Visualizing Quaternions

Course Notes for SIGGRAPH 2005

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Abstract

This intermediate-level tutorial provides a comprehensive approach to the visualization of quaternions and their relationships to computer graphics and scientific visualization. The introduction focuses on a selection of everyday phenomena involving rotating objects whose explanation for an audience that is technically trained but not pure mathematicians is essentially impossible without a quaternion visualization. The course will then pursue selected examples of quaternion-based visualization methods to help explain the behavior of *quaternion manifolds*: quaternion representations of orientation frames attached to curves, surfaces, and volumes.

Presenter's Biography

Andrew J. Hanson is a professor of computer science at Indiana University, and has regularly taught courses in computer graphics, computer vision, and scientific visualization. He received a BA in chemistry and physics from Harvard College in 1966 and a PhD in theoretical physics from MIT in 1971. Before coming to Indiana University, he did research in theoretical physics at the Institute for Advanced Study, Stanford, and Berkeley, and then in computer vision at the SRI Artificial Intelligence Center in Menlo Park, CA. He has published in IEEE Computer, CG&A, TVCG, ACM Computing Surveys, and has over a dozen papers in the IEEE Visualization Proceedings. He has also contributed three articles to the Graphics Gems series dealing with user interfaces for rotations and with techniques of N-dimensional geometry. Previous experience with conference tutorials includes a Siggraph '98 tutorial on N-dimensional graphics, a Visualization '98 course on Clifford Algebras and Quaternions, and tutorials on Visualizing Quaternions presented at Siggraph '99, Siggraph 2000, and again at Siggraph 2001 in tandem with a course on Visualizing Relativity for a graphics audience. Major research interests include scientific visualization, machine vision, computer graphics, perception, and the design of interactive user interfaces for virtual reality and visualization applications. Particular visualization applications currently being studied include an astrophysical treatment of the local galactic neighborhood of the sun, the exploitation of constrained navigation for visualization environments. Mathematical visualization interests include the depiction of Calabi-Yau spaces and the general problems of graphics and visualization in dimensions greater than three and their applications to mathematics and theoretical physics.

Contents

General Information on the Tutorial

Course Syllabus

Summary: This tutorial will deal with visualizable representations of quaternions, their features, technology, folklore, and applications. The introduction will focus on visually understanding quaternions themselves by exploiting parallels to complex variables and 2D rotations. Starting from this basis, the tutorial will proceed to give visualizations of advanced quaternion applications.

Prerequisites: Participants should be comfortable with and have an appreciation for conventional mathematical methods of 3D computer graphics and geometry used in geometric transformations and polygon rendering. The material will be of most interest to those wishing to deepen their intuitive understanding of moving coordinate frames and quaternion-based animation techniques.

Objectives: Participants will learn the basic facts relating quaternions to ordinary 3D rotations, as well as methods for examining the properties of quaternion constructions using interactive visualization methods. A variety of applications, including the use of quaternions to smaples coordinate frames for curves, surfaces, and volumes, will be explored.

Outline: This tutorial will last approximately one and a half hours plus time for questions and discussion. The material will be arranged as follows:

I. *(45 min)* **Twisting Belts, Rolling Balls, and Locking Gimbals:** *Explaining Rotation Sequences with Quaternions*

Sequences of orientations are manifestly evident in our everyday lives. While we can immediately observe strange things that happen when we twist a leather belt, roll a baseball, or push on a gyroscope, if you ask "why" and expect a real explanation, most of us hit a dead end. Quaternion visualization provides satisfying answers to such questions.

II. *(45 min)* **Quaternion Fields:** *Curves, Surfaces, and Volumes*

Once we have mastered the visualization of quaternion paths, we have the tools to take a fresh look at many problems in graphics and visualization. The quaternion *field* is a continuous map from a set of orientation frames such as framed curves, surfaces, and volumes into the corresponding quaternions. We examine a family of examples showing how quaternion curves, surfaces, and volumes can solve new problems and reveal new properties.

1 Overview

Practitioners of computer graphics and animation frequently represent 3D rotations using the quaternion formalism, a mathematical tool that originated with William Rowan Hamilton in the 19th century, and is now an essential part of modern analysis, group theory, differential geometry, and even quantum physics. Quaternions are in many ways very simple, and yet there are enormous subtleties to address in the process of fully understanding and exploiting their properties. The purpose of this Tutorial is to construct an intuitive bridge between our intuitions about 2D and 3D rotations and the quaternion representation.

The Tutorial will begin with an introduction to various natural phenomena that can be understood using quaternions. Rotations in 2D, which will be found to have surprising richness, will lead the way to the construction of the relation between 3D rotations and quaternions. Quaternion visualization methods of various sorts will be introduced, followed by applications of the quaternion frame representation to problems of interest by graphicists and visualization scientists. An extensive bibliography of related literature is included, as well as several relevant reprints and technical reports, a Mathematica implementation of the Quaternion Frenet Equations, and a basic GLUT quaternion visualization application.

2 Twisting Belts, Rolling Balls, and Locking Gimbals

We will begin with a basic introduction to the ways in which sequences of rotations enter our lives in surprising ways. We will then proceed to look at a variety of methods for understanding quaternions and making meaningful pictures of constructs involving them. These methods will range from some of the concepts pointed out by Hart, Francis, and Kauffman [54] to theoretical methods given in [47, 48, 40, 51].

Traditional treatments of quaternions range from the original works of Hamilton and Tait [35, 85] to a variety of recent studies such as those of Altmann, Pletincks, Juttler, and Kuipers [2, 73, 63, 67].

In our pedagogical treatment, we will focus on the use of 2D rotations as a rich but algebraically simple proving ground in which we can see many of the key features of quaternion geometry in a very manageable context. The relationship between 3D rotations and quaternions is then introduced as a natural extension of the 2D systems. Quaternion visualization itself utilizes a basic trick: since a four-vector quaternion $q = (q_0, \mathbf{q})$ obeying $q \cdot q = 1$, then the four-vector lies on the three-sphere S^3 and has only three independent components: if we display just $\mathbf{q}$, we can in principle *infer* the value of $q_0 = \sqrt{1 - \mathbf{q} \cdot \mathbf{q}}$.

3 Quaternion Fields

After the conceptual introduction, we proceed to study the nature of quaternions as representations of frames in 3D. The now-traditional application of quaternion animation splines was introduced to the graphics community originally by Shoemake [77]. Our visualizations of these and other applications exploit the fact that quaternions are points on the three-sphere embedded in 4D; the three-sphere (S^3) is analogous to an ordinary ball or two-sphere (S^2) embedded in 3D, except that the three-sphere is a solid object instead of a surface. To manipulate, display, and visualize rotations in 3D, we may convert 3D rotations to 4D quaternion points and treat the entire problem in the framework of 4D geometry.

We pursue three main applications, which involve the identification of quaternion frames with sampled curves, surfaces, and volumes. The curve methods follow closely techniques introduced in Hanson and Ma [47, 48] for representing families of coordinate frames on curves in 3D as curves in the 4D quaternion space. The extension to surfaces and the corresponding induced surfaces in quaternion space follow the treatment by Hanson [40, 51], and volumetric quaternions are studied using the methods of Herda, et al. [56, 57, 55].

4 Demonstration Software

We provide an elementary OpenGL-based interactive quaternion visualization application, *QuatRot*, that should be essentially system-independent and run on any platform. In addition, we supply our own version, `quatutils.nb`, of some basic Mathematica routines for quaternions (which serve as the basis for a number of the illustrations in the notes), as well as a Mathematica notebook `qfrmint.nb` that explicitly implements a numerical integration of the Frenet frame equations in quaternion form, vastly improving the exactly equivalent calculation for the standard Frenet equations implemented by Gray [33].

Acknowledgments

Portions of the course notes are adapted from the book, *Visualizing Quaternions*, by Andrew J. Hanson, published by Morgan Kaufmann Publishers, Copyright 2006 by Elsevier Inc. We thank the publisher for permission to use this material; all rights not explicitly assigned to Siggraph for the purpose of providing these course notes are reserved.

Republished in the Course Notes are two key papers from IEEE Transactions on Visualization and Computer Graphics [48], and from the Proceedings of IEEE Visualization [40]; we thank the IEEE Computer Society Press for permitting us to include this material.

Finally we would like to thank the National Science Foundation for their support: the research incorporated in portions of this work was supported in part by NSF grant CCR-0204112.

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