Visualizing Relativity

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Siggraph 2001 Tutorial

GRAND PLAN

I: Introduction: *Hanson, 50 min*

II: Visualization Methods: *Hanson, 40 min*

< 15 minute Break >

III: Light: *Weiskopf, 30 min*

IV: Rendering: *Weiskopf, 30 min*

V: Interaction Techniques: *Weiskopf, 30 min*

VI: Conclusion and Questions: *15 min*

I: Introduction to Special Relativity

- **Motivation**
- **2D Euclidean vs Minkowski:** Build Relativity concepts from 2D Graphics concepts.
- **Spacetime Points and the Twin Paradox.**
- **Relativistic Objects and Cameras:** What happens to graphics modeling near the speed of light.

II: Visualization Methods in 3D and 4D

- **2 Space + 1 Time:** Transformations.
- **Rolling the Relativistic Ball**:

  Thomas Precession
- **Aberration of Light:**
- **Object Viewing:** Occlusion, IBR, and the Terrell Cube
- **4D = 3 space + 1 time:**

III: Light

- Directions in Relativity
- Frequency Transformations
- Relativistic Radiance Transforms
- Bending Light with General Relativity

IV: Rendering

- **From the Z buffer to the T buffer**
- **Special Relativistic Ray Tracing**
- **Texture and Relativistic IBR**
- **Gravitational Lensing**

V: Interaction Techniques

VI: Conclusion

Visualizing Relativity

Part I: Introduction to Special Relativity

Andrew J. Hanson

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I: Introduction to Special Relativity

- **Motivation**
- **2D Euclidean vs Minkowski:** Build Relativity concepts from 2D Graphics concepts.
- **Spacetime Points and the Twin Paradox.**
- **Relativistic Objects and Cameras:** What happens to graphics modeling near the speed of light.

Motivation

WHY ARE YOU HERE? Let's guess:

- $\Rightarrow$ You know about Graphics
- $\Rightarrow$ You know about Visualization
- $\Rightarrow$ You DO NOT know much about Relativity.
- $*$ You WOULD LIKE to know how these three things are CONNECTED. . .

Motivation, contd.

What is Graphics?

- **Graphics:** is the art of *simulating* the *physics* of the interaction of material and light.

What is Visualization?

- **Visualization:** is the art of *creating insights* into non-self-explanatory data and geometry using graphics.

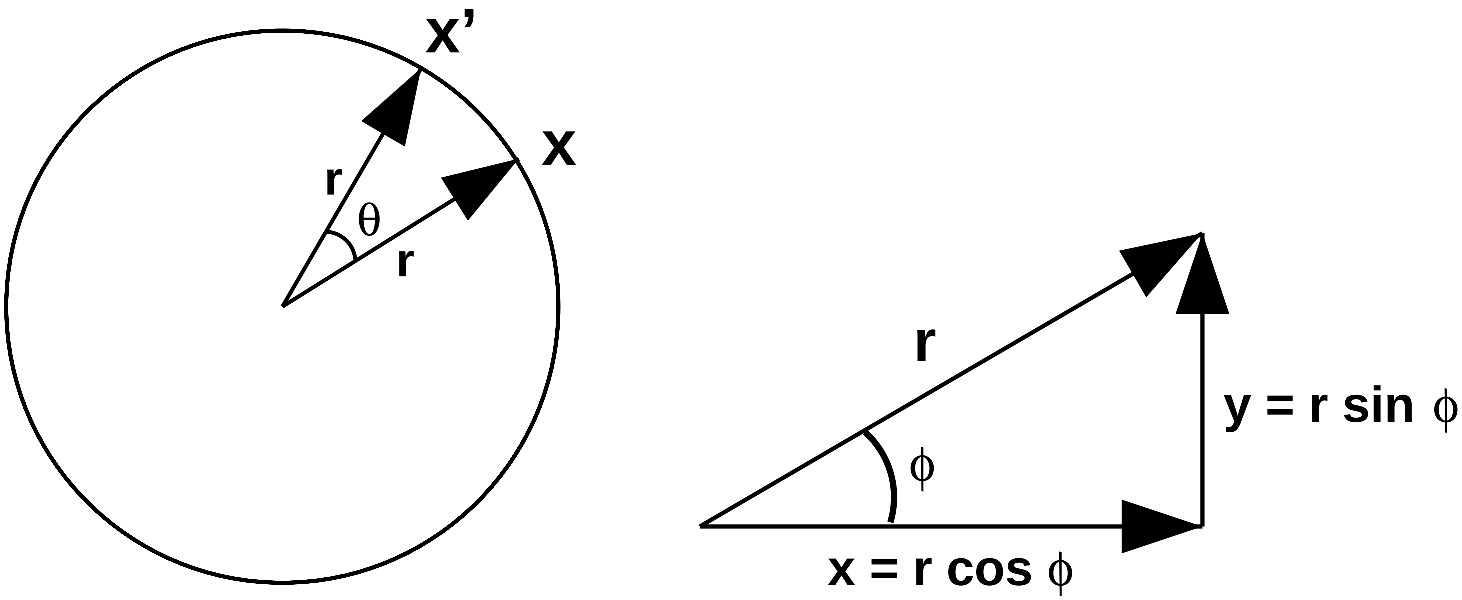
What is Relativity?

- **Relativity:** is the mathematics describing the interaction of material and light UNDER EXTREME PHYSICAL CONDITIONS.

Therefore, this course is the logical extension of everything graphicists and visualizers *already do!*

Euclidean Transformations

We begin with what we all know — 2D Rotations.



$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$

Euclidean Transformations, contd.

Explicit 2D rotations are realized by a 2D matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where

$$R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R(\theta)^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

because $\boxed{(\cos\theta)^2 + (\sin\theta)^2 = 1}$

Euclidean Transformations, contd.

Main feature: The *Radius* is unchanged under $[\mathbf{x}'] = R(\theta) \cdot [\mathbf{x}]$:

$$r = \sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}$$

In other words, $\boxed{\text{Euclidean distances}}$ do not vary under the action of rotations.

Euclidean Transformations, contd.

Similarly, the *Euclidean Inner Product* is unchanged under $[\mathrm{x}'] = R(\theta)\cdot[\mathrm{x}]$, $[\tilde{\mathrm{x}}'] = R(\theta)\cdot[\tilde{\mathrm{x}}]$

$$\mathrm{x}\cdot\tilde{\mathrm{x}} = \mathrm{x}'\cdot\tilde{\mathrm{x}}' \;=\; [\,x \;\; y\,]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}$$

$$=\; x\tilde{x} + y\tilde{y} = r\tilde{r}\cos(\phi - \tilde{\phi})$$

In other words, $\boxed{\text{Euclidean angles}}$ do not vary under the action of rotations.

Euclidean Transformations, contd.

Properties we know and love:

- Rotations have a **fixed point** at origin.
- Rotations leave **segment lengths and inner products** unchanged.
- Rotations are **orthogonal** $\Rightarrow \; R\,I\,R^t = I$
- **NOTE:** The PROJECTIONS may change, yet we "know" the segment length is constant.

Lorentz Transformations

Special Relativity is just "Rotations with hyperboloids instead of circles."

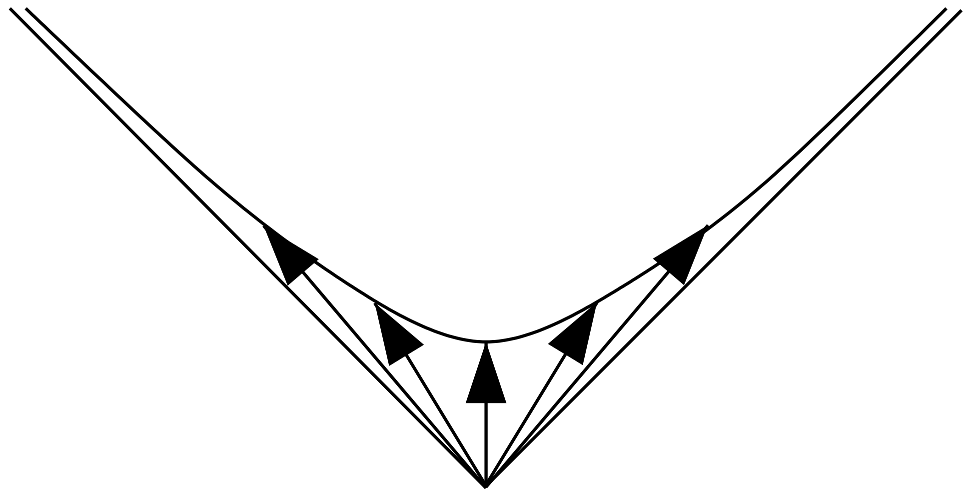
Euclidean Rotations $\Rightarrow$ Lorentz Transformations.

Let x be a space interval and t be a time interval:

$$\begin{aligned} x' &= x \cosh \xi + t \sinh \xi \\ t' &= x \sinh \xi + t \cosh \xi \end{aligned}$$

Lorentz Transformations, contd.

When we apply this transform to a vector from the origin to a point (x, t), the new point (x', t') lies on a $\boxed{\text{hyperboloid}}$ instead of a circle!



Lorentz Transformations, contd.

Explicit 1-space + 1-time Lorentz transformations are realized by a 2D “boost” matrix

$$B(\xi) = \begin{bmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{bmatrix}.$$

where

$$B(\xi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B(\xi)^t = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$B(\xi)$ preserves the length of **proper time** due to $\boxed{(\cosh\xi)^2 - (\sinh\xi)^2 = 1}$

Lorentz Transformations, contd.

Compare Euclidean and Lorentz functions:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\cos^2 + \sin^2 \ = \ 1$$

$$\cosh\xi = \frac{1}{2}\left(e^{\xi} + e^{-\xi}\right) \qquad \sinh\xi = \frac{1}{2}\left(e^{\xi} - e^{-\xi}\right)$$

$$\cosh^2 - \sinh^2 \ = \ 1$$

where the MINUS SIGN is all-important!

Lorentz Transformations, contd.

Main feature of Lorentz-transformed vectors is very close to rotations: Instead of the *Radius*, depending on sign inside root,

- THE PROPER TIME is unchanged.

$$\tau = \sqrt{t^2 - x^2} = \sqrt{t'^2 - x'^2}$$

- Alternatively, THE PROPER DISTANCE is unchanged.

$$\delta = \sqrt{x^2 - t^2} = \sqrt{x'^2 - t'^2}$$

Lorentz Transformations, contd.

- . . . and instead of the *Euclidean dot product*, the

THE MINKOWSKI SPACE INNER PRODUCT

$$\mathbf{x} \cdot \tilde{\mathbf{x}} = \begin{bmatrix} x & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} = x\tilde{x} - t\tilde{t}$$

IS UNCHANGED.

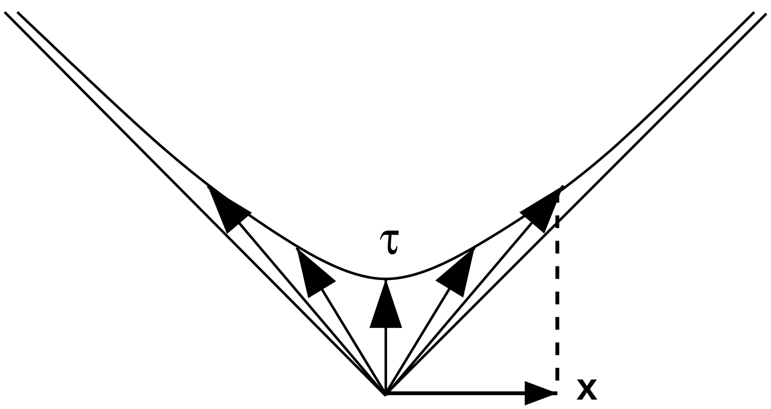
Lorentz Transformations, contd.

Now let's visualize a typical invariant:

$$\tau^2 = t^2 - x^2 = t'^2 - x'^2$$

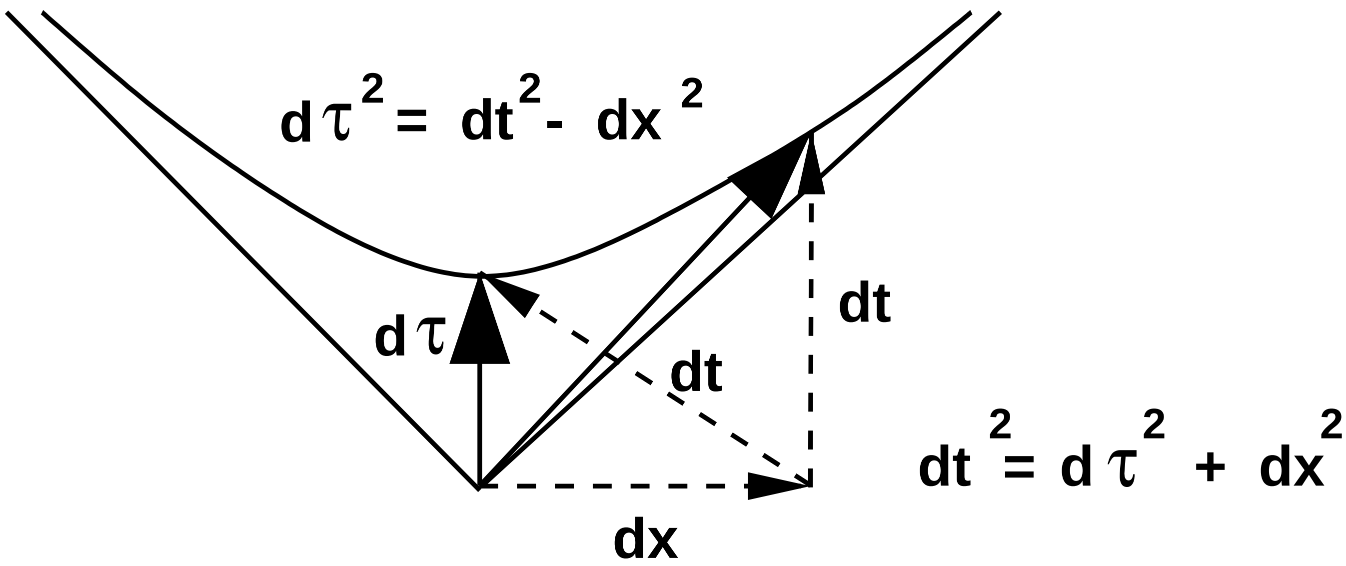
describes a $\boxed{\text{hyperbola}}$, $x = 0 \Rightarrow t = \tau$:

$$x \neq 0 \;\; \Rightarrow \;\; t = \sqrt{\tau^2 + x^2}$$



Lorentz Transformations, contd.

An alternative view showing geometry of proper time, emphasizing *interval property*.



Lorentz Transformations, contd.

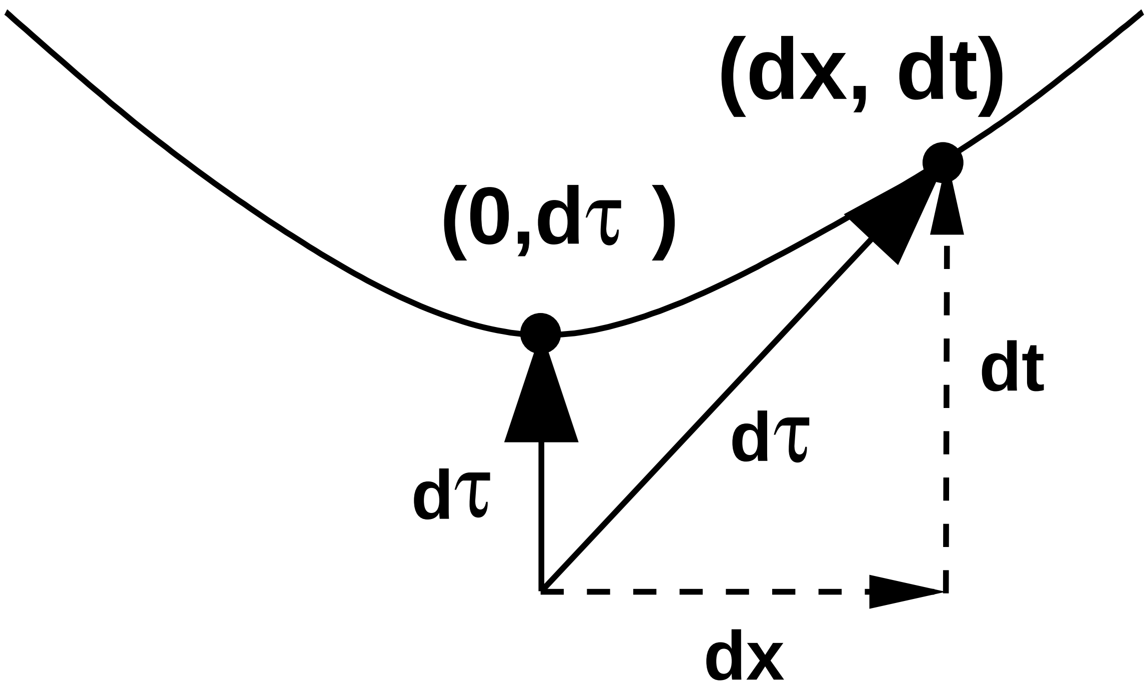
What *are* $\cosh\xi$ and $\sinh\xi$ anyway?

Suppose $t_0 = 1.0$ and $x_0 = 0$:

$$
\begin{aligned}
dx &= x_0\cosh\xi + t_0\sinh\xi = \sinh\xi \\
dt &= x_0\sinh\xi + t_0\cosh\xi = \cosh\xi .
\end{aligned}
$$

Lorentz Transformations, contd.

Thus $\boxed{(dx/dt) = \sinh\xi/\cosh\xi}$ is the *inverse slope* of the interval $(0.0,\, d\tau)$ after the transformation:



Lorentz Transformations, contd.

We identify this slope as the

$$\boxed{\text{velocity} \;=\; v \;=\; \frac{\sinh\xi}{\cosh\xi} \;=\; \tanh\xi}$$

Simple algebra and $\boxed{\cosh^2 - \sinh^2 = 1}$ give us:

$$\cosh\xi \;=\; \frac{1.0}{\sqrt{1.0 - v^2}}$$

$$\sinh\xi \;=\; \frac{v}{\sqrt{1.0 - v^2}}$$

OOps! Where did the velocity of light go?

Simple answer: we set it to unity to make (x, t) plots work!

Better answer: $\boxed{\text{Replace } v \Rightarrow v/c}$ whenever you need it.

What happens as $\boxed{c \Rightarrow \infty}$?? This is ORDINARY GALILEAN SPACETIME, where NO mixing of space and time can occur!

Lorentz Transformations and velocity of light

Check Galilean limit: as $\boxed{c \Rightarrow \infty}$

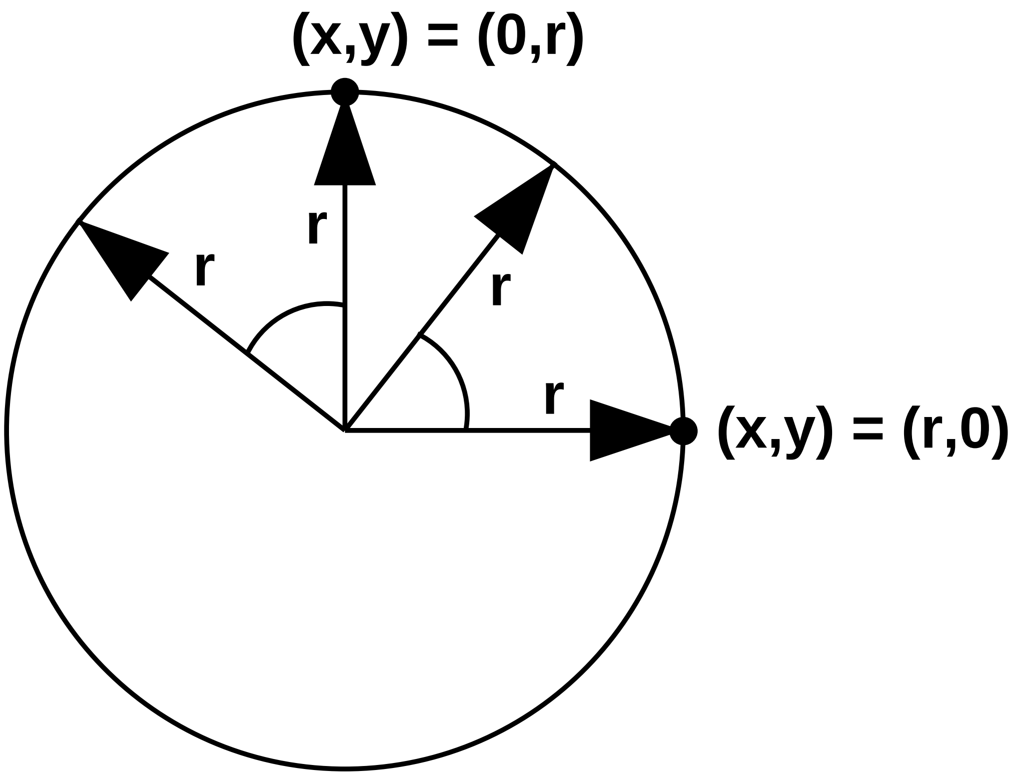
$$\cosh \xi = \frac{1.0}{\sqrt{1.0 - (v/c)^2}} \Rightarrow 1$$

$$\sinh \xi = \frac{v/c}{\sqrt{1.0 - (v/c)^2}} \Rightarrow 0$$

So we get $B(\xi) \Rightarrow$ **identity matrix** and the effects of the Lorentz transform disappear!

Lorentz Transformations, contd

Note: Euclidean intervals do not care if you start with $(x, y) = (r, 0)$ or $(x, y) = (0, r)$ before you rotate: $\Rightarrow r$ is always positive.



Lorentz Transformations, contd

Relativistic intervals **do care**:

$$(x,\, t) = (0,\, \tau),\;\; t^2 - x^2 > 0 \qquad \textbf{= Timelike interval}$$
$$(x,\, t) = (\tau,\, \tau),\;\; t^2 - x^2 \equiv 0 \qquad \textbf{= Lightlike interval}$$
$$(x,\, t) = (\delta,\, 0),\;\; t^2 - x^2 < 0 \qquad \textbf{= Spacelike interval}$$

Furthermore, these distinctions are $\boxed{\textbf{invariant}}$ under the Lorentz transform!

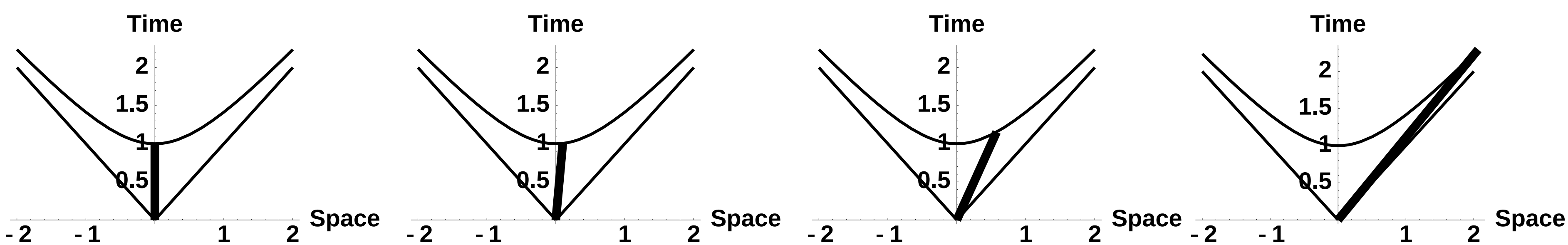
$$x' = x \cosh \xi + t \sinh \xi \qquad t' = x \sinh \xi + t \cosh \xi$$

Lorentz Transformations for timelike intervals

Define a **timelike interval**, with $x = 0.0$ and $t = 1.0$, and transform:

$$x' = x \cosh \xi + t \sinh \xi \qquad t' = x \sinh \xi + t \cosh \xi$$

$$x' = \sinh \xi \qquad t' = \cosh \xi$$



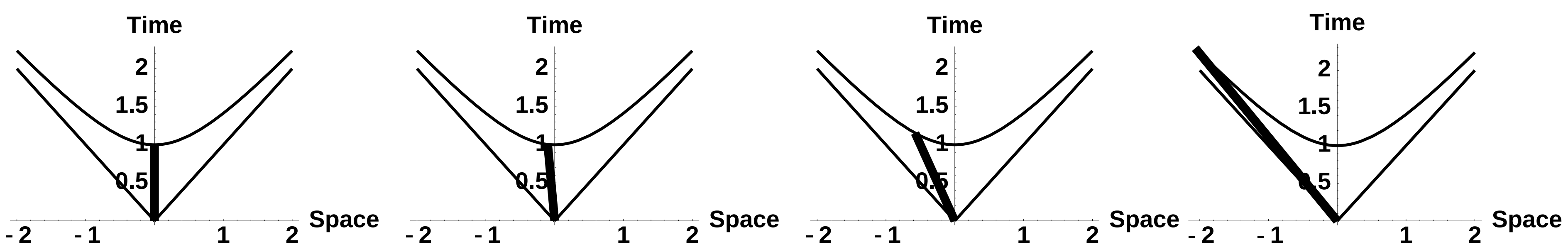
$v = 0.0$ $\quad v = 0.1$ $\quad v = 0.5$ $\quad v = 0.9$

Lorentz Transformations for time-like intervals

Let $t = 1.0$, $x = 0.0$ as before, but let **velocity be negative:**

$$x' = x \cosh \xi - t \sinh \xi \qquad t' = -x \sinh \xi + t \cosh \xi$$

$$x' = -\sinh \xi \qquad t' = \cosh \xi$$



$v = 0.0$ $\quad v = -0.1$ $\quad v = -0.5$ $\quad v = -0.9$

Lorentz Transformations: different velocity signs

You already know this difference:

Euclidean: **angle** > 0 means	**object interval is rotated**
Euclidean: **angle** < 0 means	**viewer is rotated**
Lorentz: **velocity** > 0 means	**object interval is boosted**
Lorentz: **velocity** < 0 means	**viewer is boosted**

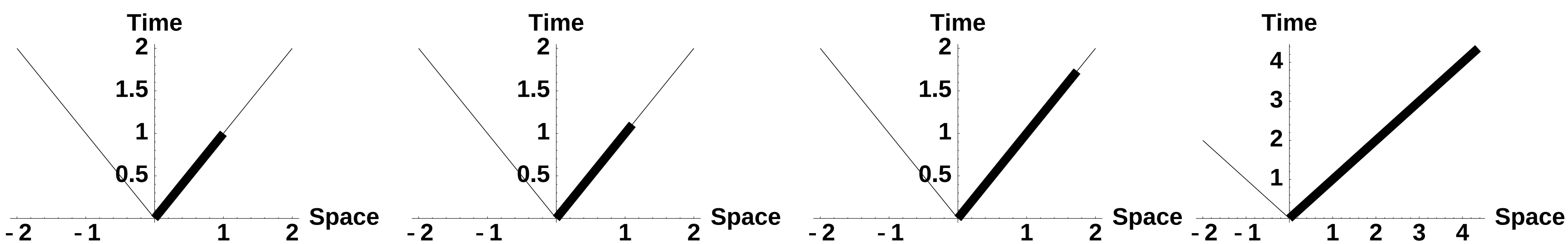
Lorentz Transformations for lightlike intervals

Define a lightlike interval,

with $x = 1.0$ and $t = 1.0$,

and observe that $x^2 - t^2 = x'^2 - t'^2 \equiv 0$:

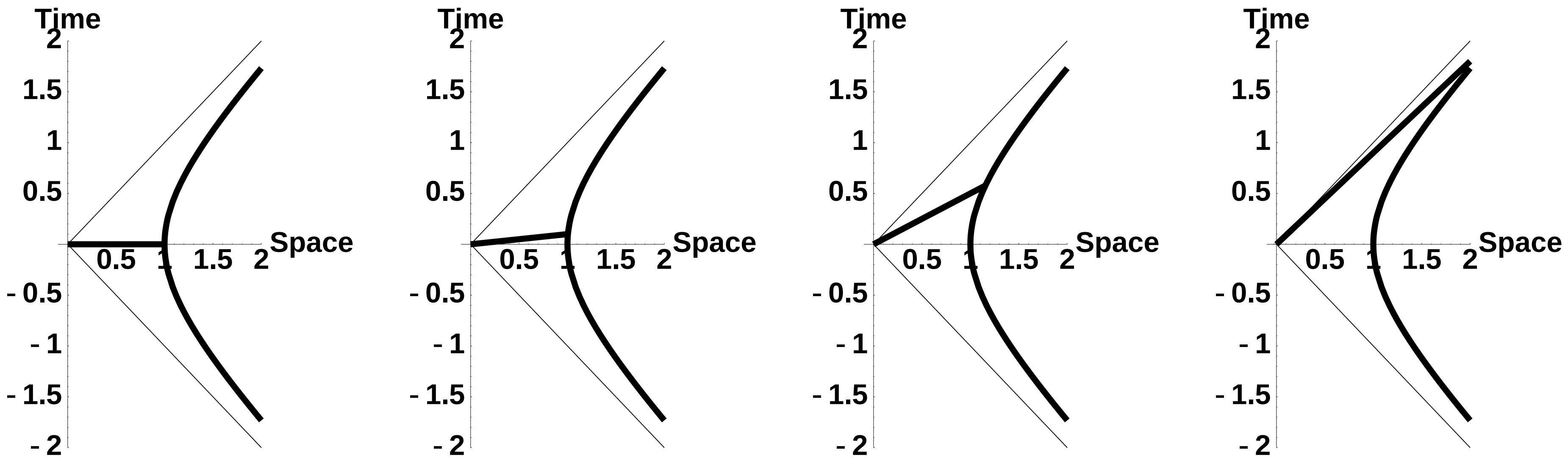
$$x' = \cosh\xi + \sinh\xi \qquad t' = \sinh\xi + \cosh\xi$$



$v = 0.0$ $\quad$ $v = 0.1$ $\quad$ $v = 0.5$ $\quad$ $v = 0.9$

Lorentz Transformations for spacelike intervals

Define a spacelike interval:
with $(x = 1.0, t = 0) \Rightarrow x^2 - t^2 > 0$
so $x' = \cosh\xi,\ t' = \sinh\xi$.



$v = 0.0$ $v = 0.1$ $v = 0.5$ $v = 0.9$

Lorentz Transformations: fixed points

Every graphicist knows that $\boxed{\mathrm{x}' = R \cdot \mathrm{x}}$ has a **fixed point at** $\mathrm{x} = 0$.

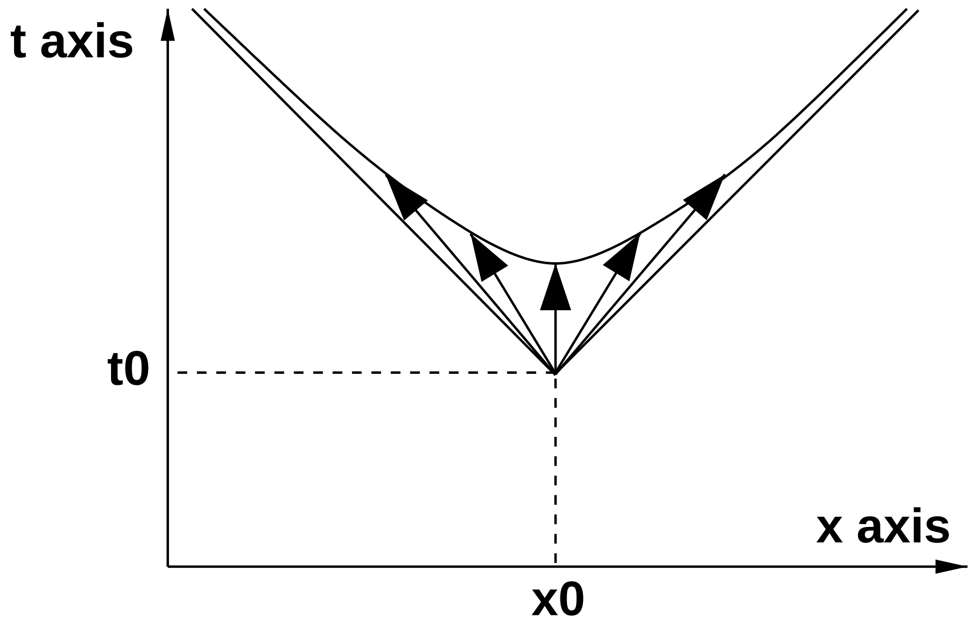
Relativity is the same: translate to $t = 0.0$ and $x = 0.0$ before transforming:

$$\begin{aligned} x' &= x_0 + (x - x_0)\cosh\xi + (t - t_0)\sinh\xi \\ t' &= t_0 + (x - x_0)\sinh\xi + (t - t_0)\cosh\xi \end{aligned}$$

Lorentz Transformations: fixed points

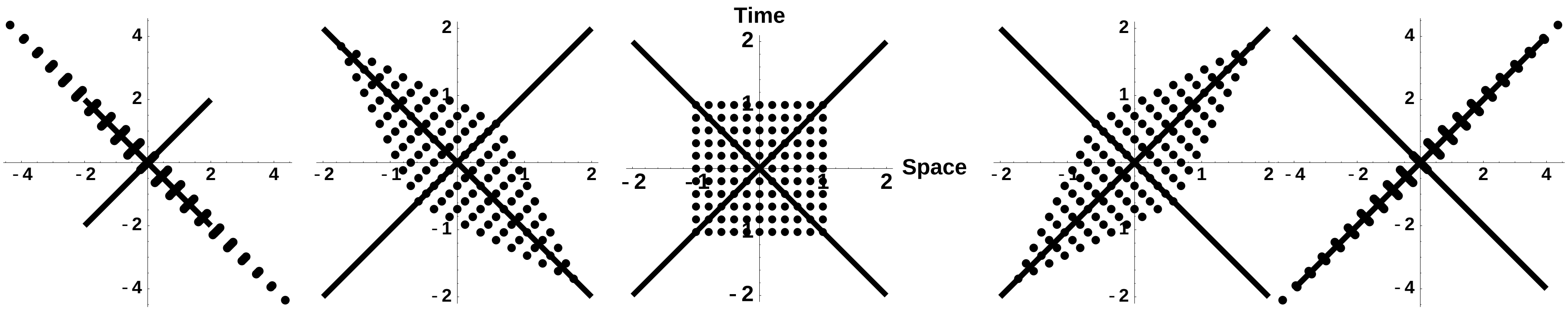
Transform with **Lorentz Fixed Point at $\mathbf{x}_0 = (x_0,\, t_0)$**:

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = T(+x_0, +t_0) \cdot B(\xi) \cdot T(-x_0, -t_0) \cdot \begin{bmatrix} x \\ t \end{bmatrix}$$



Lorentz Transformations: whole plane

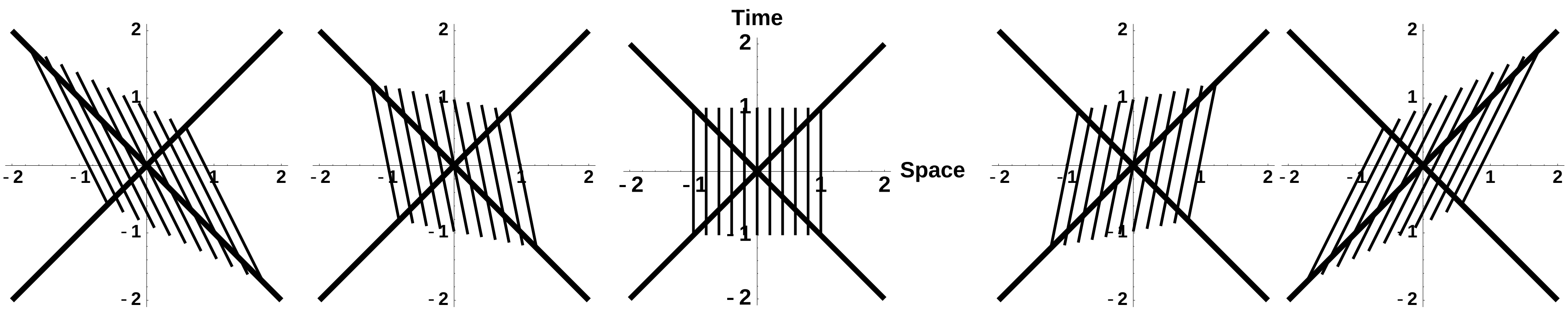
Every point in the (x, t) plane Lorentz transforms to one light cone or the other along a hyperboloid as $v \to \pm 1$:



$v = -0.9$ $v = -0.5$ $v = 0.0$ $v = 0.5$ $v = 0.9$

Lorentz Transformations: world lines

Every timelike line in the (x, t) plane Lorentz transforms to a slanted line as $v \to 1$:



$v = -0.5$ $\quad$ $v = -0.2$ $\quad$ $v = 0.0$ $\quad$ $v = 0.2$ $\quad$ $v = 0.5$

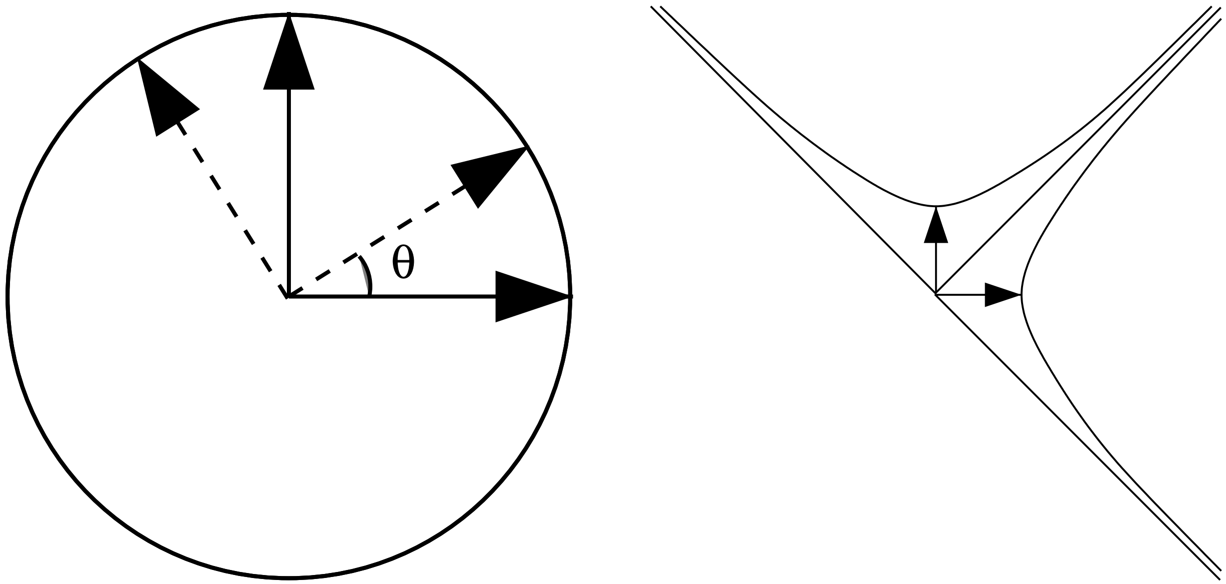
What is a Minkowski frame?

Let $\hat{\mathrm{x}}_0, \hat{\mathrm{t}}_0$ be the basis vectors of a Minkowski-space frame:

- **Space-Like:** $\hat{\mathrm{x}}_0 = (1, 0)$ whose *length* is $\hat{\mathrm{x}}_0 \cdot \hat{\mathrm{x}}_0 = 1$.
- **Time-Like:** $\hat{\mathrm{t}}_0 = (0, 1)$ whose *length* is $\hat{\mathrm{t}}_0 \cdot \hat{\mathrm{t}}_0 = -1$.

What is a Minkowski frame?

Compare a Euclidean frame to a Minkowski frame:



The Euclidean axes stay at right angles under rotations. What happens to the Minkowski axes under Lorentz transforms??

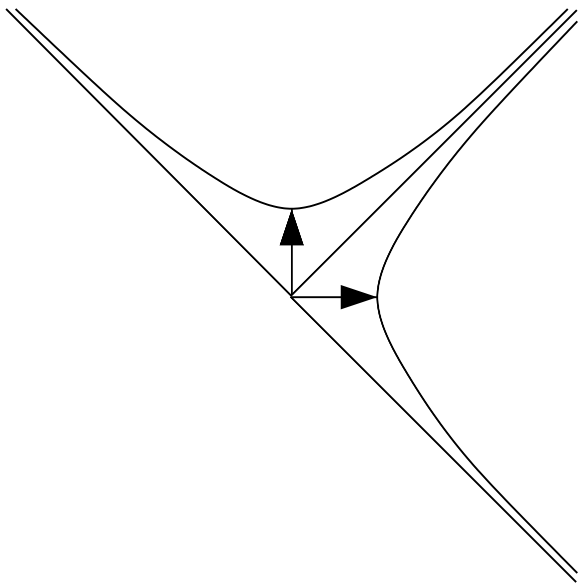
How do the frame axes transform?

The usual **Three Othonormality Conditions** are preserved **in any coord system.**

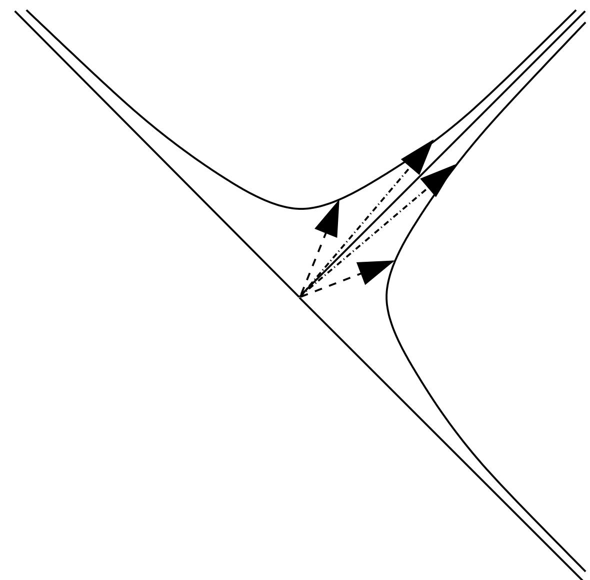
- **Space-Like:** $\hat{\mathbf{x}}_0 = (1,0)$ has *unit length*: $\hat{\mathbf{x}}_0 \cdot \hat{\mathbf{x}}_0 = 1$.
- **Time-Like:** $\hat{\mathbf{t}}_0 = (0,1)$ has *unit length*: $\hat{\mathbf{t}}_0 \cdot \hat{\mathbf{t}}_0 = -1$.
- **Orthogonality:** $\hat{\mathbf{x}}_0 = (1,0)$ and $\hat{\mathbf{t}}_0 = (0,1)$ have *vanishing inner product*: $\hat{\mathbf{t}}_0 \cdot \hat{\mathbf{x}}_0 = 0$.

Frame axis transforms, contd

The **picture** seems to show axes coming together, but **orthonormality** is automatically **preserved**:



Boost $\Rightarrow$



Lorentz Frame Axes

If we did not know about $\cosh^2\xi - \sinh^2\xi = 1$, we might represent the frame differently, e.g., as:

$$\begin{bmatrix} \hat{\mathrm{x}}_0 & \hat{\mathrm{t}}_0 \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

where the constraint $A^2 - B^2 = 1$ guarantees orthonormality in the the Minkowski space; the columns are orthogonal, and of length +1 and -1, respectively.

Lorentz Frame axes, contd

As for 2D rotations, we can define a **double-valued** parameterization $(a,\, b)$ of the frame:

$$\left[\, \hat{\mathbf{x}}_0 \;\; \hat{\mathbf{t}}_0 \,\right] = \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix}.$$

where $A^2 - B^2 = 1$ IF $a^2 - b^2 = 1$, and $(a,\, b)$ is precisely the *same frame* as $(-a,\, -b)$.

These are *hyperbolic half angle* formulas,
$a = \cosh(\xi/2)$, $b = \sinh(\xi/2)$!

1+1 "Quaternion" Frames!

Differentiating *both* $\dot{\hat{\mathbf{x}}}_0$ and $\dot{\hat{\mathbf{t}}}_0$, our eqns reduce to

$$\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

This is the square root *of Lorentz frame equations.*

(Quaternion frame equations have $\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$.)

Lorentz Transformations, summarized.

Properties we **will** know and love:

- Boosts have **fixed point** at origin.
- Boosts leave **proper times, proper lengths, and Minkowski inner products** unchanged.
- Boosts are **orthogonal on a negative signature identity matrix** $\Rightarrow \; B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^t = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- **As in Euclidean space:** The PROJECTED PARTS OF A VECTOR may change, yet we know the inner product lengths are CONSTANT.

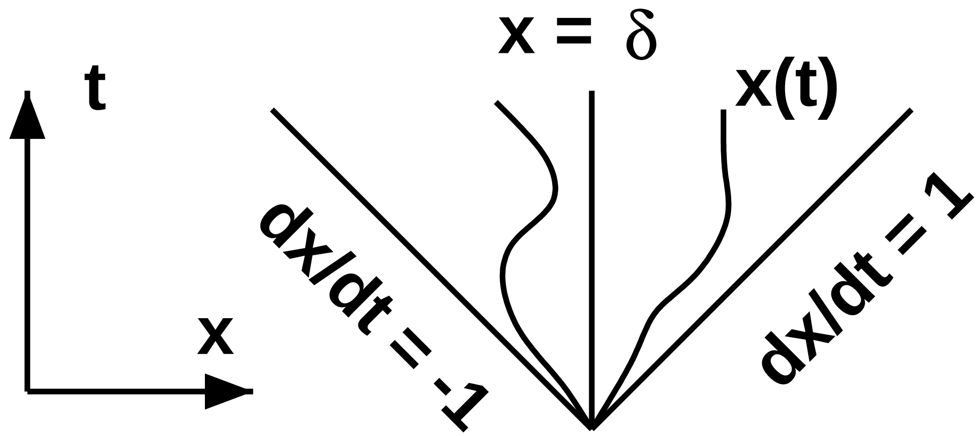
What is an object?

In Relativity, a point object is a **world line**.

- *Standing still* at one point: world line still ticks away: Equation $\Rightarrow$ $(\delta = \text{const},\ t)$.
- *Moving curve* $x(t)$ must obey $|dx/dt| < 1$.
- **Communication** can only occur using light or slower media.
- So all possibility of **image data** is restricted essentially to rays with paths having $|dx/dt| = 1$.

Point Objects . . .

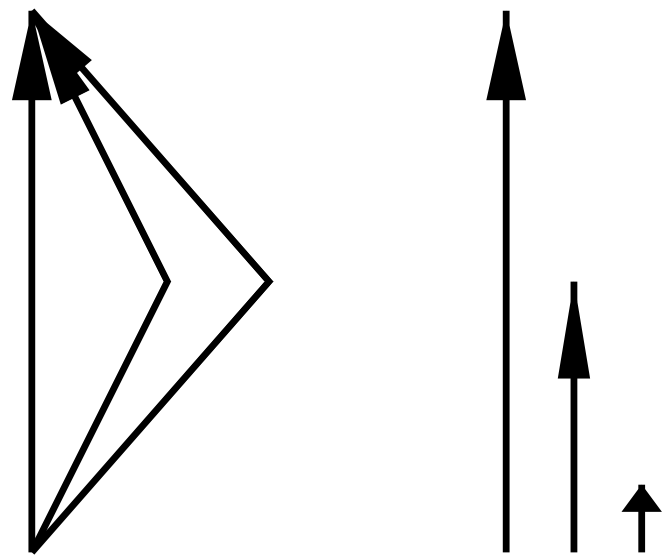
What do point objects look like in spacetime?



Relativistic equations have **space and time** components, so think of a **static point** as the **parametric line** (δ, t).

Twin Paradox

A **world line** represents an object, e.g., a person, evolving in time, possibly moving through space.



x(t) paths **corresponding proper times**

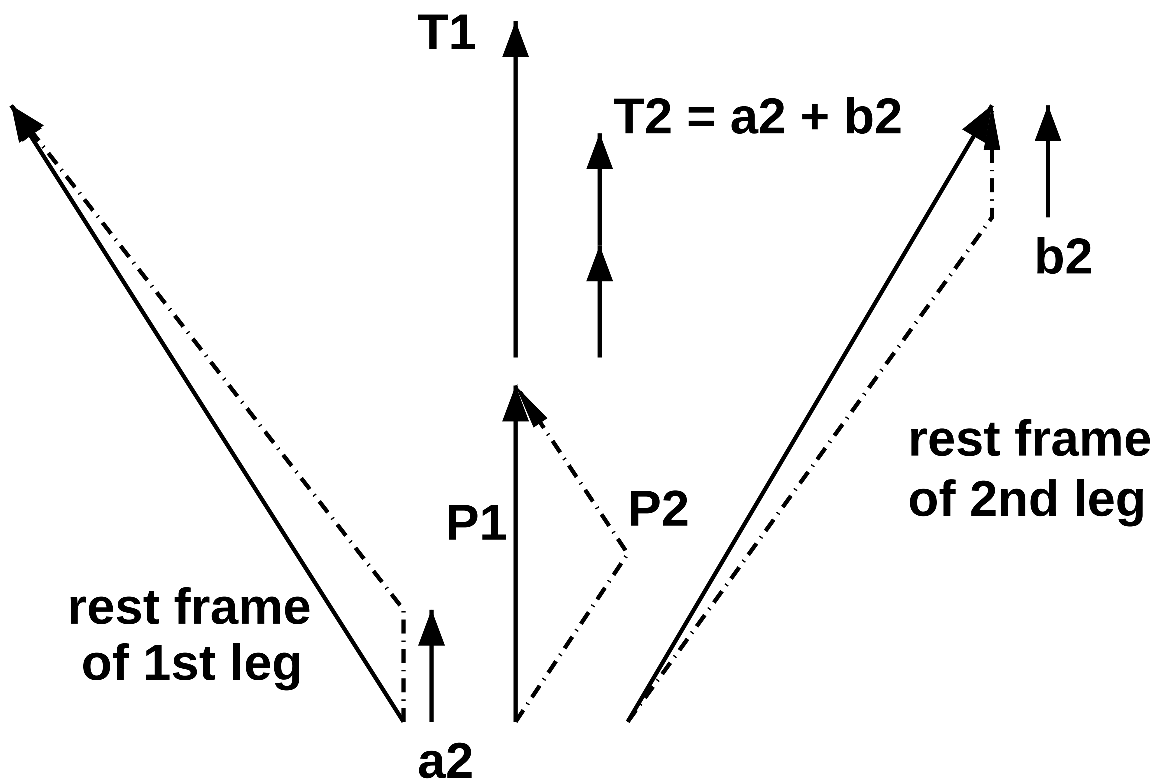
Twin Paradox, contd.

Consider $\boxed{\textbf{two twins}}$, one living on path P_1, the other on path P_2. Their ages $\boxed{\textbf{in any frame}}$ are the *proper lengths* of their world lines:

$$\textbf{Age 1} = T_1 = \int_{P_1} d\tau$$
$$\textbf{Age 2} = T_2 = \int_{P_2} d\tau$$

Twin Paradox, contd.

Graphical picture of twin ages: go to rest frame of each leg of journey to visualize true proper time:



Time Dilation of Point Clocks

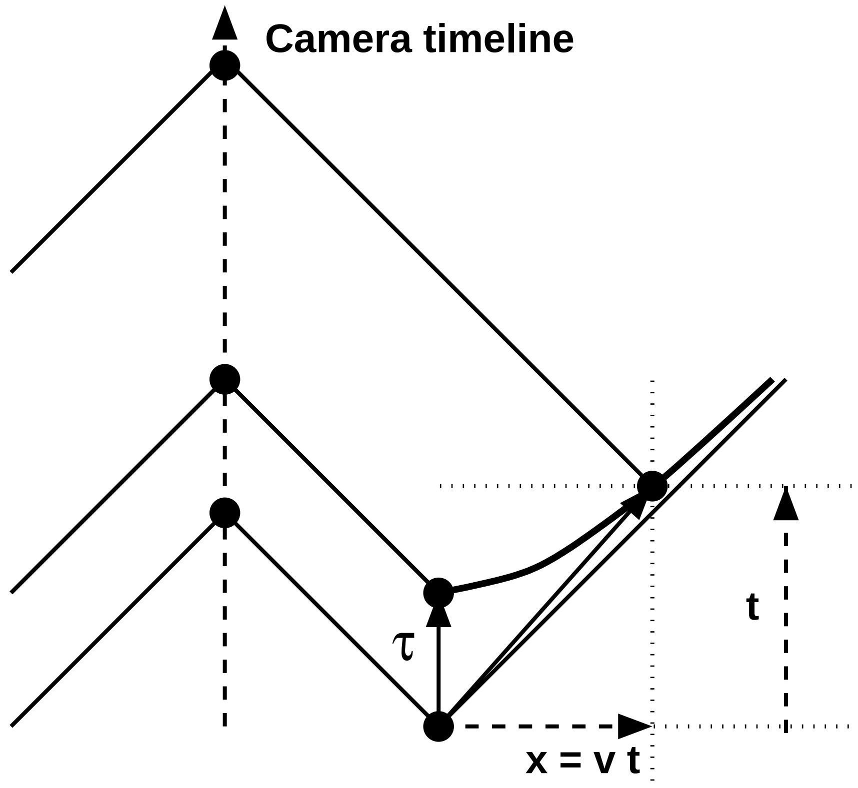
Since the point $(0, \tau)$ is transformed to $x = \tau \sinh \xi$, $t = \tau \cosh \xi$, we can solve for τ, yielding $x = vt$, so the **invariant proper time** can be written:

$$\tau = \sqrt{t^2 - x^2} = t\sqrt{1 - v^2}$$

Since the measured time $t = \tau/\sqrt{1 - v^2} > \tau$, this is $\boxed{\textbf{Time Dilation}}$.

Time Dilation, contd.

Now visualize change in apparent tick rate of **moving clock**, as well as **how you would measure it**:



Lorentz Contraction of Spacelike Intervals

For spacelike intervals, the situation is trickier. Let

$$x_1(t) = (0, t)$$
$$x_2(t) = (\delta, t)$$

be the ends of a line segment.

Lorentz Contraction, contd.

Under a Lorentz transform, the origin stays fixed, but

$$\begin{aligned} x_2'(t) &= (X(t), T(t)) \\ &= (\delta \cosh \xi + t \sinh \xi, \delta \sinh \xi + t \cosh \xi) \end{aligned}$$

becomes a curve with the old $(\delta, 0)$ pushed far up the hyperboloid to

$$X(0) = \delta \cosh \xi \qquad T(0) = \delta \sinh \xi$$

for large $v = \sinh \xi / \cosh \xi$.

Lorentz Contraction, contd.

We must take the line $(X(t), T(t)$ and **extrapolate backwards to $T(t) = 0$** to find the new interval as seen by the observer. Solving

$$T(t) = \delta \sinh \xi + t \cosh \xi = 0$$

for $t = t_0$, we find

$$t_0 = -\delta \sinh \xi / \cosh \xi$$

Lorentz Contraction, contd.

Thus t_0 is **negative** and we must have a **length reduction**.

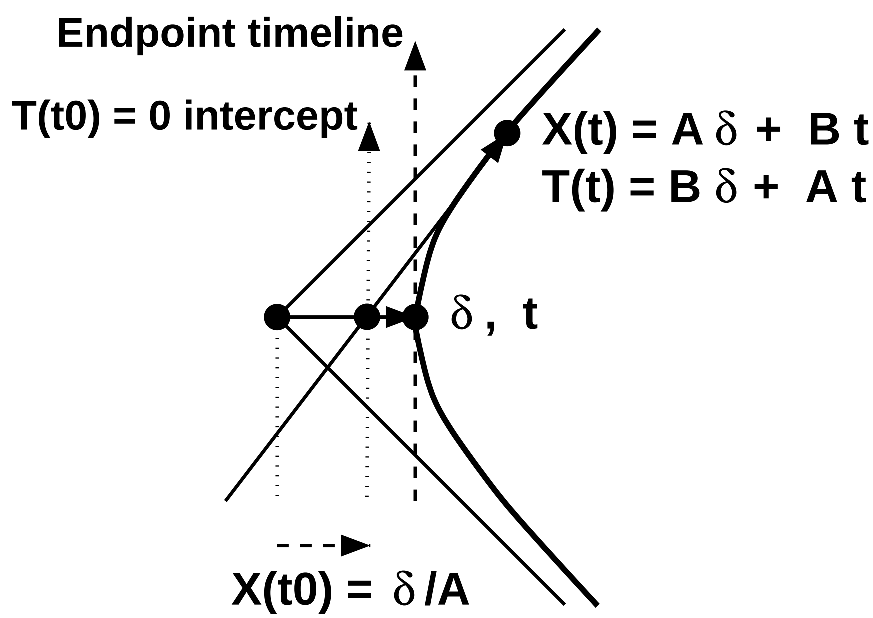
The numbers come out to be:

$$\begin{aligned} X(t_0) &= \delta \cosh \xi + t_0 \sinh \xi \\ &= \delta \cosh \xi - \delta \frac{\sinh^2 \xi}{\cosh \xi} \\ &= \frac{\delta}{\cosh \xi} \left(\cosh^2 \xi - \sinh^2 \xi \right) \\ &= \frac{\delta}{\cosh \xi} = \delta \sqrt{1 - v^2} \end{aligned}$$

Therefore the observed interval $X(t_0) -$ **origin** $= \delta\sqrt{1-v^2}$ is **Lorentz Contracted** in the moving frame relative to the rest frame interval δ.

Lorentz Contraction, contd.

We may visualize the Lorentz contraction as a backwards sliding of the intercept of the Lorentz transformed worldline, $X(t_0) = \delta / \cosh \xi = \delta\sqrt{1 - v^2}$:



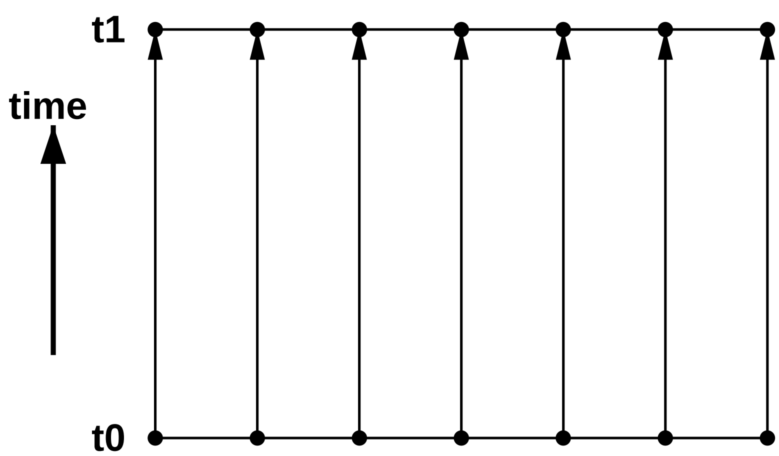
What is a solid object?

In 2D relativity, a solid object is a **line segment**.

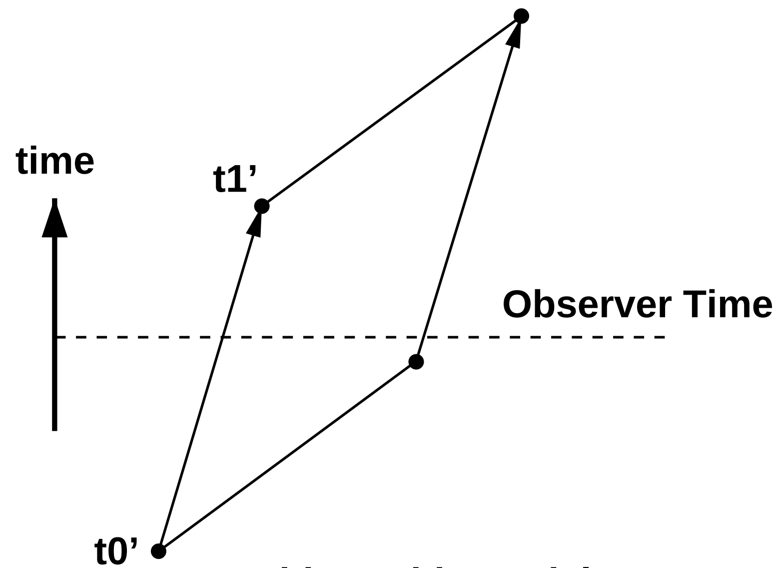
- Each end tracks **timelike world line.**
- Segment itself is **spacelike interval.**
- **Simultaneity** is tricky; after Lorentz transform, observer time cuts a skewed slice.

What is an object, contd

Watch the points — spacelike and unable to communicate sideways — as they each evolve on a timelike worldline.



Object with spatial extent



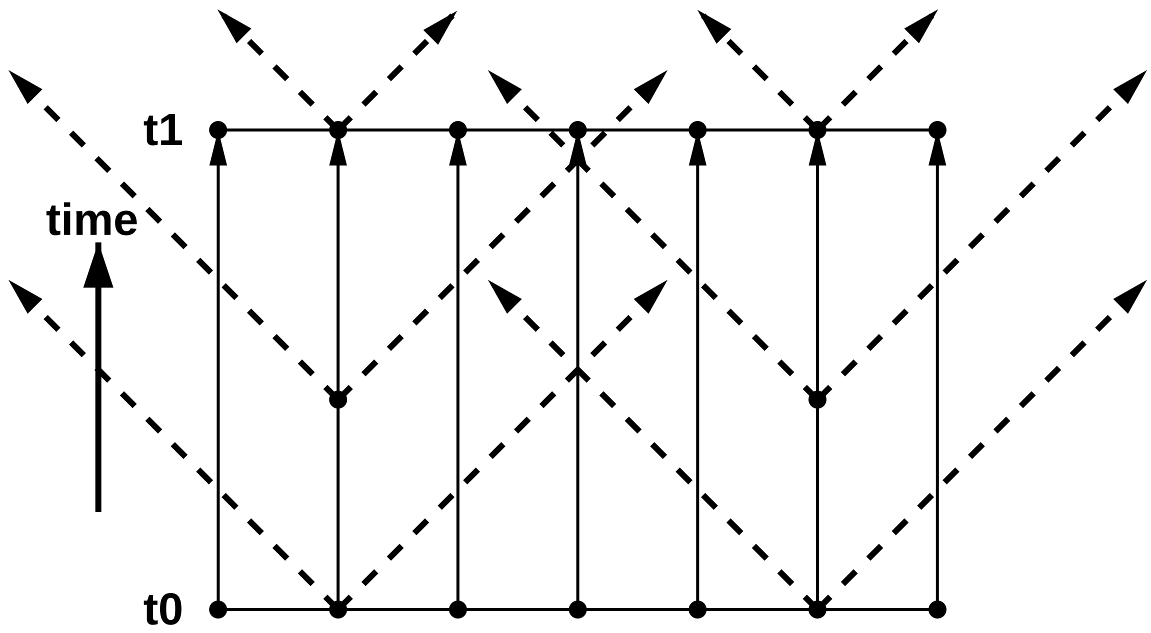
Object with spatial extent after Lorentz transform

After Lorentz transform, **Simultaneity** is modified.

What is an observation?

Observation of object is only possible via lightlike rays striking **CAMERA.**

These rays must strike observing camera's **world line** at SAME TIME!



Object with spatial extent

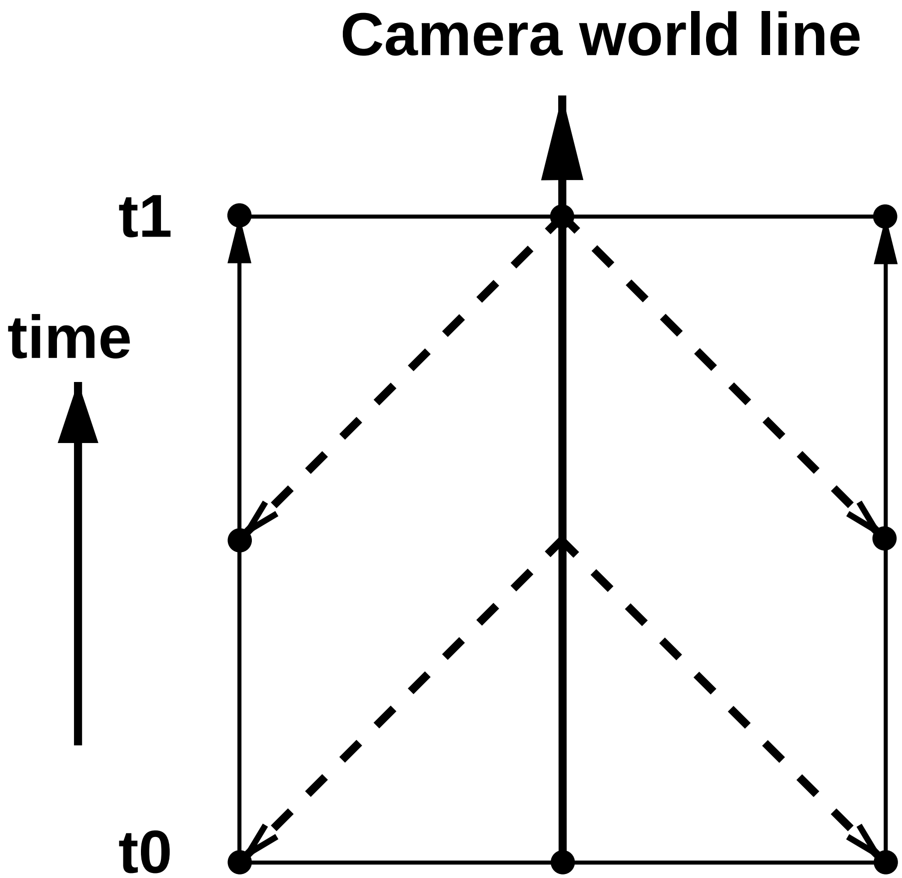
What is an observation, contd.

Since emitted rays must arrive simultaneously at camera on forward light cone to create a snapshot, we have an alternate method:

Shoot a light cone of rays **backwards** from camera

All relativistic pictures then come from time-reversed ray tracing:

What is an observation, contd



Object with spatial extent seen by camera rays at earlier times

Summary So Far:

- $\cos$ **to** $\cosh$ and $\sin$ **to** $\sinh$ make rotations change to Lorentz transformations.
- **Invariants** are inner products with minus sign.
- **Slope** $=$ $\tan$ **to Velocity** $=$ $\tanh$**:** helps visualize the meaning of the Lorentz parameters.
- **Objects:** spacelike intervals, endpoints track timelike worldlines, emitting lightlike signals.
- **Cameras:** construct images by back-tracing light rays to intersect object worldlines.

Visualizing Relativity

Part II: Visualization Methods for Special Relativity in 3D and 4D

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Part II: Visualization Methods for Special Relativity in 3D and 4D

- **2 Space + 1 Time:** Transformations.
- **Rolling the Relativistic Ball**:
  Thomas Precession
- **Aberration of Light**
- **Object Viewing:** Occlusion, IBR, Terrell
- **4D = 3 space + 1 time**

From 2D (1+1) to 3D (2+1)

We need at least **two space dimensions** to make interesting pictures. In 2 space + 1 time:

- **Objects** are polygons (at one time)
- Polygon **vertices** sweep out proper-time lines.
- Whole **spacetime object** is **tube-like.**
- **Cameras** see **cones** intersecting these tubes.

$\Longrightarrow$ $\boxed{\text{First, revisit transforms:}}$

2 + 1 Spacetime Boost Matrices

What happens to good old $\begin{bmatrix} \cosh & \sinh \\ \sinh & \cosh \end{bmatrix}$ in 2+1?

$B(\mathbf{v}) =$

$$\begin{bmatrix} 1 + {v_x}^2(\cosh\xi - 1) & v_x v_y(\cosh\xi - 1) & v_x \sinh\xi \\ v_x v_y(\cosh\xi - 1) & 1 + {v_y}^2(\cosh\xi - 1) & v_y \sinh\xi \\ v_x \sinh\xi & v_y \sinh\xi & \cosh\xi \end{bmatrix}$$

Note: $\hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = v_x v_x + v_y v_y = 1$ and we define velocity as $\boxed{\mathbf{v} = \hat{\mathbf{v}} \tanh\xi}$ (units: velocity of light = 1), and $\det B = 1$.

Pursue 3D space analogy:

Interesting things happen when you perform *sequences of rotations* in Euclidean 3D space:

$$R(\epsilon, \widehat{\mathbf{x}})R(\epsilon, \widehat{\mathbf{y}}) - R(\epsilon, \widehat{\mathbf{y}})R(\epsilon, \widehat{\mathbf{x}}) =$$

$$(\epsilon^2 + \mathcal{O}(\epsilon^3)) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This generates an infinitesimal Z**-axis rotation!**

3D space analogy:

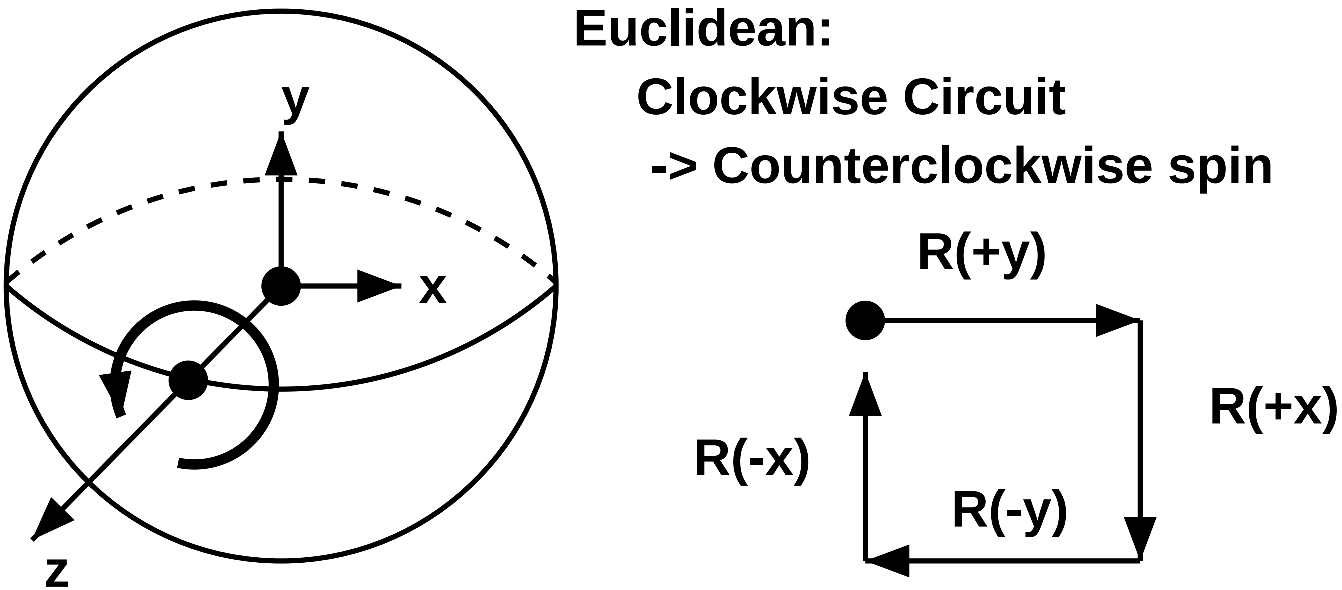
Sequences of rotations in Euclidean 3D space

counter-rotate:

Euclidean:

Clockwise Circuit

-> Counterclockwise spin



This is the **Rolling Ball** effect.

2 + 1 spacetime: properties

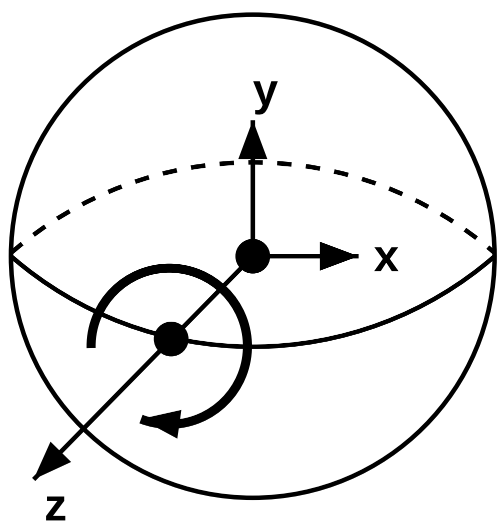
$\boxed{\textbf{Very Interesting}}$ things happen when you perform *sequences of Boosts* in 2 space + 1 time:

$$B(\widehat{\mathbf{x}})B(\widehat{\mathbf{y}}) - B(\widehat{\mathbf{y}})B(\widehat{\mathbf{x}}) \;=\; (\epsilon^2 + \mathcal{O}(\epsilon^3)) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

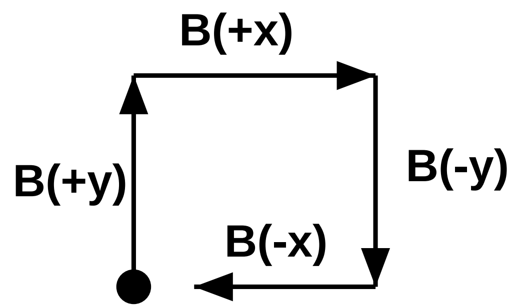
This is an infinitesimal **negative Z-axis rotation!**

2 + 1 spacetime: Thomas Precession

This observed $\boxed{\textbf{Spatial Rotation}}$ is the origin of *Thomas precession:* in 3D:



Minkowski:
Clockwise Circuit
-> Clockwise spin



This is a Relativistic Rolling Ball Effect.

Thomas Precession, contd.

Thomas Precession is the exact analog of the Euclidean 3D "Rolling Ball" effect.

This relativistic effect modifies magnetic coupling of atomic electrons in accelerated circular motion by causing an angular velocity

$$\omega = -(\cosh \xi - 1)\frac{v \times \dot{v}}{v^2} \approx -\frac{1}{2} v \times \dot{v}$$

to be applied to the rest frame of an orbiting electron.

. . . recall 3D Euclidean Quaternion Frames . . .

- **Quaternion Correspondence.** The unit quaternions q and $-q$ correspond to a single 3D rotation $R_3(q)$:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

- **Rotation Correspondence.**
  If $q = (\cos\frac{\theta}{2}, \hat{\mathbf{n}}\sin\frac{\theta}{2})$, with $\hat{\mathbf{n}}$ a unit 3-vector, $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$, then $R(\theta, \hat{\mathbf{n}})$ is usual 3D rotation by θ in the plane perpendicular to $\hat{\mathbf{n}}$.

2 + 1 spacetime quaternion-like form

In 2 space + 1 time, we can construct exactly the same type of quadratic form for the **boost**:

$$B(\mathbf{v}) \;=\; \begin{bmatrix} h_0^2+h_x^2-h_y^2 & 2h_xh_y & 2h_0h_x \\ 2h_xh_y & h_0^2+h_y^2-h_x^2 & 2h_0h_y \\ 2h_0h_x & 2h_0h_y & h_0^2+h_x^2+h_y^2 \end{bmatrix}.$$

If $\boxed{\mathbf{h}=(h_0,h_x,h_y)=(\cosh\xi/2,\hat{\mathbf{v}}\sinh\xi/2)}$
with $v=\sinh\xi/\cosh\xi$ and $|\hat{\mathbf{v}}|=1$, then this is exactly the standard 2+1 Lorentz transformation!

2 + 1 spacetime quaternion-like form

Caveat: Because of the Thomas Precession, even though $\mathbf{h} = (\cosh\xi/2, \hat{\mathbf{v}}\sinh\xi/2)$ generates $B(\mathbf{v})$, the full group of 2+1 transformations is not quite there, and the algebra is incomplete.

No time for details here, but the full treatment is straightforward using Clifford Algebra to generate $\mathbf{Spin}(2,1)$.

Features of Light in 2+1 Spacetime

Lorentz transforming a light ray can **change its direction.** Let

$$x' = x \cosh \xi + t \sinh \xi \qquad t' = x \sinh \xi + t \cosh \xi$$

Thus even if $x < 0$,

$$\boxed{x' > 0 \ \text{ if } \ t \sinh \xi > x \cosh \xi!}$$

Light in 2+1, contd

Let θ describe an isotropic distribution of light-like vectors with $(x, y, t) = (\cos\theta,\ \sin\theta,\ 1)$, and Boost with $\widehat{\mathbf{v}}$ in x direction:

$$
\begin{aligned}
x' &= \cos\theta\cosh\xi + \sinh\xi \\
y' &= \sin\theta \\
t' &= \cos\theta\sinh\xi + \cosh\xi
\end{aligned}
$$

Slice t in *observer frame*, so observed $\boxed{\tan\theta' = y'/x'.}$

Light Aberration: summary

Aberration Formulas we know and love:

After boosting to $v = \sinh / \cosh$ in units of $c = 1$, the isotropic light ray distribution

$(x, y, t) = (\cos\theta,\ \sin\theta,\ 1)$ deforms to:

$$\sin\theta' = \frac{\sin\theta}{(1 + v\cos\theta)\cosh\xi}$$

$$\cos\theta' = \frac{v + \cos\theta}{1 + v\cos\theta}$$

$$\tan\theta' = \frac{\sin\theta}{(v + \cos\theta)\cosh\xi}$$

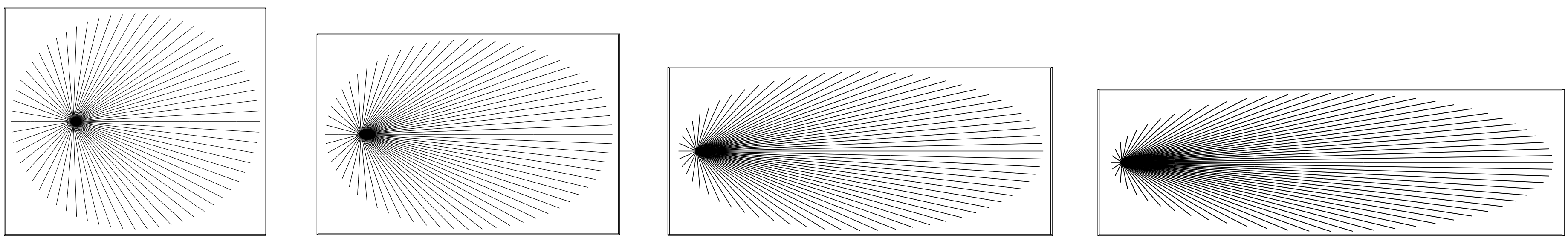
Light in 2+1, contd

Observations on relativistic light distortion:

- $\tan\theta' \propto 1/\cosh\xi = \sqrt{1-v^2}$.
- So, as $v = \sinh/\cosh \to 1 \ldots$
- …the $\boxed{\textbf{aberration of light}}$ (resembling a **search-light**) swings all the rays to the **forward direction!**

Visualizing aberration: light cones

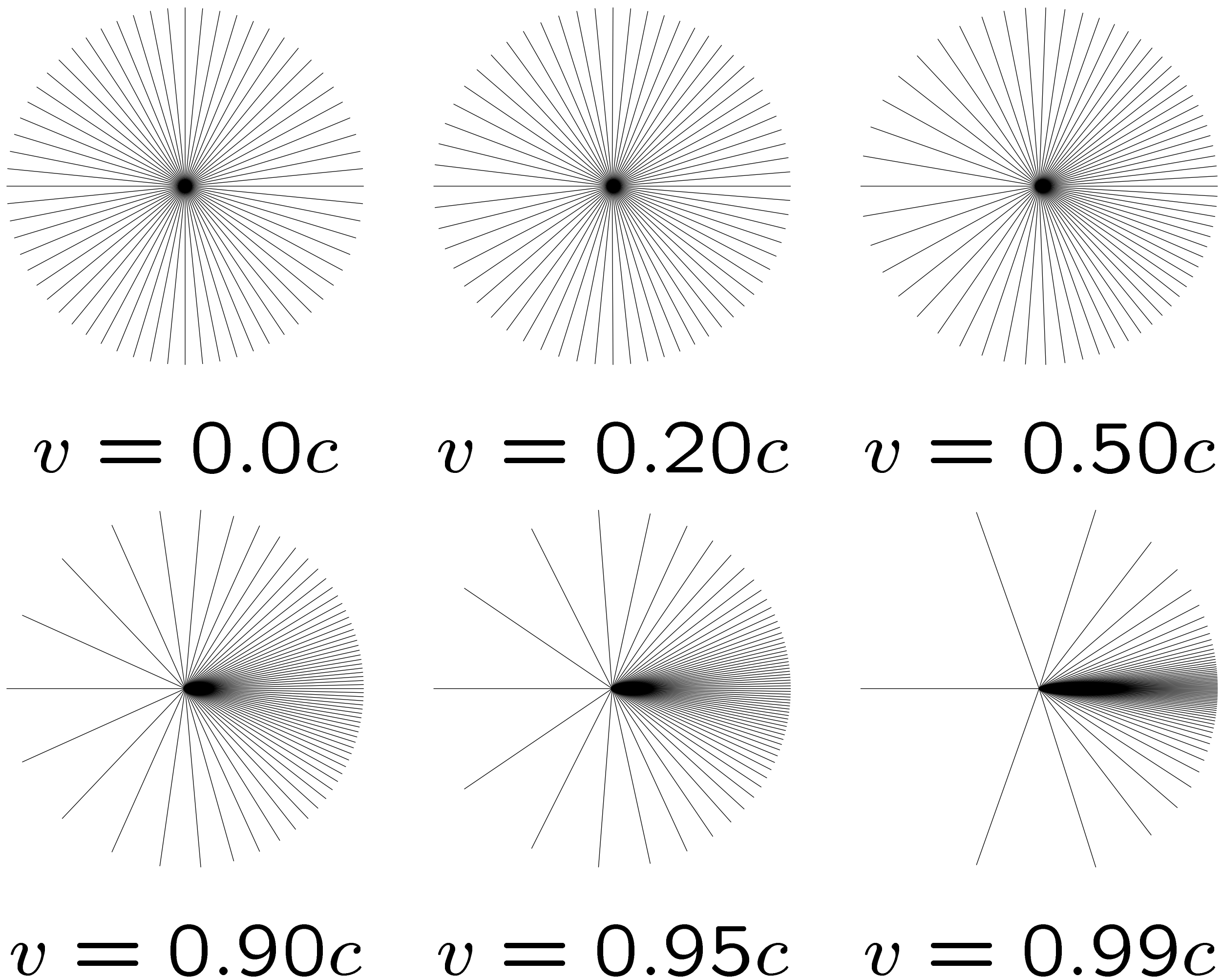
Looking down on boosted spacetime cones representing symmetric Light Ray distributions:



$v = 0.5c$ $v = 0.75c$ $v = 0.9c$ $v = 0.95c$

Visualizing aberration: circular distrib.

Looking down on boosted 2D symmetric Light Ray distributions:

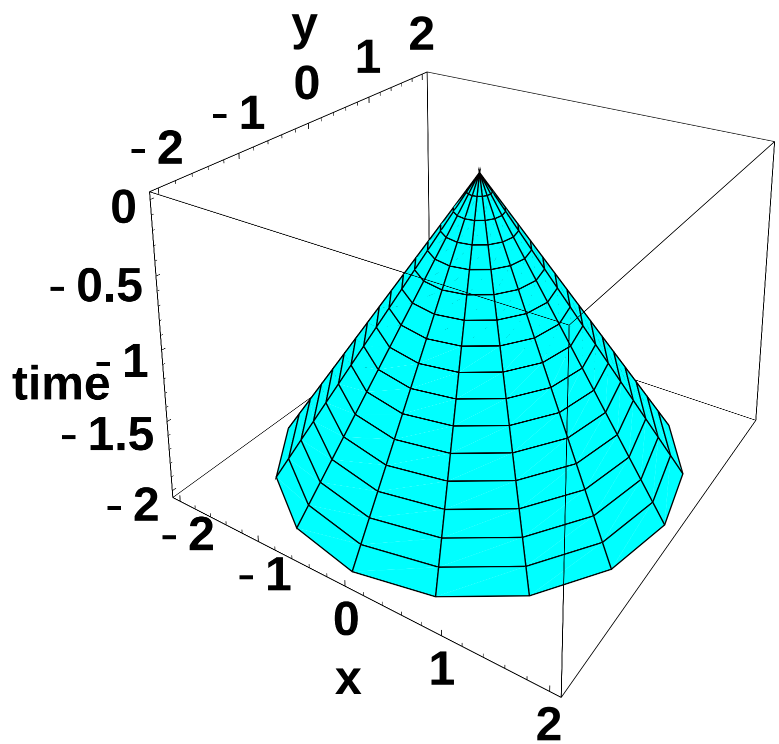


Seeing 2+1 Spacetime

- Points: Still World Lines tracing Proper Time
- Objects: Segments (slabs) $\Rightarrow$ Polygons (tubes)
- Light: Diagonals $\Rightarrow$ Cones
- Images/Cameras: Trace inverse Cones
- Transformations: Completely new features, analogous to 3D rotations

2 + 1 Spacetime Image Construction

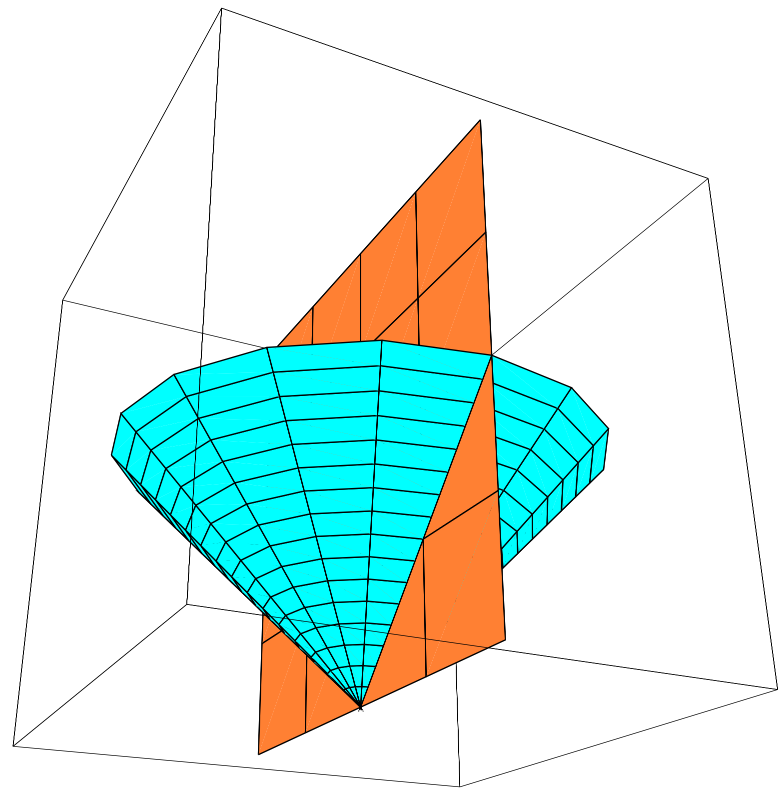
At one instant, camera receives back-traced light from a single inverted cone in 2+1 spacetime:



TIME advances UP to zero at the apex, the camera focal point.

2 + 1 Spacetime Object Viewing

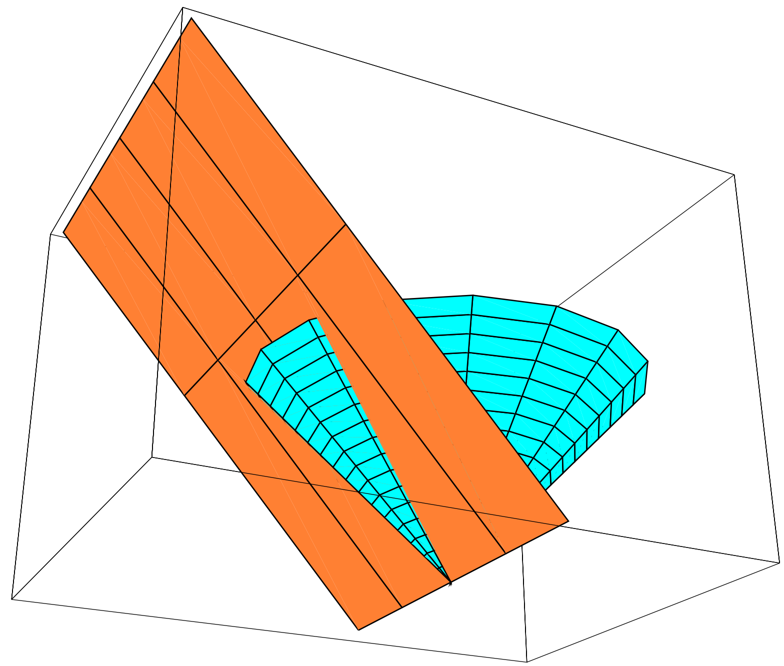
How front and back of polygon side emit light towards camera:



Now vary velocity . . .

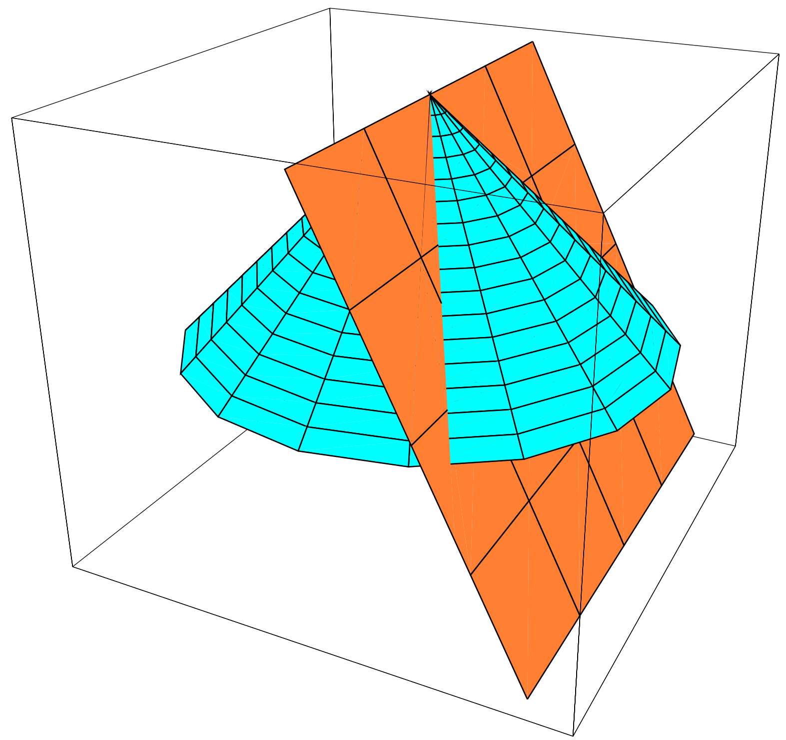
2 + 1 spacetime object viewing

When velocity is **0.90** times the speed of light, light escapes from back side in a **almost a full circle:**



2 + 1 Spacetime Object Viewing

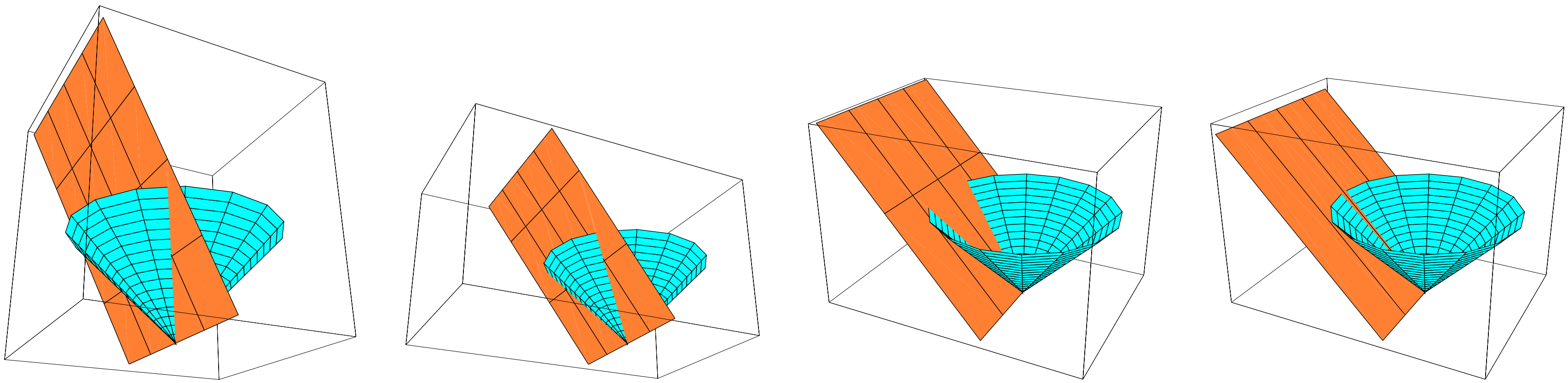
How is light from a moving slab distributed to the camera?



Light cone is invariant but world-sheet of a polygon tilts: visibility of front and back sides varies drastically with velocity.

2 + 1 Spacetime Object Viewing

How face's light distribution changes with velocity:



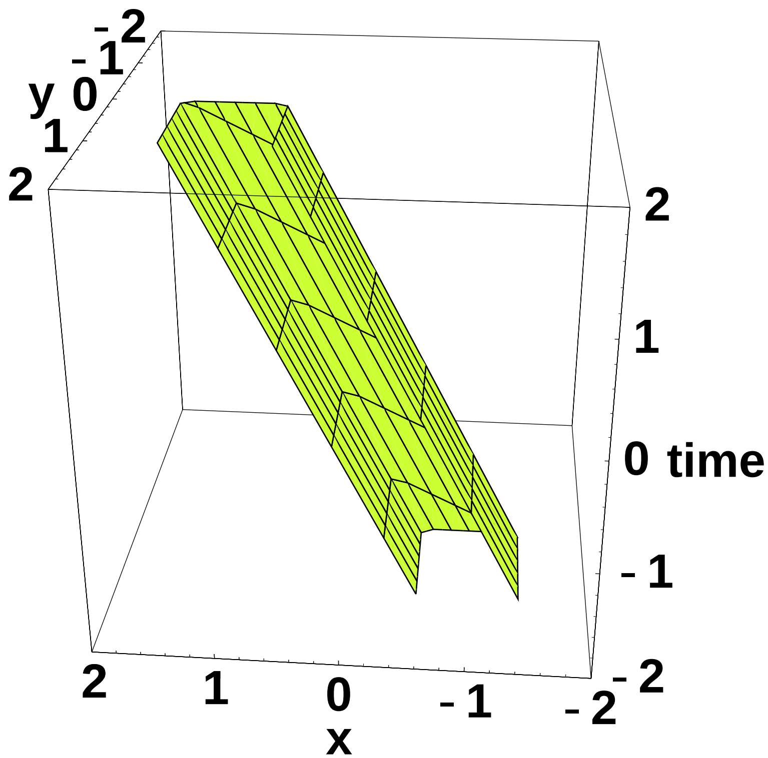
$v = 0.50$ $v = 0.75$ $v = 0.90$ $v = 0.99$

The front side is visible only under more and more restricted conditions.

The back side becomes visible from practically EVERYWHERE as $v \to 1$!

2 + 1 spacetime object viewing

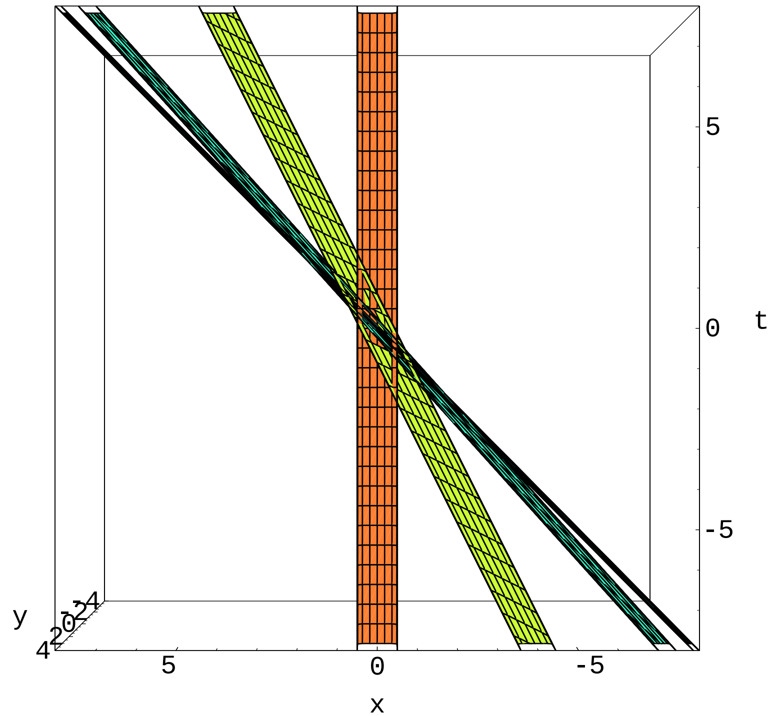
Simple model: square in 2+1 spacetime: with one side removed so we can see inside:



Here, velocity $v = 0.50$ times the speed of light.

2 + 1 spacetime object viewing

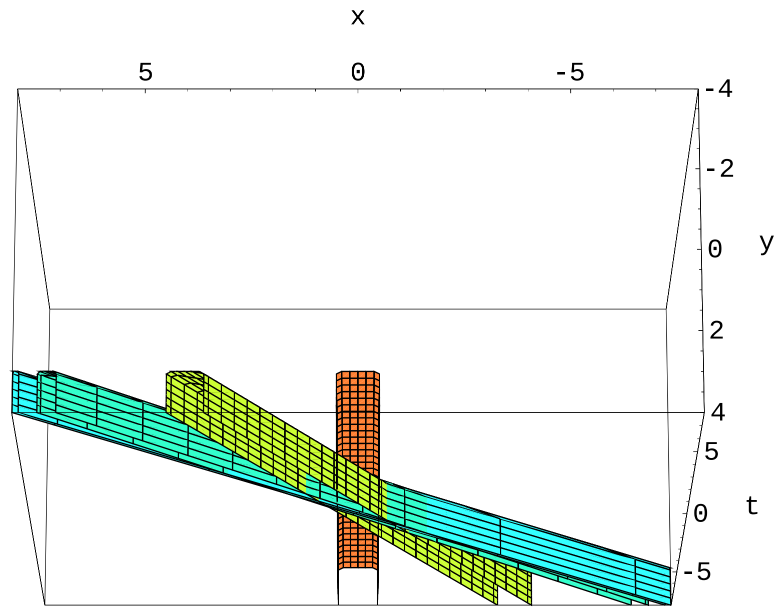
Simple model: square in 2+1 spacetime: with one side removed so we can see inside:



Velocities: **0.00, 0.50, 0.90, 0.99** times the speed of light.
Note Lorentz Contraction.

2 + 1 spacetime object viewing

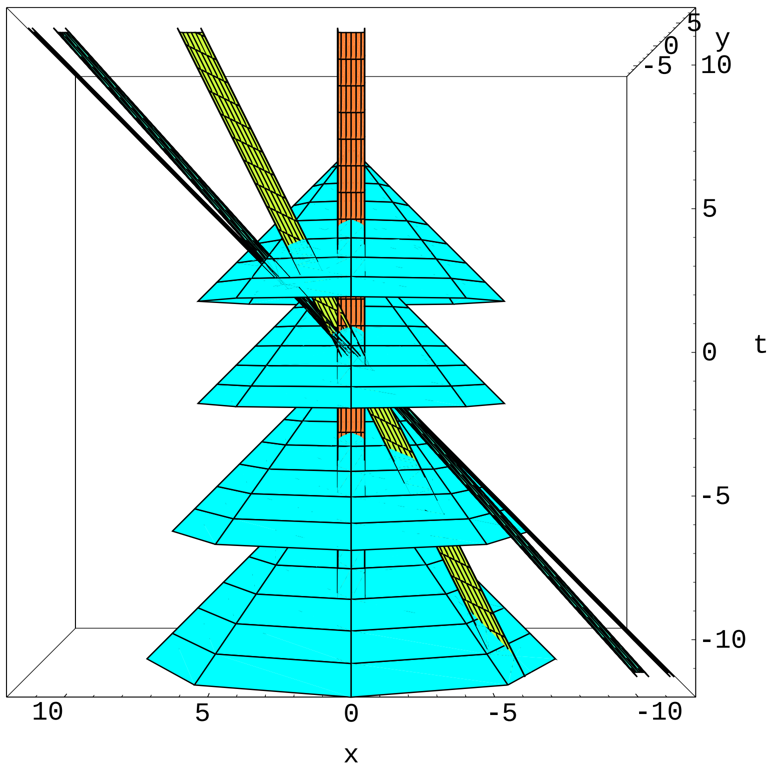
Looking down from the camera's spacetime viewpoint:



Velocities: **0.00, 0.50, 0.90, 0.99** times the speed of light.

2 + 1 spacetime object viewing

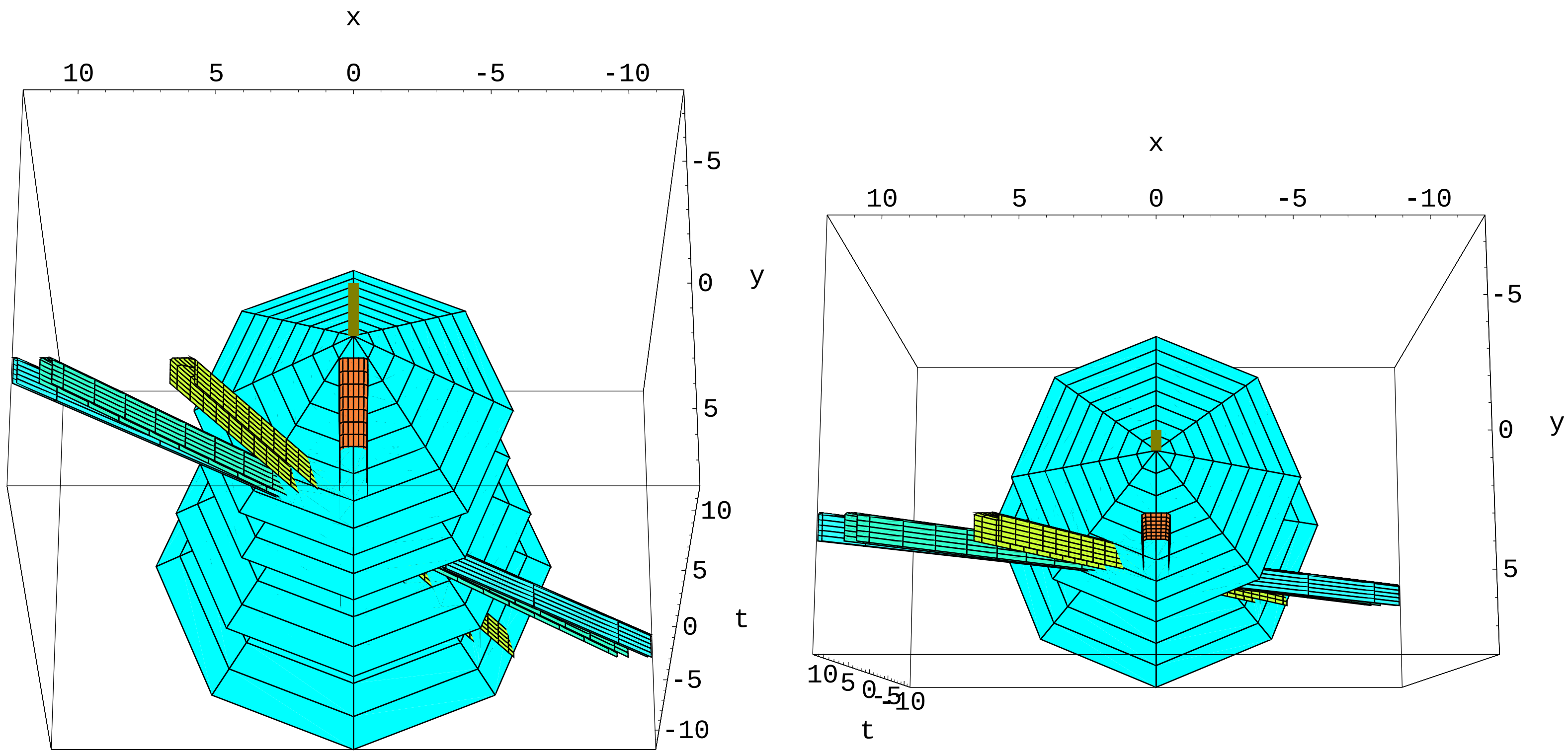
Add a stationary camera: at each time step, the camera sees what the cone intersects:



Velocities: **0.00, 0.50, 0.90, 0.99** times the speed of light.

2 + 1 spacetime object viewing

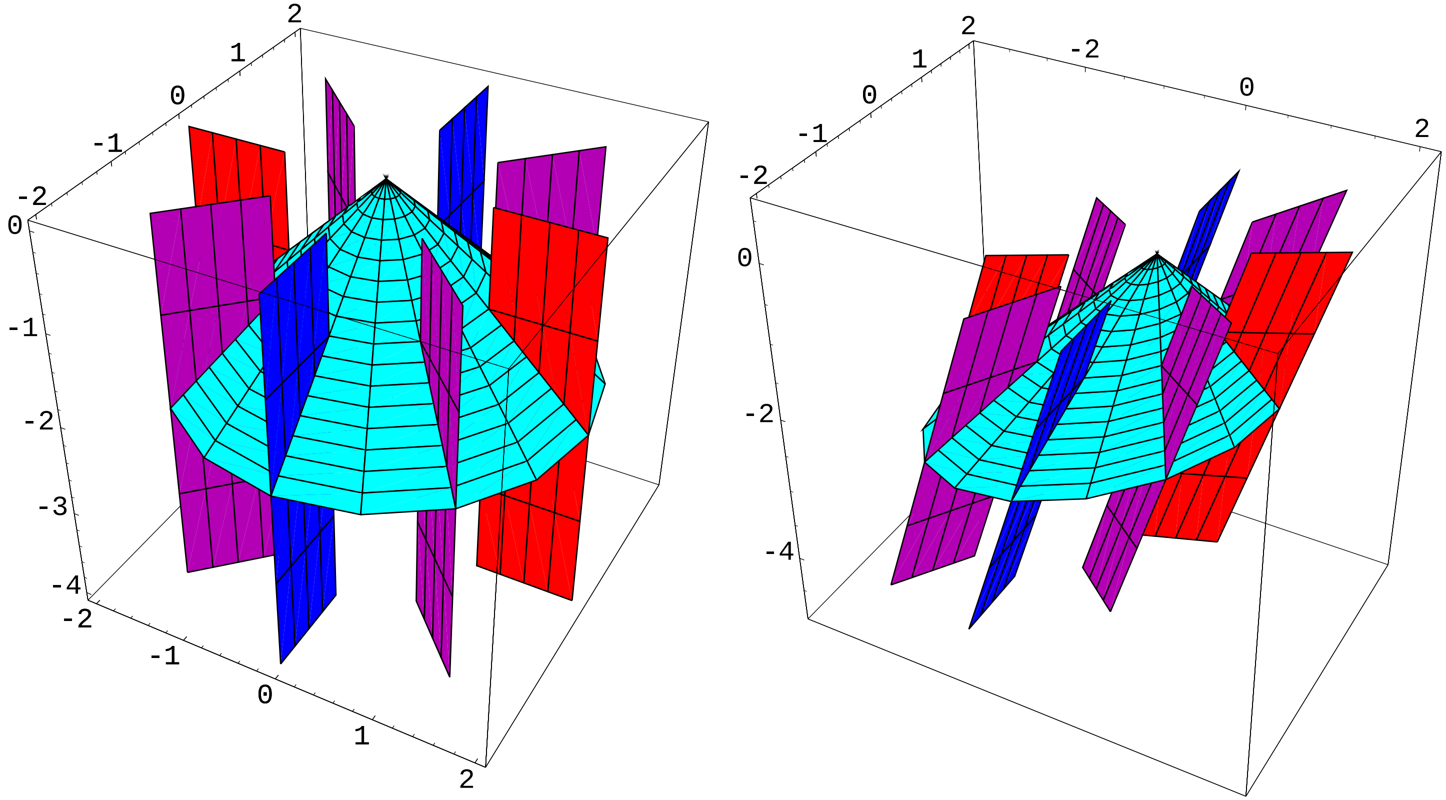
Stationary camera, looking down on the camera's spacetime viewpoint:



Velocities: **0.00, 0.50, 0.90, 0.99** times the speed of light.

Occlusion in Relativistic Scenes

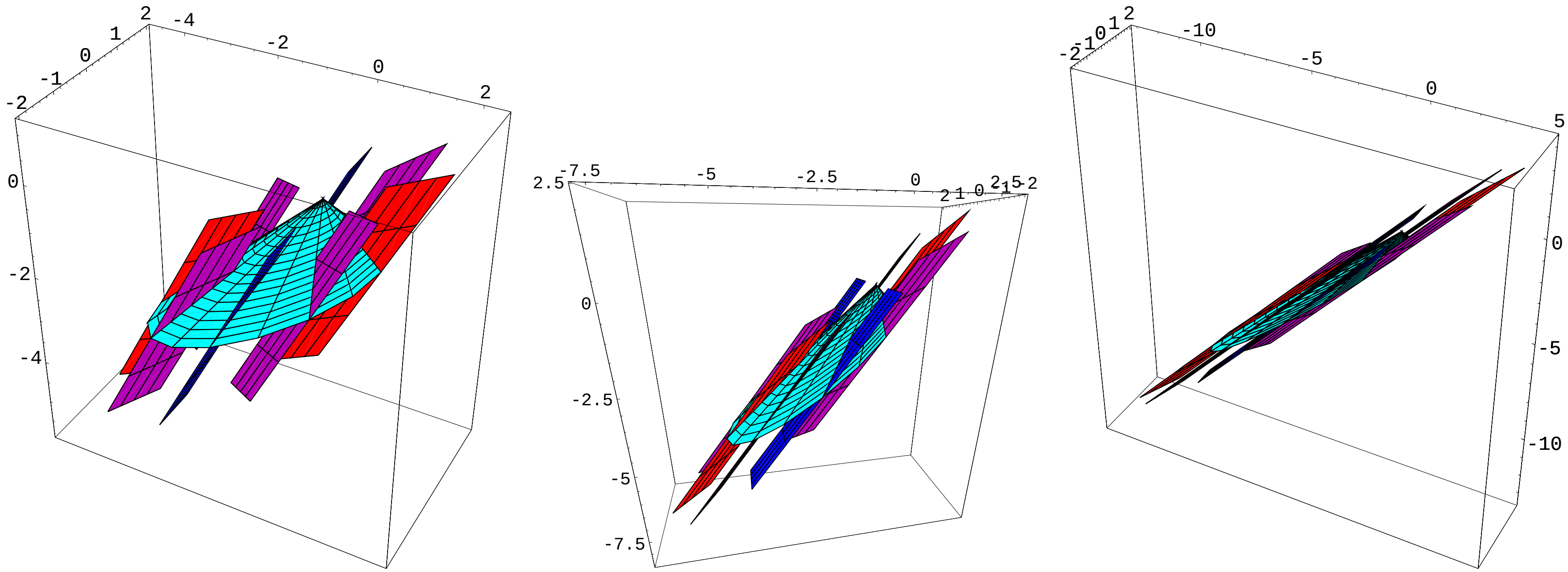
Study **occlusion** using polygons **aligned with camera rays:**



Observe: Once an occlusion edge, ALWAYS an occlusion edge!

2 + 1 occlusion, contd

Even at extreme velocities, occluding edges persist, so boosts will never add face material to a static scene.



Velocity: **0.50, 0.75, 0.90** times the speed of light.

Static Scenes and Image-Based Rendering

As long as a scene is STATIC, you can take the light distribution in any frame, and use that to make a relativistically distorted scene.

THIS IS THE BASIS OF RELATIVISTIC IMAGE-BASED RENDERING! (See later in Weiskopf lectures).

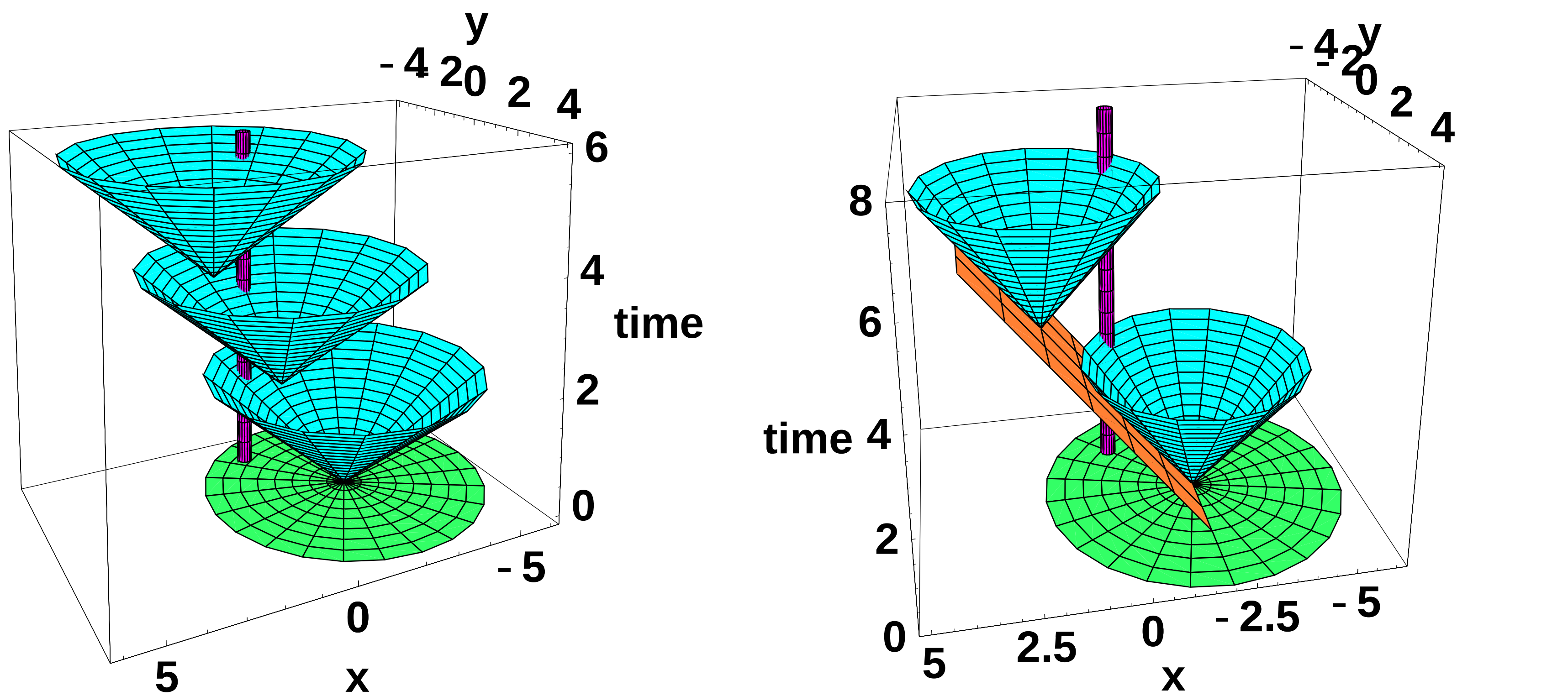
- The angles and frequencies may change, but the geometric transformations conspire to keep all invisible polygon faces perpetually invisible.

2 + 1 Moving Scenes and the Terrell Effect

In **moving scenes**, the **delay** of light rays reaching us from a rapidly moving object causes bizarre effects

$\boxed{\textbf{Only the back side}}$ of a cube moving towards us at $v \approx 1$ is seen under normal conditions.

Moving Scenes and the Terrell Effect



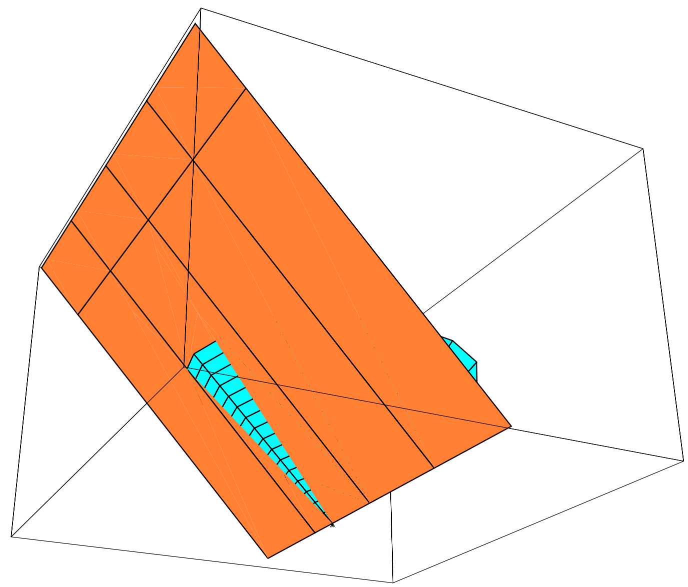
Tube: camera world line.
Disk: ∞ light velocity would make **FRONT** visible.
Cones: finite light velocity shows only BACK.

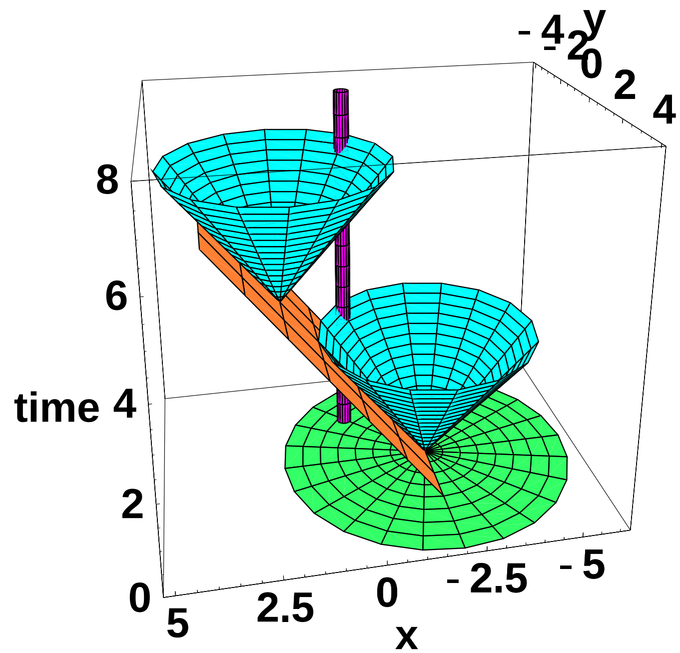
2 + 1 Moving Scenes and the Terrell Effect

This effect went virtually unnoticed until Terrell (1959) pointed it out. Intuitively, it arises as follows:

- As $v \Rightarrow 1$, aberration reduces the visibility of front edge to a **single ray**.
- Simultaneously, back edge becomes visible *at some time* to any camera in the world.

2 + 1 Moving Scenes and the Terrell Effect





Front only visible along single ray for finite light velocity.

Would be visible everywhere in a half-plane with infinitelight velocity!

3 Space + 1 Time: The Real World!

Goal so far: build intuition in 1+1 and 2+1 dimensions of spacetime. Now do 3 Space and 1 Time:

- **Transformations:** SIX Parameters: 3 boosts $(\mathbf{v})$, 3 Euler angles $(\theta, \hat{\mathbf{n}})$. Most significant features occurred already in 2+1.
- **Aberration:** Same form, spun about boost axis.

3 Space + 1 Time: The Real World!

- **Imaging:** Still the light cone, but now harder to draw; think of as a growing sphere surrounding light source.

- **IBR, Terrell effect, etc:** All just about the same as in 2 space + 1 time, only **objects are like swept spheres** instead of tubes = swept circles.

3 + 1 spacetime Full Boost

In real-world spacetime, a Lorentz transform with velocity $\mathbf{v} = \hat{\mathbf{v}}(\sinh\xi / \cosh\xi)$ becomes:

$$B(\mathbf{v}) = \begin{bmatrix} 1 + v_x^2\,\mathrm{C} & v_x v_y\,\mathrm{C} & v_x v_z\,\mathrm{C} & v_x \sinh\xi \\ v_x v_y\,\mathrm{C} & 1 + v_y^2\,\mathrm{C} & v_y v_z \mathrm{C} & v_y \sinh\xi \\ v_x v_z\,\mathrm{C} & v_y v_z\,\mathrm{C} & 1 + v_z^2\,\mathrm{C} & v_z \sinh\xi \\ v_x \sinh\xi & v_y \sinh\xi & v_z \sinh\xi & \cosh\xi \end{bmatrix}$$

where $\mathrm{C} = (\cosh\xi - 1)$. Here $\det[B] = 1$ and $B(\mathbf{v})$ leaves the matrix $\mathrm{diag}(1, 1, 1, -1)$ invariant.

3 + 1 spacetime quaternion-like form

Defining $D_x = h_0^2 + h_x^2 - h_y^2 - h_z^2$, cyclic, 4D boosts acquire a quaternion-like form:

$$B(\mathbf{v}) =$$

$$\begin{bmatrix} D_x & 2h_xh_y & 2h_xh_z & 2h_0h_x \\ 2h_xh_y & D_y & 2h_yh_z & 2h_0h_y \\ 2h_xh_z & 2h_yh_z & D_z & 2h_0h_z \\ 2h_0h_x & 2h_0h_y & 2h_0h_z & h_0^2 + h_x^2 + h_y^2 + h_z^2 \end{bmatrix}$$

where $\mathbf{h} = (h_0, h_x, h_y, h_z) = (\cosh\xi/2, \hat{\mathbf{v}}\sinh\xi/2)$ with $|\hat{\mathbf{v}}| = 1$ generates a standard Lorentz transformation!
Note: $\det[B] = (\cosh^2 - \sinh^2)^4 \equiv 1$.

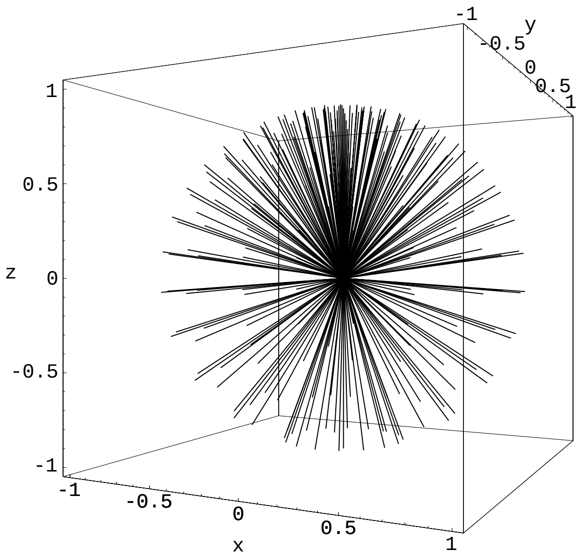
3 + 1 spacetime quaternion-like form

Caveat: Even though $\mathbf{h} = (\cosh\xi/2, \hat{\mathbf{v}}\sinh\xi/2)$ generates $B(\mathbf{v})$, this is also incomplete, since rotations (e.g., Thomas precession) must be merged in with boosts in the full theory of 3+1 spacetime.

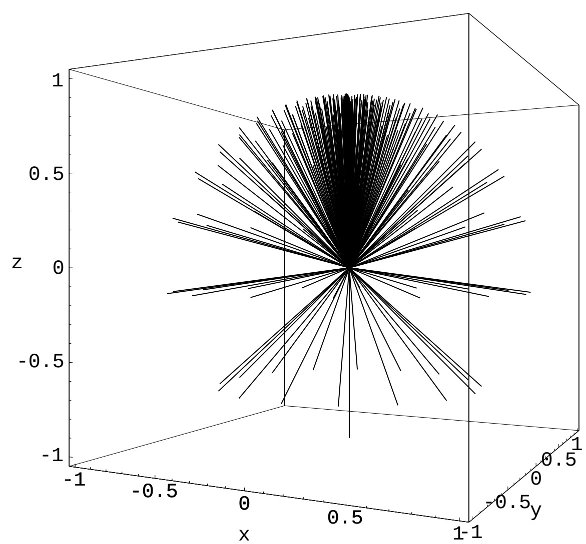
Footnote: The full group $\mathbf{SO}(3,1)$ has a quadratic form corresponding to its "double covering group." This group is directly derivable from Clifford algebra methods, and is written $\mathbf{Spin}(3,1)$. It corresponds to the six parameter group of complex 2×2 matrices $\mathbf{SL}(2,\mathbf{C})$, and eventually leads to the Dirac Equation for the relativistic spin 1/2 electron.

Seeing 3+1 Spacetime

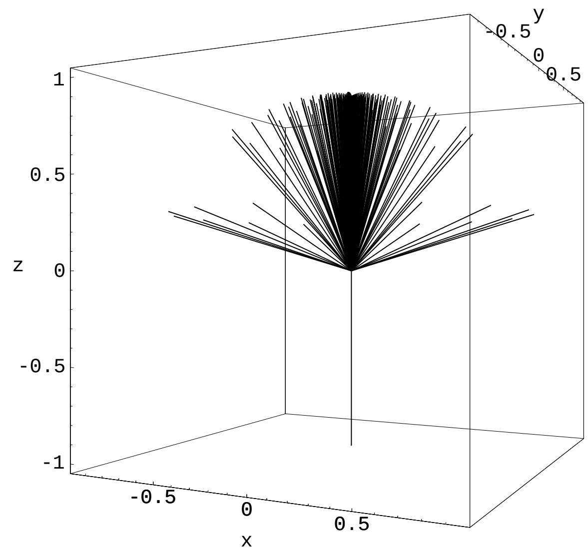
3D **spatial** light ray distributions for a symmetric source are very similar to the 2D spatial distributions:



$v = 0.5c$



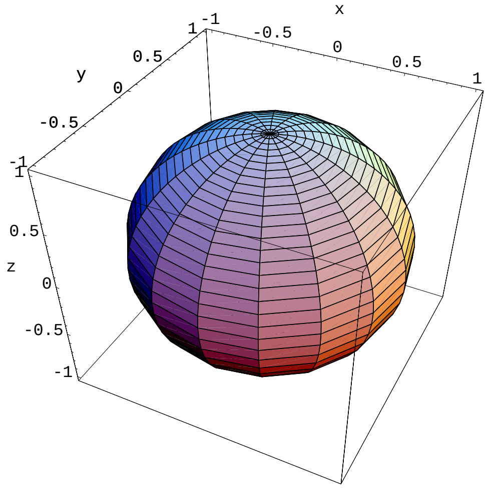
$v = 0.90c$



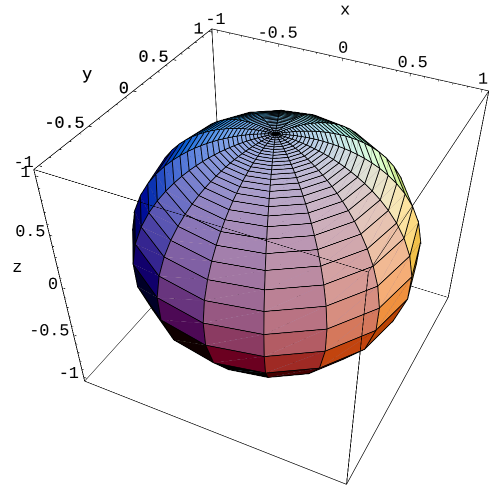
$v = 0.99c$

Seeing 3+1 Spacetime

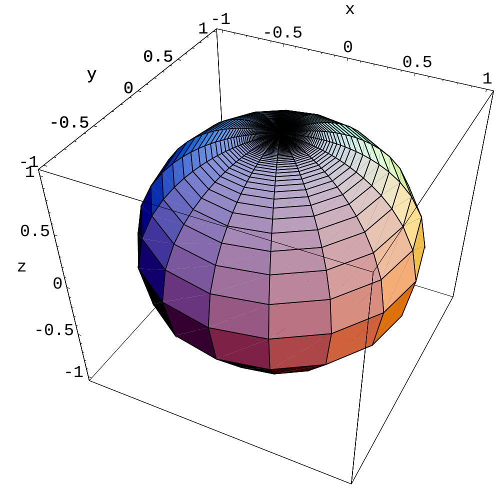
Alternative Visualization: Solid sphere plot of 3D space light ray distributions for symmetric source:



$v = 0.5c$



$v = 0.90c$



$v = 0.99c$

Texture Maps on these distorted spheres provide an implementation of **Relativistic IBR**.

Summary of 3+1 effects:

- $B(\mathbf{v})$ is an orthogonal 4×4 matrix, mostly $\cosh$'s and $\sinh$'s as usual!
- Quaternion-like forms exist, rigorously corresponding to the representations and algebra of $\mathbf{SL}(2, \mathbf{C})$.
- Occlusion invariance and light aberration allow relativistic IBR to be implemented.
- Objects are made up of vertices tracing world lines, linked into edges, polygons, and polyhedra.
- Camera images can be formed by tracing light rays backward in time on negative light cone until they hit scene objects.

Intuition Overview

- **Orthogonal Matrices:** Did you understand that $\cos$, $\sin$ matrices leave dot products unchanged?

  If so, NOW you understand that $\cosh$, $\sinh$ matrices leave proper-time dot products unchanged!

- **Rigidity:** Did you understand that 3D rotations change 2D length of projected components, yet radius is constant?

  If so, NOW you understand that Lorentz matrices change $(\mathbf{x}, t)$ coordinate components, yet proper-lengths are unchanged!

Intuition Overview, contd.

- **Non-Commuting Matrices:** Did you understand that x, y 3D rotation matrices generate *extra z-spin*?

  If so, NOW you understand that circular Lorentz transformations generate *Thomas Precession.*

- **Relativistic IBR Theorem:** Did you understand that occlusion of light rays by polygons is *relativistically invariant* due to invariance of dot product?

  If so, NOW you understand how relativistic IBR is possible with **real world image sources.**

Transition:

- **Algebraic thinking** was the focus of the course so far, learning to understand **behavior of light, geometry, and matter** under relativistic conditions.
- **Rendering Virtual Relativistic Reality** will be demonstrated in the final part of the course.
- Together, the two techniques combine to let you **SEE and UNDERSTAND** how Relativity works.

Time for a 15 minute break!