])</th>
<th>([A, B])</th>
<th>([B, C])</th>
<th>([A, C])</th>
<th>([A, B, C])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(freq)</td>
<td>1.000</td>
<td>0.75</td>
<td>0.75</td>
<td>0.25</td>
<td>0.50</td>
<td>0.00</td>
<td>0.25</td>
</tr>
<tr>
<td>(rare)</td>
<td>0.00</td>
<td>0.25</td>
<td>0.25</td>
<td>0.75</td>
<td>0.50</td>
<td>1.00</td>
<td>0.75</td>
</tr>
</tbody>
</table>

**Table 3**

A list of baskets over the item set \(\{A, B, C\}\) and the corresponding values for the frequency and rarity functions.

\(\mathcal{B} = \{(A, B), [A, B], [B], [A, C]\}\)

\(|freq| = |\mathcal{B}(X)|/|\mathcal{B}|\). The frequency function \(freq\) is anti-isotone, since \(freq(X) \geq freq(X \cup Y)\), and supermodular, since \(|\mathcal{B}(Y \cap Z)| \geq |\mathcal{B}(Y) \cup \mathcal{B}(Z)| = |\mathcal{B}(Y)| + |\mathcal{B}(Z)| - |\mathcal{B}(Y) \cap \mathcal{B}(Z)| = |\mathcal{B}(Y)| + |\mathcal{B}(Z)| - |\mathcal{B}(Y \cup Z)|\). We note that Calders and Paredaens [19] consider rarity functions. By definition, \(rare = freq^{1/8}\). Hence, rarity is an isotone, submodular measure.
willing to pay the amount $b_1$ for it, and player $B_2$ is a potential buyer willing to pay $b_2$ for it. (This example is a slight generalization of a game introduced by Brandenburger [20].) It is natural to assume that $s \leq \min(b_1, b_2)$, and without loss of generality we assume that $b_1 \leq b_2$. Hence, we have $s \leq b_1 \leq b_2$. The worth function of the game is depicted in Table 4: (a) the worth values $v((S))$, $v((B_1))$, and $v((B_1, B_2))$ are 0, because, for these coalitions, no seller–buyer transaction can take place; (b) for the coalition $\{S, B_1\}$, the total gain (worth) is the difference between player $B_1$’s willingness to pay amount $b_1$ for the commodity and seller $S$’s willingness to receive amount $s$ for it, i.e., $v(\{S, B_1\}) = b_1 - s$, (c) similarly, $v(\{S, B_2\}) = b_2 - s$; and (d) $v(\{S, B_1, B_2\})$ is the maximum of the gains associated with the two possible seller–buyer transactions, i.e., $v(\{S, B_1, B_2\}) = \max(b_1 - s, b_2 - s) = b_2 - s$. It is easy to verify that this game is monotone, and, in the case that $s = b_1$, also convex. (When $s < b_1$, the game is neither convex nor concave.)

Assuming that $v(\emptyset) = 0$, we may conclude that the worth function of a monotone game is an isotone measure, and that the worth function of a monotone, convex (respectively concave) game is an isotone, supermodular (respectively submodular) measure.

### 2.5. Reasoning about uncertainty—Dempster–Shafer theory

Dempster and Shafer introduced a theory of reasoning about uncertainty as a sub-theory of the theory of evidential reasoning [4,5]. More formally, let $S$ denote a finite set of exclusive and exhaustive hypotheses. Beliefs can be assigned to each subset of $S$ based on the evidence assigned to each of its subsets. Specifically, if $ev : 2^S \rightarrow \mathbb{R}^{\geq 0}$ is an evidence function\(^7\) (or mass function), then its associated belief function $\text{Bel}_{ev} : 2^S \rightarrow \mathbb{R}^{\geq 0}$ is defined by

$$\text{Bel}_{ev}(X) = \sum_{U \subseteq X} ev(U).$$

\(^7\)It is customary to assume that $ev(\emptyset) = 0$ and $\sum_{U \subseteq S} ev(U) = 1$.

The plausibility function $\text{Plaus}_{ev}$ associated with $ev$, is defined as the dual of $\text{Bel}_{ev}$: $\text{Plaus}_{ev}(X) = \text{Bel}_{ev}^{\text{co neg}}(X) = 1 - \text{Bel}_{ev}(\bar{X})$, or

$$\text{Plaus}_{ev}(X) = \sum_{U \subseteq X, U \neq \emptyset} ev(U).$$

It is clear that $\text{Bel}_{ev} \leq \text{Plaus}_{ev}$, and in the Dempster–Shafer theory, the range $[\text{Bel}_{ev}, \text{Plaus}_{ev}]$ represents the uncertainty interval for the true likelihood of $X$.

The function $\text{Bel}_{ev}$ is an isotone, supermodular measure, since,

$$\sum_{U \subseteq X} e(U) \leq \sum_{U \subseteq X \cup Y} e(U),$$

and

$$\sum_{U \subseteq X \cap Y} e(U) + \sum_{U \subseteq X \cup Y} e(U) = \sum_{U \subseteq X} e(U) + \sum_{U \subseteq X \cup Y} e(U) \geq \sum_{U \subseteq X} e(U) + \sum_{U \subseteq X \cap Y} e(U) + \sum_{U \subseteq X \cup Y} e(U) = \sum_{U \subseteq X} e(U) + \sum_{U \subseteq Y} e(U),$$

respectively. Consequently, by Proposition 2, $\text{Plaus}_{ev} = \text{Bel}_{ev}^{\text{co neg}}$ is an isotone, submodular measure.

It is also customary in the Dempster–Shafer theory to consider doubt, nonbelief, and commonality. Given the evidence function $ev$, the doubt function $\text{Dbt}_{ev}$, the nonbelief function $\text{NonBel}_{ev}$, and the commonality function $\text{Com}_{ev}$ associated with $ev$ are defined by

$$\text{Dbt}_{ev}(X) = \sum_{U \subseteq S, U \cap X = \emptyset} ev(U);$$

$$\text{NonBel}_{ev}(X) = \sum_{U \subseteq S, U \cap X \neq \emptyset} ev(U);$$

$$\text{Com}_{ev}(X) = \sum_{U \subseteq S, U \supseteq X} ev(U).$$

<table>
<thead>
<tr>
<th>Coalition</th>
<th>$\emptyset$</th>
<th>${S}$</th>
<th>${B_1}$</th>
<th>${B_2}$</th>
<th>${S, B_1}$</th>
<th>${S, B_2}$</th>
<th>${B_1, B_2}$</th>
<th>${S, B_1, B_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worth</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b_1 - s$</td>
<td>$b_2 - s$</td>
<td>0</td>
<td>$b_2 - s$</td>
</tr>
<tr>
<td>Worth ($s = b_1$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b_2 - s$</td>
<td>0</td>
<td>$b_2 - s$</td>
</tr>
</tbody>
</table>
It easily follows that $Db_{te}(X) = Be^{co}_{te}(X)$ and that $NonBel_{te}(X) = Bel^{co}_{te}(X)$, whence, by Proposition 2, $Db_{te}$ is an anti-isotone, supermodular measure, and $NonBel_{te}$ is an anti-isotone, submodular measure. Furthermore, we infer from

$$Com_{ev}(X) = \sum_{U \subseteq X} ev(U) = \sum_{U \subseteq X} ev^{co}(U)$$

that $Com_{ev}$ is also an anti-isotone, supermodular measure.

The Dempster–Shafer theory generalizes the theory of uncertainty based on probability distributions. Indeed, consider an evidence function $ev$ for which $ev(X) = 0$ if $X$ is not a singleton subset of $S$. Then the belief and plausibility functions associated with $ev$ coincide. If we denote $P := Bel_{ev} = Plaus_{ev}$, then $P$ satisfies $P(X) = \sum_{x \in X} P(\{x\})$, as well as the boundary conditions $P(\emptyset) = 0$ and $P(S) = 1$. Hence, $P$ is a probability distribution. Moreover, the uncertainty interval in the true likelihood of $X$ collapses to the singleton $\{P(X)\}$, which is as expected. Finally, $Db_{te}(X) = NonBel_{te}(X) = 1 - P(X)$, which is usually referred to as the co-probability distribution associated with $P$.

Example 8. Consider the tossing of a coin. Then, $S$ could consist of the basic events that, upon a toss, the coin will show head ($H$), tail ($T$), or will land on its edge ($E$), i.e., $S = \{H, T, E\}$. Then the set of all events, $2^S$, describes the hypotheses false ($\emptyset$), head ($\{H\}$), tail ($\{T\}$), edge ($\{E\}$), head or tail ($\{H, T\}$), head or edge ($\{H, E\}$), tail or edge ($\{T, E\}$), and true ($\{H, T, E\}$).

In a probabilistic interpretation, it would be natural to assign probabilities between 0 and 1 to the basic hypotheses $\{head\}$, $\{tail\}$, and $\{edge\}$ whose sum adds up to 1. In the Dempster–Shafer theory, one could, for example, associate the evidences acquired for the hypotheses shown in Table 5. Table 5 also shows the resulting belief, plausibility, doubt, nonbelief, and commonality values.

2.6. Reasoning about uncertainty—possibility theory

Possibility theory [7] is another theory for reasoning about uncertainty that aims to quantify degrees of possibility, and by duality, degrees of necessity of propositions (hypotheses). At its core are possibility and necessity functions. A possibility function is a special case of a fuzzy measure [21]. A function $F : 2^S \rightarrow [0, 1]$ is a fuzzy measure if $F(\emptyset) = 0$, $F(S) = 1$, and $F(X \cup Y) \geq \max(F(X), F(Y))$. A function $Poss : 2^S \rightarrow [0, 1]$ is a possibility function if it is a fuzzy measure, and $Poss(X \cup Y) = \max(Poss(X), Poss(Y))$ when $X \cap Y = \emptyset$. A function $Nec : 2^S \rightarrow [0, 1]$ is a necessity function if $Nec(\emptyset) = 0$, $Nec(S) = 1$, and $Nec(X \cap Y) = \min(Nec(X), Nec(Y))$ when $X \cup Y = S$. (Obviously, the dual of a possibility function $Poss$, $Poss^{co}$, is a necessity function, and the dual of a necessity function $Nec$, $Nec^{co}$, is a possibility function.) From these definitions, it follows that the possibility function can be defined in terms of the possibility values of its basic hypotheses: $Poss(X) = \max\{Poss(\{x\}) | x \in X\}$. By duality, $Nec(X) = \min\{Nec(\{x\}) | x \in X\}$.

Clearly, if $Nec = Poss^{co}$, then $Nec(X) \leq Poss(X)$. Hence, the interval $[Nec(X), Poss(X)]$ can be thought of as bounding the degree to which the proposition $X$ is true, relative to the possibility model implied by $Poss$. In particular, if $Nec(X) = Poss(X)$, then the proposition $X$ is true with probability $Nec(X) = Poss(X)$ in this possibility model.

It can be shown that a possibility function is a plausibility function, and, consequently, a necessity function.

Table 5

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>$\emptyset$</th>
<th>${H}$</th>
<th>${T}$</th>
<th>${E}$</th>
<th>${H, T}$</th>
<th>${T, E}$</th>
<th>${H, E}$</th>
<th>${H, T, E}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evidence</td>
<td>0.00</td>
<td>0.25</td>
<td>0.25</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.10</td>
<td>0.40</td>
</tr>
<tr>
<td>Belief</td>
<td>0.00</td>
<td>0.25</td>
<td>0.25</td>
<td>0.00</td>
<td>0.50</td>
<td>0.25</td>
<td>0.35</td>
<td>1.00</td>
</tr>
<tr>
<td>Plausibility</td>
<td>0.00</td>
<td>0.75</td>
<td>0.65</td>
<td>0.50</td>
<td>1.00</td>
<td>0.75</td>
<td>0.75</td>
<td>1.00</td>
</tr>
<tr>
<td>Doubt</td>
<td>1.00</td>
<td>0.25</td>
<td>0.35</td>
<td>0.50</td>
<td>0.00</td>
<td>0.25</td>
<td>0.25</td>
<td>0.00</td>
</tr>
<tr>
<td>Nonbelief</td>
<td>1.00</td>
<td>0.75</td>
<td>0.75</td>
<td>1.00</td>
<td>0.50</td>
<td>0.75</td>
<td>0.65</td>
<td>0.00</td>
</tr>
<tr>
<td>Commonality</td>
<td>1.00</td>
<td>0.75</td>
<td>0.65</td>
<td>0.50</td>
<td>0.40</td>
<td>0.40</td>
<td>0.50</td>
<td>0.40</td>
</tr>
</tbody>
</table>
function is a belief function [6]. The underlying evidence function can be shown to be consonant, meaning that, if two sets have strictly positive evidence values, then one of them is contained in the other. Thus, possibility functions are isotone, submodular measures, and necessity functions are anti-isotone, supermodular measures.

**Example 9.** Let \( S = \{A, B, C\} \), and let \( ev \) be the consonant evidence function such that \( ev(\emptyset) = 0 \), \( ev(A) = 0.25 \), \( ev(A, B) = 0.75 \), and \( ev(A, B, C) = 0 \). (Since \( ev \) is consonant, the evidence values of all other sets are 0.) In Table 6, we show the possibility and the necessity values for all subsets of \( S \). Notice that, since \( \text{Nec}(A, B) = \text{Poss}(A, B) = 1 \), the hypothesis \( \{A, B\} \) is true, in accordance with the given evidence function. Also notice that the hypothesis \( \{C\} \) is false, since \( \text{Nec}(\{C\}) = \text{Poss}(\{C\}) = 0 \).

**Table 6**
Evidence and resulting possibility and necessity values for the hypotheses corresponding with the events in \( S \) of Example 9

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \emptyset )</th>
<th>( {A} )</th>
<th>( {B} )</th>
<th>( {C} )</th>
<th>( {A, B} )</th>
<th>( {B, C} )</th>
<th>( {A, C} )</th>
<th>( {A, B, C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evidence</td>
<td>0.00</td>
<td>0.25</td>
<td>0.00</td>
<td>0.00</td>
<td>0.75</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Possibility</td>
<td>0.00</td>
<td>1.00</td>
<td>0.75</td>
<td>0.00</td>
<td>1.00</td>
<td>0.75</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Necessity</td>
<td>0.00</td>
<td>0.25</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>0.00</td>
<td>0.25</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 7 divides all examples of measures discussed above over the four families we consider. We denote these by \( \mathcal{A}^s \), \( \mathcal{A}^k \), \( \mathcal{A}^s \), and \( \mathcal{A}^s \), respectively. An “s” in the subscript position means “submodular,” and an “k” in the superscript position means “supermodular.” Likewise, an “i” in the subscript position means “isotone,” and an “i” in the superscript position means “anti-isotone.”

### 3. Measure differentials

A natural problem that arises for measures as for other functions is to calculate their rate of change and the rate of change of their rate of change. For continuous functions, these quantities are captured by the first and second derivatives, respectively. We adapt these notions to measures.

**Definition 10.** Let \( F : 2^S \to \mathbb{R} \) be a (finite) set-based function into the real numbers. The first right-differential and the first left-differential of \( F \) are the functions from \( 2^S \times 2^S \) to \( \mathbb{R} \) defined by

\[
F^r_>(X) = F(X) - F(X \cup Y);
\]

\[
F^r_<(X) = F(X) - F(X \cap Y).
\]

For a fixed value of \( Y \), the function \( F^r_>: 2^S \to \mathbb{R} \) and the function \( F^r_<: 2^S \to \mathbb{R} \) are again (finite) set-based functions into the real numbers, of which we can again take the first finite differentials. These are then the second differentials of \( F \).

**Definition 11.** Let \( F : 2^S \to \mathbb{R} \) be a (finite) set-based function into the real numbers. The second finite right-differential and the second finite left-differential of \( F \) are the functions from \( 2^S \times 2^S \times 2^S \) to \( \mathbb{R} \) defined by

\[
F^{r,r}_>(X) = (F^r_>)^2(X);
\]

\[
F^{r,r}_<(X) = (F^r_<)^2(X).
\]
It is useful to expand these differentials.

\[ F^{YZ}_{-} (X) = F(X \cup Y \cup Z) - F(X \cup Y) - F(X \cup Z) + F(X); \]

\[ F^{YZ}_{+} (X) = F(X \cap Y \cap Z) - F(X \cap Y) - F(X \cap Z) + F(X). \]

The first and second differentials were introduced by Choquet [9] in his studies on capacities. He called these the first and second successive differences, respectively.

The following proposition gives useful characterizations of measures in terms of their second differentials. We remind the reader that \( X, Y, \) and \( Z \) range over all subsets of \( S. \)

**Proposition 12.** Let \( M : 2^S \to \mathbb{R}_{\geq 0} \) be a nonnegative real function over \( S. \) Then

1. \( M \) is isotone and submodular if and only if \( M^{YZ}_{-} (X) \leq 0; \)
2. \( M \) is isotone and supermodular if and only if \( M^{YZ}_{-} (X) \geq 0; \)
3. \( M \) is anti-isotone and submodular if and only if \( M^{Y\{X\}_{-}} (X) \leq 0; \) and
4. \( M \) is anti-isotone and supermodular if and only if \( M^{Y\{X\}_{-}} (X) \geq 0. \)

**Proof.** Statement (1) was proved by Choquet [9]. The other statements can be derived from this one using Proposition 2. \( \square \)

4. Measure constraints

In this section, we consider the situations in which the first or second differentials of a measure are zero. This leads us to introduce primary and secondary constraints. Satisfaction of such constraints by particular measures discussed in Section 2 yields conditions in terms of satisfaction of certain natural constraints in databases, data mining, cooperative game theory, the Dempster–Shafer theory of reasoning about uncertainty, and possibility theory.

**Definition 13.** A measure \( M \) satisfies

- the right primary constraint \( X \to Y \) if \( M^{Y}_{-} (X) = 0; \)
- the left primary constraint \( X \leftarrow Y \) if \( M^{Y}_{+} (X) = 0; \)
- the right secondary constraint \( X \to Y|Z \) if \( M^{YZ}_{-} (X) = 0; \)
- the strong right secondary constraint \( X \to Y \) if \( M^{Z,Y}_{-} (X) = 0; \)
- the left secondary constraint \( X \leftarrow Y|Z \) if \( M^{Z,Y}_{+} (X) = 0; \) and
- the strong left secondary constraint \( X \leftarrow Y \) if \( M^{Z,Y}_{+} (X) = 0. \)

Right and left versions of the constraints in Definition 13 are connected, as follows:

**Proposition 14.** Let \( M \) be a measure. Then the following four statements are equivalent:

1. \( M \) satisfies the right primary constraint \( X \to Y, \) respectively the right secondary constraint \( X \to Y|Z, \) respectively the strong right secondary constraint \( X \to Y; \)
2. \( M^{\text{neg}} \) satisfies the right primary constraint \( X \to Y, \) respectively the right secondary constraint \( X \to Y|Z, \) respectively the strong right secondary constraint \( X \to Y; \)
3. \( M^{\text{co}} \) satisfies the left primary constraint \( \overline{X} \to \overline{Y}, \) respectively the left secondary constraint \( \overline{X} \leftarrow \overline{Y}|Z, \) respectively the strong left secondary constraint \( \overline{X} \leftarrow \overline{Y}; \) and
4. \( M^{\text{co \ neg}} \) satisfies the left primary constraint \( \overline{X} \to \overline{Y}, \) respectively the left secondary constraint \( \overline{X} \leftarrow \overline{Y}|Z, \) respectively the strong left secondary constraint \( \overline{X} \leftarrow \overline{Y}. \)

We illustrate primary and secondary constraints for the various measures discussed in Section 2.

4.1. Databases—aggregate functions

It can easily be seen that the \( \text{sum} \) and \( \text{count} \) measures satisfy the right primary constraint \( X \to Y \) if and only if \( Y \subseteq X. \) Likewise, \( \text{max} \) satisfies the right primary constraint \( X \to Y \) if and only if \( \max(X) \geq \max(Y). \) Finally, \( \text{min} \) satisfies the right
primary constraint \( X \rightarrow Y \) if and only if \( \min(X) \leq \min(Y) \).

For the secondary constraints, we restrict ourselves to the measures \( \text{count} \) and \( \text{max} \).

By a straightforward calculation, \( \text{count} \) satisfies the right secondary constraint \( X \rightarrow Y|Z \) if and only if \( Y \cap Z \subseteq X \). In particular, in the case where \( X = \emptyset \), it follows that \( Y \) and \( Z \) must be disjoint.

Likewise, \( \text{max} \) satisfies the right secondary constraint \( X \rightarrow Y|Z \) if and only if \( \max(X) \geq \max(Y) \) or \( \max(X) \geq \max(Z) \). Observe that, in the case where \( X = \emptyset \), either \( Y \) or \( Z \) must be the empty set.

4.2. Databases—data uniformity

Database theory is arguably the field of computer science in which constraints were most extensively studied. For the readers not familiar to database theory, we briefly recall the definition of one of the most important types of database constraints. Then, let \( S \) be a relation schema, let \( X \) and \( Y \) be subsets of \( S \), and \( q \) be a relation over \( S \). Then \( q \) satisfies the functional dependency [22] \( X \xrightarrow{fd} Y \) if, for any two tuples \( t_1 \) and \( t_2 \) in \( q \), \( t_1[X] = t_2[X] \) implies that \( t_1[Y] = t_2[Y] \). (For example, the relation shown in Table 1 satisfies the functional dependency \( BC \xrightarrow{fd} A \), but it does not satisfy \( B \xrightarrow{fd} A \).)

Right primary constraints for the Gini-index and the Shannon entropy measure over relations describe functional dependencies over the corresponding relations [23,24], in the following sense. Let \( S \) be a relation schema, let \( X \) and \( Y \) be subsets of \( S \), and let \( q \) be a relation over \( S \). Let \( p \) be a probability distribution over \( q \) satisfying \( p(t) \neq 0 \) for all \( t \) in \( q \). The corresponding Gini-index \( G \) and the Shannon entropy measure \( H \) satisfy the constraint \( X \rightarrow Y \) if and only if \( G \) satisfies the functional dependency \( X \xrightarrow{fd} Y \).

Under the same conditions on \( p \), the Gini-index \( G \) satisfies the strong right secondary constraint \( X \rightarrow Y \) if and only if, for all \( x \) in the projection \( \pi_X(q) \), the selection\( ^8 \) \( q_x = \sigma_{X=x}(q) \) satisfies either the functional dependency \( X \xrightarrow{fd} Y \) or the functional dependency \( X \xrightarrow{fd} Y \).

To see this, we first assume that \( X \) and \( Y \) are disjoint.

First, observe that \( G \) satisfies \( X \rightarrow Y \) if and only if \( G(X \cup Y \cup Z) + G(X) - G(X \cup Y) - G(X \cup Z) = 0 \), with \( Z = \overline{X \cup Y} \).

Next, for all \( x \) in \( \pi_X(q) \), let \( q^x = \pi_{S \setminus X}(q_x) \). We define the probability distribution \( p^x \) on \( q^x \) by \( p^x(t) = p(xt)/p_X(x) \) for all \( t \) in \( q^x \). Notice that \( p^x \) satisfies \( p^x(t) \neq 0 \) for all \( t \) in \( q^x \). Let \( G^x \) be the corresponding Gini-index.

Now, let \( U \) be an arbitrary subset of \( S \) disjoint from \( X \). We have that

\[
G(X \cup U) = 1 - \sum_{xu \in \pi_{X \cup U}(q)} p^2_{X \cup U}(xu) \\
= 1 - \sum_{x \in \pi_X(q)} \sum_{u \in \pi_U(q_x)} p^2_{X}(x)p^2_{U}(u) \\
= 1 - \sum_{x \in \pi_X(q)} p^2_{X}(x) \\
+ \sum_{x \in \pi_X(q)} p^2_{X}(x) \left[ 1 - \sum_{u \in \pi_U(q_x)} p^2_{U}(u) \right] \\
= G(X) + \sum_{x \in \pi_X(q)} p^2_{X}(x)G^x(U).
\]

Hence,

\[
G(X \cup Y \cup Z) + G(X) - G(X \cup Y) - G(X \cup Z)
= \sum_{x \in \pi_X(q)} [G^x(\overline{X}) + G^x(\emptyset) - G^y(Y) - G^z(Z)].
\]

Notice that, for all \( x \) in \( \pi_X(q) \), \( G^x(\overline{X}) + G^x(\emptyset) - G^y(Y) - G^z(Z) \leq 0 \), since \( G^x \) is an isotone measure. Consequently, \( G(X \cup Y \cup Z) + G(X) - G(X \cup Y) - G(X \cup Z) = 0 \) if and only if, for all \( x \) in \( \pi_X(q) \), \( G^x(\overline{X}) + G^x(\emptyset) - G^y(Y) - G^z(Z) = G^x(\overline{X}) - G^y(Y) - G^z(Z) = 0 \), or, equivalently,

\[
1 - \sum_{t \in q^x} p^x(t) - \sum_{y \in \pi_Y(q^x)} p^2_{Y}(y) - \sum_{z \in \pi_Z(q^x)} p^2_{Z}(z) = 0.
\]

If we rewrite the above equality using

\[
\sum_{y \in \pi_Y(q^x)} p^2_{Y}(y) = \sum_{y \in \pi_Y(q^x)} \left[ \sum_{t \in q^x, \|Y\|=y} p^x(t) \right]^2,
\]

\[
\sum_{z \in \pi_Z(q^x)} p^2_{Z}(z) = \sum_{z \in \pi_Z(q^x)} \left[ \sum_{t \in q^x, \|Z\|=z} p^x(t) \right]^2,
\]

and

\[
\sum_{t \in q^x} p^x(t)^2 = 1,
\]

\( ^8 \)By the selection \( q_x = \sigma_{X=x}(q) \) we mean the subset of \( q \) consisting of all tuples \( t \) of \( q \) satisfying the condition \( \|X\| = x \).
we obtain, after simplification, that
\[ \sum_{t,t' \in \mathbb{C}, \ n \neq t', n \neq t[Z]} p^x(t)p^y(t') = 0. \]

Notice, however, that \( p^x(t) \neq 0 \) for all \( t \in q^x \). Hence, the above sum contains no terms. In other words, for all \( t,t' \) in \( q^x \), either \( t[Y] = t'[Y] \) or \( t[Z] = t'[Z] \). This condition can only be satisfied if either \( |\pi_Y(q^x)| = 1 \) or \( |\pi_Z(q^x)| = 1 \), or, in other words, if \( q_x \) satisfies either \( X \rightarrow Y \) or \( X \rightarrow Z \), or, equivalently, if \( q_x \) satisfies either \( X \rightarrow Y \) or \( X \rightarrow Y \).

For the case that \( X \) and \( Y \) are not disjoint, we observe that \( G \) satisfies \( X \rightarrow Y \) if and only if \( G \) satisfies \( X \rightarrow Y \) if and only if \( \pi_X(q) \), can be immediately seen by writing out the corresponding equations. Hence, for all \( x \) in \( \pi_X(q) \), \( q_x \) satisfies either \( X \rightarrow Y \rightarrow X \) or \( X \rightarrow Y \rightarrow X \), or, equivalently, if \( q_x \) satisfies either \( X \rightarrow Y \) or \( X \rightarrow Y \).

In the database literature, these constraints are called **degenerate multivalued dependencies** [25].

Let \( S \) be a relation scheme, \( X \) and \( Y \) be subsets of \( S, Z = X \cup Y \), and let \( q \) be a relation over \( S \). Then \( q \) satisfies the **multivalued dependency** [26–28] \( X \rightarrow Y \), if for each pair of tuples \( t_1 \) and \( t_2 \) of \( q \) with \( t_1[X] = t_2[X] \), there also exists a tuple \( t \) in \( q \) such that \( t[X \cup Y] = t_1[X \cup Y] \) and \( t[X \cup Z] = t_2[X \cup Z] \).

Now, let \( C \) be a set of functional and multivalued dependencies, and let \( C_d \) be the same set in which each multivalued dependency has been replaced by the corresponding degenerate multivalued dependency. Similarly, let \( c \) be a functional or multivalued dependency, and let \( C_d \) be either \( c \), if \( c \) is a functional dependency, or the degenerate multivalued dependency corresponding to \( c \), if \( c \) is a multivalued dependency. It has also been shown that \( c \models c \) if and only if \( C_d \models c_d \) [25]. This justifies the term “degenerate multivalued dependency.”

**Example 15.** We revisit Example 4. For the relation \( q \) shown in Table 1 and \( p \) the uniform distribution on the tuples of \( q \), Table 2 also lists the values of the corresponding Gini-index. We observe that \( G((C)) = G((A, C)) \), whence \( G \) satisfies the right secondary constraint \( C \rightarrow A \). Hence, \( q \) must satisfy \( C \rightarrow A \), which is indeed the case.

We also notice that \( G((A, B, C)) + G((A)) = G((A, B)) + G((A, C)) \), whence \( G \) satisfies the strong right primary constraint \( A \rightarrow B \). Hence, \( q \) must satisfy the corresponding degenerate multivalued dependency, which is indeed the case: \( \sigma_{A=0}(q) \) satisfies \( A \rightarrow B \), and \( \sigma_{A=1}(q) \) satisfies \( A \rightarrow C \).

The above results for the Gini-index hold for any probability distribution \( p \) with \( p(t) \neq 0 \) for all \( t \in q \). If, in particular, \( p \) is the uniform probability distribution over \( q \), then the corresponding Shannon entropy measure \( H \) satisfies the strong right secondary constraint \( X \rightarrow Y \) if and only if \( q \) satisfies the multivalued dependency \( X \rightarrow Y \). This fact was shown by Malvestuto [23] and Lee [24].

**Example 16.** We revisit Example 5. For the relation \( q \) shown in Table 2 and \( p \) the uniform distribution on the tuples of \( q \), Table 2 also lists the values of the corresponding Shannon entropy measure. We observe that \( H((A, B, C)) + H((A)) = H((A, B)) + H((A, C)) \), whence \( H \) satisfies the strong right secondary constraint \( A \rightarrow B \). Hence, \( q \) must satisfy \( A \rightarrow B \), which is indeed the case.

Finally, we observe that Gyssens and Paredaens [29] considered relational database constraints that we shall refer to here as **functional** and **multivalued domain dependencies**. For a finite relation \( q \) over the relation schema \( S \), the active domain of \( X \), denoted by \( \text{adom}(X) \), is the set of all domain entries occurring under some attribute of \( X \) in some tuple of \( q \), i.e., \( \text{adom}(X) = \{ t[A] \mid t \in q \land A \in X \} \). The relation \( q \) satisfies the **functional domain dependency** \( X \rightarrow Y \) if \( \text{adom}(Y) \subseteq \text{adom}(X) \). The relation \( q \) satisfies the **multivalued domain dependency** \( X \rightarrow Y \) if \( \text{adom}(Y) \cap \text{adom}(Y) \subseteq \text{adom}(X) \). It has been shown that the implication of functional and multivalued dependencies, on the one hand, and the implication of functional and multivalued domain dependencies, on the other hand, obey the same sound and complete set of inference rules [29].

So, if \( C \) is a set of functional and multivalued dependencies, and \( C_{\text{adom}} \) is the corresponding set of functional and multivalued domain dependencies, and if \( c \) is a functional or multivalued dependency, and \( c_{\text{adom}} \) is the corresponding functional or multivalued domain dependency, it follows that \( c \models c \) if and only if \( C \models C_{\text{adom}} \).

Since functional and degenerate multivalued database dependencies can be obtained by expressing that the Gini-index satisfies the corresponding right primary and secondary constraints, and functional and multivalued database dependencies can be obtained by expressing that the Shannon entropy measure for the uniform probability distribution satisfies the corresponding right primary...
and secondary constraints, one may wonder whether functional and multivalued domain dependencies can also be characterized in terms of some measure satisfying the corresponding right primary and secondary constraints.

It turns out that this is indeed the case. Define the nonnegative real function $D : S \rightarrow \mathbb{R}^{\geq 0}$ by $D(X) = |\text{dom}(X)|$. It is easily verified that $D$ is an isotone, submodular measure. Now, $D$ satisfies $X \rightarrow Y$ if and only if $|\text{dom}(X) \cup Y| = |\text{dom}(X)|$ if and only if $|\text{dom}(Y) \subseteq \text{dom}(X)|$, or, in other words, if $q$ satisfies $X \leftrightarrow Y$. Similarly, $D$ satisfies $X \rightarrow Y$ if and only if $|\text{dom}(S)| + |\text{dom}(X)| = |\text{dom}(X \cup Y)|$ if and only if $|\text{dom}(X) \cup \text{dom}(Y)|$, or, equivalently, since the left-hand side set is contained in the right-hand side set, $|\text{dom}(X) \cup \text{dom}(Y)| = |\text{dom}(X \cup Y) \cap \text{dom}(X \cup Y)|$, or, equivalently, since the left-hand side set is contained in the right-hand side set, $|\text{dom}(X) \cup \text{dom}(X \cup Y)| = |\text{dom}(X \cup Y) \cap \text{dom}(X \cup Y)|$, or, in other words, if $q$ satisfies $X \leftrightarrow Y$.

4.3. Data mining—frequent item sets

The $\text{freq}$ measure satisfies the right primary constraint $X \rightarrow Y$ in a list of baskets $B$ if and only if $\text{freq}(X \cup Y) = \text{freq}(X)$ if and only if $\text{freq}(B(X \cup Y) \cup \text{dom}(Y))$, or, equivalently, since the left-hand side set is contained in the right-hand side set, $|\text{dom}(X) \cup \text{dom}(Y)| = |\text{dom}(X \cup Y) \cap \text{dom}(X \cup Y)|$, or, in other words, if $q$ satisfies $X \leftrightarrow Y$.

The right secondary constraints of the $\text{freq}$ measure can be characterized as follows. For item sets $X$, $Y$, and $Z$, and basket list $B$, $X \rightarrow Y \mid Z$ holds for $\text{freq}$ if and only if $\text{freq}(X \cup Y) = \text{freq}(X)$ if and only if $\text{freq}(B(X \cup Y) \cup \text{dom}(Y))$, or, equivalently, since the left-hand side set is contained in the right-hand side set, $|\text{dom}(X) \cup \text{dom}(Y)| = |\text{dom}(X \cup Y) \cap \text{dom}(X \cup Y)|$, or, in other words, if $q$ satisfies $X \leftrightarrow Y$.

Example 17. We revisit Example 6. Table 3 shows a list of baskets over the set of items $S = \{A, B, C\}$ and the values of the frequency and rarity functions. We observe that $\text{freq}(A) = \text{freq}(A, C)$, whence $\text{freq}$ satisfies the right primary constraint $C \rightarrow A$. Indeed, we see that the only basket containing $C$ containing $(A, C)$ also contains $A$. Conversely, we see that there is a basket in $B$ containing $B$, but not $A$ $(\{B\})$. Hence, $\text{freq}$ cannot satisfy the right primary constraint $B \rightarrow A$. Indeed, $\text{freq}(B) \neq \text{freq}(AB)$.

We also observe that $\text{freq}(A, B, C) + \text{freq}(A) = \text{freq}(A, B) + \text{freq}(A, C)$, whence $\text{freq}$ satisfies the strong right secondary constraint $A \rightarrow B = A \rightarrow B \mid C$. Indeed, we see that each basket containing $A$ contains either $B$ or $C$. Conversely, we see that the basket $(B)$ contains neither $A$ nor $C$. Hence, $\text{freq}$ cannot satisfy the right secondary constraint $B \rightarrow A \mid C$. Indeed, $\text{freq}(A, B, C) + \text{freq}(B) \neq \text{freq}(A, B) + \text{freq}(B, C)$.

4.4. Cooperative game theory

Let $(N, v)$ be a cooperative game. The first right-differential $v^{X \mid \{i\}}(X)$, which, by definition, is $v(X) - v(X - \{i\})$ is called the marginal contribution (or the added worth) of player $i$ when he joins the coalition $X - \{i\}$ [33]. Intuitively, it measures the amount with which the overall worth of coalition $X - \{i\}$ would grow if $i$ were to join it. In particular, when $v(X) - v(X - \{i\}) = 0$, $v$ satisfies the left primary constraint $X \leftarrow X - \{i\}$. In cooperative game theory, this situation is described using the notion of a null player [33]. In this case, $i$ is a null player, relative to $X - \{i\}$. Intuitively, this says that the act of player $i$ joining the coalition $X - \{i\}$ has no impact in strengthening the worth of that coalition.

The second right-differential $v^{X \cup \{i,j\}}(X \cup \{i\})$, which, by definition, is $v(X \cup \{i,j\}) - v(X \cup \{i\}) - v(X \cup \{j\}) + v(X)$, was first considered by Grabisch and Roubens [33] (see also [34]). (Instead of calling this a “differential,” they called it a “derivative.”) Intuitively, this derivative measures the degree of interaction between players $i$ and $j$ in the context of them joining or not joining coalition $X$. When this derivative is (a) positive, the players have an interest in joining the coalition; (b) negative, the players are better off joining the coalition individually; and (c) zero, the players can act without interference. Case (c) can be captured in our theory by saying that $v$ satisfies the secondary left constraint $X \cup \{i,j\} \leftrightarrow X \cup \{i\} | X \cup \{j\}$.
Revisiting Example 7, Table 4, we observed that the worth function, when \( s = b_1 \), is an isotone, supermodular measure. In this case, we observe that \( v(\{s, b_1, b_2\}) = v(\{s, b_1\}) \), whence \( v \) satisfies the left primary constraint \( \{s, b_1, b_2\} \prec \{s, b_2\} \). Thus, \( b_1 \) is a dummy player relative to \( \{s, b_2\} \). This makes sense since the joining of \( b_1 \) to the coalition \( \{s, b_2\} \) does nothing to the total gain that can be expected when the seller and player \( b_2 \) were to engage in a seller–buyer transaction. We also observe that \( v(\{s, b_1, b_2\}) + v(\{s\}) = v(\{s, b_1\}) + v(\{s, b_2\}) \), whence \( v \) satisfies the secondary left constraint \( \{s, b_1, b_2\} \prec \{s, b_1\} \cup \{s, b_2\} \). Again this makes sense since clearly \( b_1 \) and \( b_2 \) can act without interference in simultaneously joining in a coalition with \( \{s\} \). Obviously, in this example, this is a consequence of the fact that \( b_1 \) is a dummy player, but in general this phenomenon can occur when neither of the two players is a dummy player.

4.5. Reasoning about uncertainty—Dempster–Shafer theory

A belief function \( \text{Bel} \) based on the evidence function \( e \) satisfies the left primary constraint \( X \prec Y \) if and only if \( \text{Bel}(X) = \text{Bel}(X \cap Y) \) (i.e., \( \sum_{U \subseteq X} e(U) = \sum_{U \subseteq X \cap Y} e(U) \)) if and only if all the evidence supporting the belief in the event \( X \) is already present among the evidence supporting the belief in the event \( Y \) if and only if the belief in \( X \) is weaker than or equal to the belief in \( Y \).

The belief function \( \text{Bel} \) satisfies the left secondary constraint \( X \prec Y \mid Z \) if and only if all the evidence supporting the belief in \( X \) is already present among the combined evidence supporting the belief in \( Y \) and the belief in \( Z \), i.e., if and only if the belief in \( X \) is weaker than or equal to the combined belief in \( Y \) and \( Z \). An interesting left secondary constraint in this respect is \( S \prec Y \mid Z \), with \( Y \cap Z = \emptyset \). In this case, \( \text{Bel}(Y) + \text{Bel}(Z) = 1 \). One can therefore interpret \( Y \) and \( Z \) as exclusive and exhaustive events relative to \( \text{Bel} \).

A plausibility function \( \text{Plaus} \) based on the evidence function \( e \) satisfies the right primary constraint \( X \rightarrow Y \) if and only if the belief function \( \text{Plaus}^{\text{co neg}} \) satisfies the left primary constraint \( \overline{X} \prec \overline{Y} \) (Proposition 14) if and only if all the evidence supporting the belief of the event \( X \) is already present among the evidence supporting the belief of the event \( Y \) if and only if all the evidence supporting the plausibility of the event \( Y \) is already present among the evidence supporting the plausibility of the event \( X \) if and only if the plausibility of \( Y \) is weaker than or equal to the plausibility of \( X \).

The plausibility function \( \text{Plaus} \) satisfies the right secondary constraint \( X \rightarrow Y \mid Z \) if and only if the belief function \( \text{Plaus}^{\text{co neg}} \) satisfies the left secondary constraint \( \overline{X} \prec \overline{Y} \mid Z \) (Proposition 14) if and only if all the evidence supporting the belief in \( \overline{X} \) is already present among the evidence supporting the belief in \( \overline{Y} \) and the evidence supporting the belief in \( \overline{Z} \) if and only if the combined evidence supporting the plausibility of \( Y \) and the plausibility of \( Z \) is already present among the evidence supporting the plausibility of \( X \) if and only if both the combined plausibilities of \( Y \) and \( Z \) are weaker than or equal to the plausibility of \( X \).

An interesting right secondary constraint in this respect is \( \emptyset \rightarrow Y \mid Z \), with \( Y \cup Z = S \). In this case, \( \text{Plaus}(Y) + \text{Plaus}(Z) = 1 \). One can therefore interpret \( Y \) and \( Z \) as exhaustive events relative to \( \text{Plaus} \).

When a belief function is a probability distribution, the interpretation for primary and secondary constraints can be sharpened. A probability distribution \( P \) satisfies the left (right) primary constraint \( X \prec Y \) \((X \rightarrow Y)\) if and only if the event \( X - Y \) \((Y - X)\) is an impossible event. A probability distribution \( P \) satisfies the left (right) secondary constraint \( X \prec Y \mid Z \) \((X \rightarrow Y)\) if and only if the event \( X - Y \mid Z \) \((Y - X) \mid Z \) is an impossible event.

Example 18. We revisit Example 8. Table 5 shows the evidence and resulting belief, plausibility, doubt, nonbelief, and commonality values for the hypotheses of this example. In this example, we interpret the belief values.

We observe that \( \text{Bel}(\{T\}) = \text{Bel}(\{T, E\}) \), whence \( \text{Bel} \) satisfies the left primary constraint \( \{T, E\} \prec \{T\} \). Indeed, we see that all the evidence supporting the belief in \( \{E\} \) (namely, none) is already among the evidence supporting the belief in \( \{T\} \).

We also observe that \( \text{Bel}(\{H, T\} \cap \{H, E\} \cap \{T, E\}) + \text{Bel}(\{H, T\} \cap \{H, E\}) + \text{Bel}(\{H, T\} \cap \{T, E\}) \), whence \( \text{Bel} \) satisfies the left secondary constraint \( \{H, T\} \prec \{H, E\} \mid \{T, E\} \). Indeed, we see that all the evidence supporting the belief in \( \{H, T\} \) (namely, the evidence for \( H \) and the evidence for \( T \)) is already among the combined evidence supporting the belief in \( \{H, E\} \) (namely, the evidence for \( H \) and the belief in \( \{T, E\} \) (namely, the evidence for \( T \)).

Notice that, by Proposition 14, \( \text{Doubt} \) also satisfies \( \{T, E\} \prec \{T\} \) and \( \{H, T\} \prec \{H, E\} \mid \{T, E\} \), while \( \text{Plaus} \) satisfies \( \{H\} \rightarrow \{H, E\} \) and \( \{E\} \rightarrow \{T\} \mid \{H\} \) (which can
also be written as the strong right constraint \( \{E\} \to \{T\} \).

4.6. Reasoning about uncertainty—possibility theory

Since a possibility function can always be interpreted as a plausibility function in Dempster–Shafer theory and a necessity function can similarly be interpreted as a belief function, we refer to Section 4.5 for the interpretation of the corresponding constraints.

5. The implication problem for measure constraints

In this section, we will study the implication problem associated with primary and secondary constraints for the four families of measures considered in this paper.\(^{10}\) In addition, we will give sound, and sometimes complete, inference systems for this problem.

Remember that the four families of measures under consideration are denoted by \( \mathcal{A}_i \), \( \mathcal{M}_i \), and \( \mathcal{M}_s \), respectively. An ‘s’ in the subscript position means “submodular,” and an ‘i’ in the superscript position means “isotone,” and an “i” in the subscript position means “anti-isotone.”

**Definition 19.** Let \( \mathcal{C} \) be a finite set of measure constraints, and let \( c \) be a measure constraint, all over the same finite set \( S \). We will say that \( \mathcal{C} \) implies \( c \) in a family of measures, if each measure \( M \) over \( S \) in this family satisfying all the constraints in \( \mathcal{C} \) also satisfies \( c \). As the family of measures will always be clear from the context, we shall denote this by \( \mathcal{C} \models c \).

Now, suppose \( M \in \mathcal{A}_i \) satisfies the right primary constraint \( X \to Y \). Then, \( M^\neg \), which is in \( \mathcal{M}_i \) also satisfies \( X \to Y \). However, \( M^\neg \), which is in \( \mathcal{M}_s \), as well as \( M^\neg \), which is in \( \mathcal{M}_i \), satisfy \( \neg Y \to \neg Y \). The generalization of this example is captured in the next proposition.

**Proposition 20.** Let \( \mathcal{C} \) be a finite set of right measure constraints, and let \( \mathcal{C}^\neg \) be the set of the correspond-

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\(^{10}\)In another paper, we also considered the implication problem associated with primary constraints for set-based functions which are isotone/anti-isotone, but not necessarily submodular/supermodular [35]. This implication problem is of independent interest, but, due to paper-length considerations, we decided not to incorporate it in this paper.
these variables in \( \mathbb{R} \) can be interpreted as a real function over \( 2^S \), which is nonnegative provided \( x_U \geq 0 \), for all \( U \subseteq S \). This function is an isotone, submodular measure if and only if, for all \( U, V \subseteq S \), \( x_U \leq x_{U \cap V} \), and \( x_{U \cup V} + x_{U \cap V} \leq x_U + x_V \). This measure satisfies the primary right measure constraint \( X \to Y \) if and only if \( x_X = x_{X \cup Y} \) evaluates to true. Similarly, this measure satisfies the secondary right measure constraint \( X \to Y | Z \) if and only if \( x_{X \cup Y \cap Z} + x_X = x_{X \cup Y} + x_{X \cup Z} \) evaluates to true. By putting the pieces of the puzzle together, we can now easily construct a first-order sentence over the reals with addition expressing precisely that \( \forall \in c \). It is well known that such a sentence can be decided. \( \Box \)

As a corollary to Theorem 21 and Proposition 20, the implication problems for right measure constraints in the context of anti-isotone, supermodular measures, left measure constraints in the context of isotone, supermodular measures, and left measure constraints in the context of anti-isotone, submodular measures are also decidable. The same holds if we only consider measures that are rational-valued (the first-order over the rationals with addition is decidable) or integer-valued (Pressburger arithmetic is decidable).

Unfortunately, the complexity of these decision problems is high. Deciding a first-order sentence over the reals or rationals with addition known to be nondeterministic exponential-time hard, and deciding a first-order sentence in Pressburger arithmetic is double-exponential time (see, e.g., [36]). However, there are subclasses of the implication problem which can be decided much more efficiently using inference rules and systems.

### 5.2. Inference rules and systems

In this subsection, we will specify inference rules and inference systems for the implication problem for right measure constraints in the context of measures in \( \mathcal{M}_a \). To this effect, consider the inference rules for right measure constraints shown in Table 8. In this table, the upper-left box, the upper-right box, the bottom-left box, and the bottom-right box show inference rules for primary constraints, secondary constraints, strong secondary constraints, and mixed constraints, respectively.

We can now state the main results which relate the implication problem for measure constraints with the inference problem for such constraints. Our results are mixed. For primary constraints and for strong secondary constraints we obtain soundness and completeness. However, for general secondary constraints, we only obtain soundness. The existence of a finite axiomatization in this case is discussed later.

**Theorem 22.** (1) The triviality, augmentation, and transitivity rules form a sound and complete inference system for the implication problem of primary right constraints in \( \mathcal{M}_a \).

(2) The triviality, weak augmentation, weak transitivity, and symmetry rules form a sound inference system for the implication problem of right secondary constraints in \( \mathcal{M}_s \).
(3) The triviality, augmentation, transitivity, and the symmetry rules form a sound and complete inference system for the implication problem of strong right secondary constraints in $\mathcal{M}_s$.

(4) The triviality, augmentation, and transitivity rules for primary right constraints, together with the triviality, augmentation, transitivity, and symmetry rules for strong right secondary constraints, and the replication and the coalescence rules form a sound and complete inference system for the implication problem of right primary and strong right secondary constraints in $\mathcal{M}_s$.

Proof. Soundness. Throughout, we assume that $M$ is an isotone, submodular measure satisfying all constraints in the antecedent of the rule under consideration.

To avoid overly long expressions, we will omit “union” signs, e.g., $XY$ will denote $X \cup Y$.

- **Triviality for right primary constraints:** If $Y \subseteq X$, then $XY = X$, and, therefore, $M(XY) = M(X)$. Hence, $M$ satisfies $X \rightarrow Y$.

- **Augmentation for right primary constraints:** From Proposition 12, it follows that $M^{X,U}(X) \leq 0$, or $M(XY) - M(XU) \leq M(XY) - M(X)$. Since, by assumption, $M$ satisfies $X \rightarrow Y$, the right-hand side equals 0. Since $M$ is isotone, it follows that $M(XY) = M(XU)$, or, since $U \supseteq V$, that $M(XU) = M(XU)$. Hence, $M$ satisfies $XU \rightarrow YV$.

- **Transitivity for right primary constraints:** By assumption, $M$ satisfies $X \rightarrow Y$, whence $M(XY) = M(X)$. By assumption, $M$ also satisfies $Y \rightarrow Z$. By augmentation, $M$ furthermore satisfies $XY \rightarrow Z$, whence $M(XYZ) = M(XY)$. Thus, $M(XYZ) = M(X)$. By the isotonicity of $M$, $M(X) \leq M(XZ) \leq M(XYZ)$, whence $M(XZ) = M(X)$. Thus, $M$ satisfies $X \rightarrow Z$.

- **Triviality for right secondary constraints:** If $X \supseteq Y$, then $M(XYZ) - M(XZ) - M(XY) + M(X) = M(XZ) - M(XZ) - M(X) + M(X) = 0$, whence $M$ satisfies $X \rightarrow Y|Z$.

- **Weak augmentation for right secondary constraints:** We distinguish three cases.

  (1) **Case 1:** $U \subseteq XY$. Since $M$ is submodular, it suffices to show that
  $$M(XU) - M(XUY) - M(XU) \geq 0.$$ 
  Since $XY \supseteq U \supseteq V$, the left-hand side of the conjectured inequality reduces to
  $$M(XYZ) - M(XY) - M(XU) + M(XU).$$

  Since $M$ satisfies $X \rightarrow Y|Z$, this expression equals
  $$\begin{align*}
  M(XZ) - M(X) - M(XUY) + M(XU).
  \end{align*}$$

  The desired inequality now follows from the submodularity of $M$.

  (2) **Case 2:** $U \subseteq XZ$. The proof of this case is completely analogous to the proof of Case 1.

  (3) **Case 3:** $U \subseteq X$. Let $U = U \cap XY$ and $V = V \cap XY$. By Case 1, $M$ satisfies $XU \rightarrow YV|Z$. Since $U \subseteq XU|Z$, it finally follows, by Case 2, that $M$ also satisfies $XU \rightarrow YV|Z$.

- **Weak transitivity for right secondary constraints:** By assumption, we know that $M^{Y,V}(X) = M^{Y,V}(X) = 0$. Subtracting both equalities yields
  $$M(XYZ) - M(XZ) - M(XU) + M(X) = 0.$$ 

  By the submodularity of $M$, $M(XYZ) - M(XZ) - M(XU) + M(X) = 0$. Again by the submodularity of $M$, this inequality is actually an equality, whence $M$ satisfies $X \rightarrow Z|U$.

- **Symmetry for right secondary constraints:** The expansion of $M^{Y,Z}(X) = 0$ is invariant under interchanging $Y$ and $Z$.

- **Inference rule for strong right secondary constraints:** They follow directly from the corresponding rules for general right secondary constraints, except that, in the case of transitivity, we have to observe first that, by augmentation, $Y \rightarrow Z$ yields $X \rightarrow Z$.

- **Replication:** By assumption, $M(X) - M(XY) = 0$, and by the isotonicity of $M$, $M(S) - M(XT) \geq 0$. Adding these up yields $M(S) - M(XY) - M(XT) + M(X) \geq 0$. By the submodularity of $M$, this inequality is actually an equality, whence $M$ satisfies $X \rightarrow Y$.

- **Coalescence:** By an application of the triviality and transitivity rules, $Y \rightarrow Z$ yields $Y \rightarrow Z - Y$. Without loss of generality, we may therefore show

\[
\begin{align*}
X \rightarrow Y & \quad \& Y \rightarrow Z \\
\hline
X \rightarrow Z
\end{align*}
\]

under the assumption that $Y \cap Z = \emptyset$. By replication, we know that $M$ also satisfies $Y \rightarrow Z$, or

$$M(S) - M(Z) - M(YZ) + M(Z) = 0,$$
taking into consideration that \( Y \cap Z = \emptyset \). Since \( M \) satisfies \( Y \rightarrow Z \), \( M(YZ) = M(Y) \), whence \( M(S) = M(Z) \). By the isotonicity of \( M \), it follows that \( M(S) = M(XZ) \). By the transitivity rule, we know that \( M \) satisfies \( X \rightarrow Z \), or
\[
M(S) = M(XZ) = M(XZ) + M(X) = 0.
\]
Substituting \( M(S) = M(XZ) \) in this equality yields \( M(XZ) = M(X) \), whence \( M \) satisfies \( X \rightarrow Z \).

**Completeness**: We only prove the completeness part of statement 1 of Theorem 22. The other completeness proofs are completely analogous. To do so, let \( \mathcal{C} \) be a set of right primary constraints, let \( X \rightarrow Y \) be a right primary constraint, and assume that \( \mathcal{C} \models X \rightarrow Y \) in \( \mathcal{M}_{si} \). Now, let \( \varrho \) be an arbitrary relation satisfying all functional dependencies in \( \mathcal{C}_{fd} = \{ V \rightarrow Y \} \) where \( V \rightarrow Y \in \mathcal{C} \). If \( \varrho \) is empty, \( \varrho \) trivially satisfies \( X \rightarrow Y \). If \( \varrho \) is not empty, we proceed as follows. Let \( p \) be the uniform probability distribution on \( \varrho \). By the observations above, the Shannon measure for \( \varrho \) corresponding to \( p \), \( H \), which is a measure in \( \mathcal{M}_{si} \), satisfies all constraints in \( \mathcal{C} \). By assumption, \( H \) also satisfies \( X \rightarrow Y \). Hence, \( \varrho \) satisfies \( X \rightarrow Y \). We have thus proved that the functional dependency \( X \rightarrow Y \) is logically implied by the set of functional dependencies \( \mathcal{C}_{fd} \). As shown by Armstrong [37], \( X \rightarrow Y \) can be inferred from \( \mathcal{C}_{fd} \) by the reflexivity, augmentation, and transitivity rules for functional dependencies. Hence, \( X \rightarrow Y \) can be inferred from \( \mathcal{C} \) by the triviality, augmentation, and transitivity rules for right primary constraints.

The completeness proof is by reduction to the implication problem for functional and multivalued dependencies, using the Shannon entropy measure with respect to a uniform distribution. By the observations made in Section 4.2, we could as well have reduced to the implication problem for functional and degenerate multivalued dependencies, using the Gini-index, or to the implication problem for functional and multivalued domain dependencies, using the isotone, submodular measure \( D \) proposed near the end of Section 4.2.

The connections between right primary and strong right secondary constraints for measures on the one hand and, respectively, functional and multivalued dependencies for database relations on the other hand, make one wonder if there is a similar connection between general right secondary constraints and embedded multivalued dependencies\(^\text{11}\) [26,27]. This, however, is not the case, since the implication problem for general right measure constraints is decidable, by Theorem 21, whereas the implication problem for embedded multivalued dependencies is not, as was shown by Herrmann [38]. There is also a more direct way to see this. Consider the right secondary constraint \( X \rightarrow Y|Y \). An isotone submodular (respectively, anti-isotone supermodular) measure \( M \) satisfies \( X \rightarrow Y|Y \) if and only if \( M(X \cup Y) = M(X) + M(Y) \); i.e., if and only if \( M(X \cup Y) = M(X) \), i.e., if and only if \( M \) satisfies \( X \rightarrow Y \). Thus, we see that the right secondary constraint \( X \rightarrow Y|Y \) is equivalent to the primary constraint \( X \rightarrow Y \), whereas the embedded multivalued dependency \( X \rightarrow Y|Y \) is trivial. This leaves open the question whether there exists a— in some sense— finite set of inference rules that is sound and complete for general right secondary constraints (or general right secondary constraints combined with right primary constraints).

The equivalence between the implication problem for certain types of measure constraints and the implication problem for functional and multivalued dependencies in relational databases revealed in Theorem 22 has also repercussions in the area of complexity. Since the implication problems for functional dependencies, multivalued dependencies, and mixed functional and multivalued dependencies are in PTIME [39], it follows that each of the corresponding implication problems for measure constraints are also in PTIME. This is an example of how the theory of relational dependencies is applicable to the more general theory of measure constraints.

Finally, we extend Theorem 22 to the other classes of measures. For measures in \( \mathcal{M}_{si} \) or \( \mathcal{M}_{s} \), left primary and secondary constraints must be considered in conjunction with the inference rules in Table 9.

### 6. Further work

A possible direction for future work is to consider other domains to which our theory applies. For

\(^{11}\)Let \( S \) be a relation scheme, \( X, Y, Z \subseteq S \), and \( \varrho \) a relation over \( S \). Then \( \varrho \) satisfies the embedded multivalued dependency \( X \rightarrow Y|Z \) if the projection \( \pi_{X \cup Y \cup Z}(\varrho) \) satisfies the multivalued dependency \( X \rightarrow Y \).
Another direction is to consider more general constraints. For example, in a theory consistent with ours, one could consider constraints based on higher-order differentials, or on all-together different concepts. In this context, one can consider the study of measure constraints that have properties beyond those that we considered in this paper. Of particular interest are measures that can be defined in terms of mass (density) functions such as is the case for measures like frequency, belief, plausibility, possibility, etc. Work in this direction has already begun [42–45].

Finally, we like to observe that for the Dempster–Shafer and the possibility theories, our paper contributes in the area of disjunctive reasoning in these theories. This should be contrasted with the heavy focus on studying conjunctive reasoning in these theories (see, e.g., [6]).

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