Quaternion Applications

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Siggraph Asia 2012 Tutorial

OUTLINE

I: (55 min) **Introduction to Quaternions:**

What are they good for?

Understanding Rotation Sequences!

II a: (15 min) **Quaternion Tubing:**

Visualizing Framed Space Curves

...OUTLINE...

II b: (15 min) **Quaternion Protein Maps:**

Amino Acid Frame Sequences with Quaternions

II c: (20 min) **Intro to Dual Quaternions:**

Applications to Six-Degrees-of-Freedom

Part I

Introduction to Quaternions:

...Twisting Belts and Rolling Balls...

Explaining Rotation Sequences with Quaternions

Where Did Quaternions Come From?

…from the discovery of *Complex Numbers:*

- $\boxed{z = x + iy}$ Complex numbers = realization that $z^2 + 1 = 0$ cannot be solved unless you have an "imaginary" number with $i^2 = -1$.
- **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$ allows you to do most of 2D geometry.

Hamilton

The first to ask *"If you can do 2D geometry with complex numbers, how might you do 3D geometry?"* was William Rowan Hamilton, circa 1840.

Sir William Rowan Hamilton
4 August 1805 — 2 September 1865

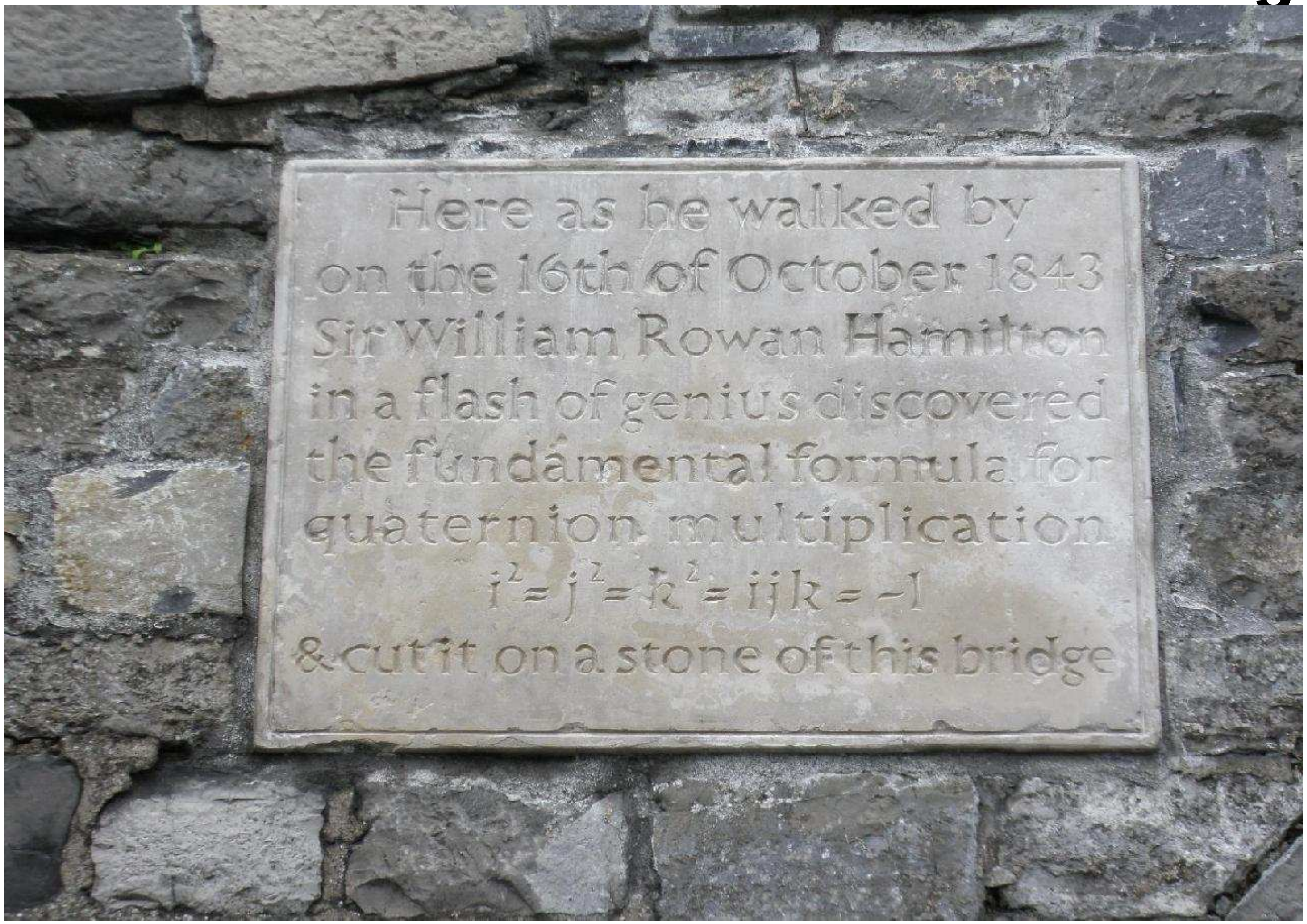
Hamilton’s epiphany: 16 October 1843

“An electric circuit seemed to close; and a spark flashed forth . . . Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,

$$i^2 = j^2 = k^2 = ijk = -1$$

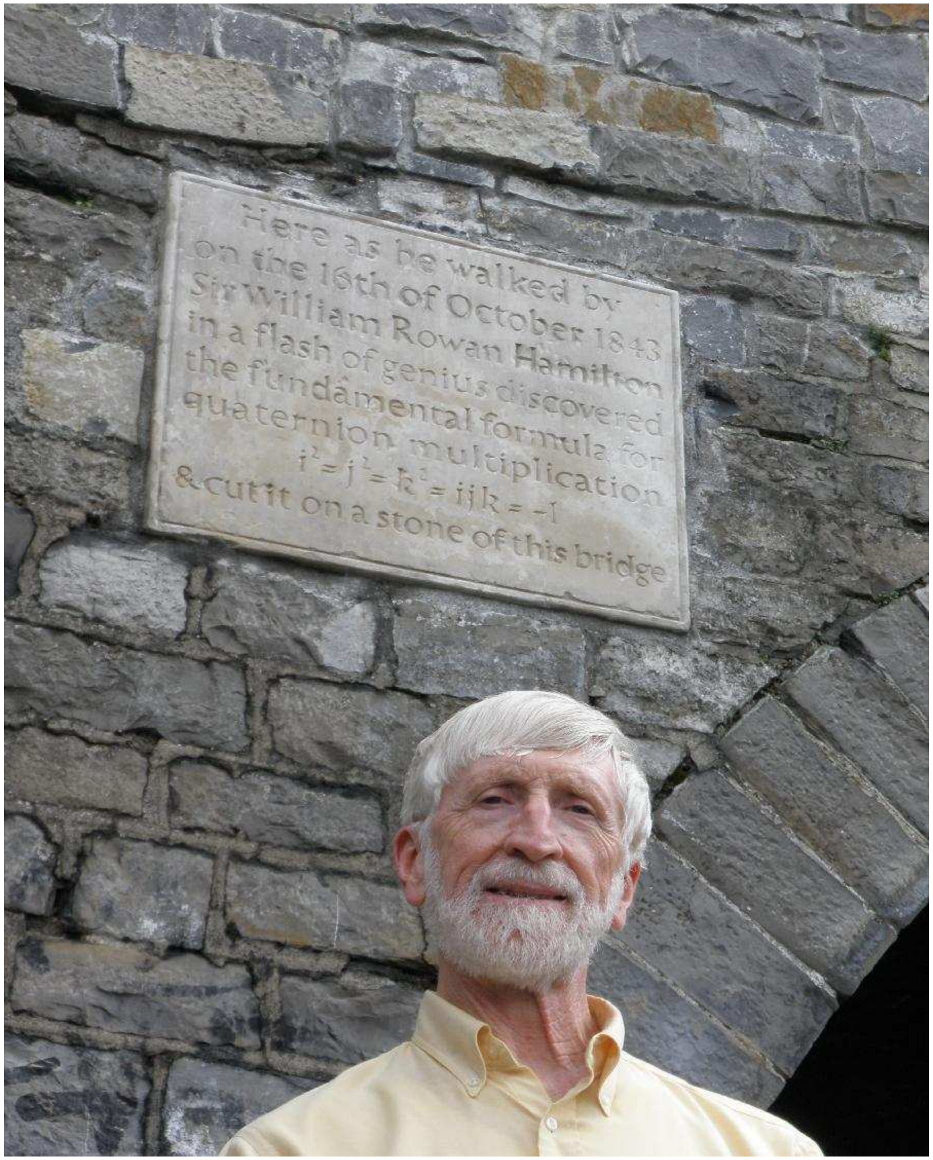
which contains the Solution of the Problem...”

...at the site of Hamilton's carving



The plaque on Broome Bridge in Dublin, Ireland, commem-orating the legendary location where Hamilton conceived of the idea of quaternions. *(Photo taken July 2012)***.**

...the author on Broome Bridge...



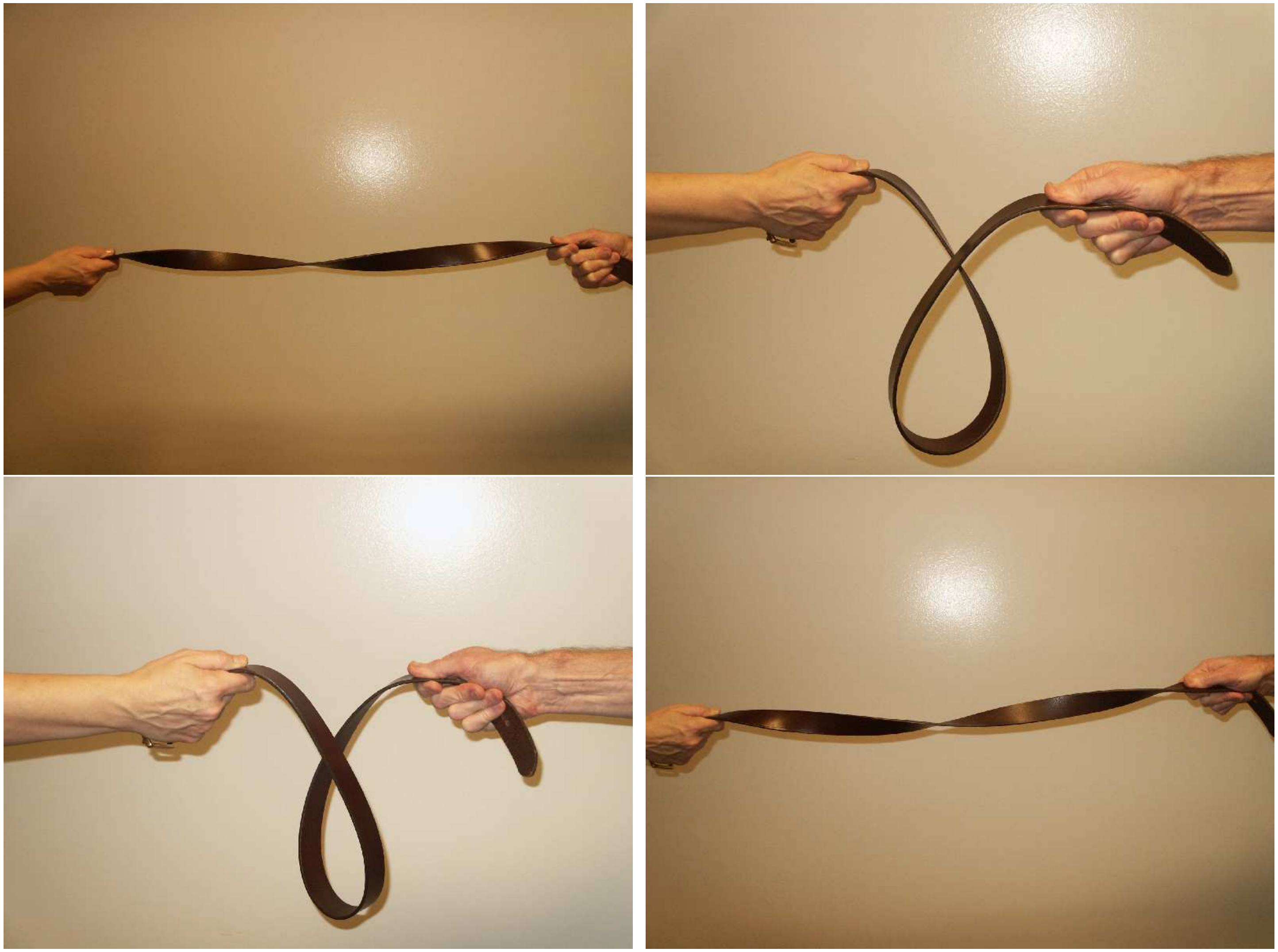
Yes, I have actually been there!

The Belt Trick

Quaternion Geometry in our daily lives

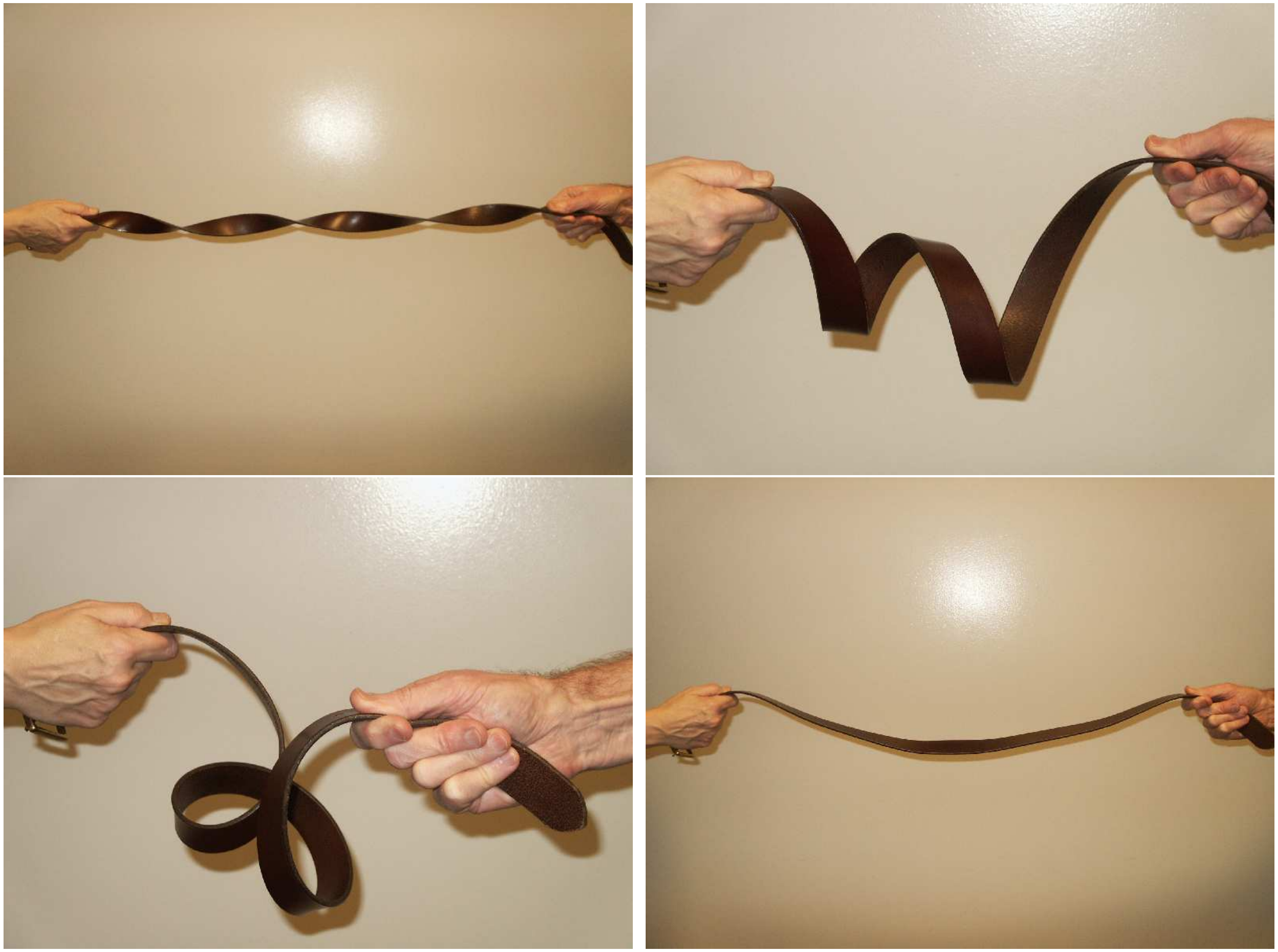
- Two people hold ends of a belt.
- Twist the belt either 360 degrees or 720 degrees.
- **Rule:** *Move belt ends any way you like but do not change orientation of either end.*
- Try to straighten out the belt.

360 Degree Belt



360 twist: stays twisted, can change DIRECTION!

720 Degree Belt



720 twist: CAN FLATTEN OUT WHOLE BELT!

The Beltless Trick

Quaternion Geometry is right in your hand!

- Hold a coffee cup (empty is a good idea) in the palm of your hand.
- Keeping the cup vertical, user your hand to twist the handle, first by 360 degrees (painful).
- *Now CONTINUE another 360 degrees,* for a total of 720 degrees.
- *Your arm is once again STRAIGHT!*

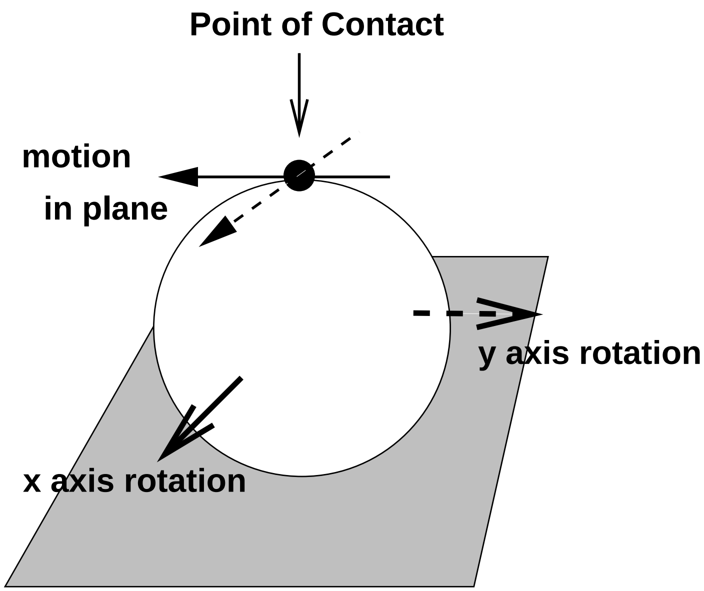
Rolling Ball Puzzle

1. Put a ball on a flat table.
2. Place hand flat on top of the ball
3. Make circular rubbing motion, as though polishing the tabletop.
4. Watch a point on the equator of the ball.
5. *small clockwise circles* $\longrightarrow$

   equator goes counterclockwise
6. *small counterclockwise circles* $\longrightarrow$

   equator goes clockwise

Rolling Ball Scenario



Gimbal Lock

Gimbal Lock occurs when a mechanical or computer system experiences an anomaly due to an (x, y, z)-based orientation control sequence.

- *Mechanical systems cannot avoid all possible gimbal lock situations .*

- *Computer orientation interpolation systems can avoid gimbal-lock-related glitches* **by using quaternion interpolation.**

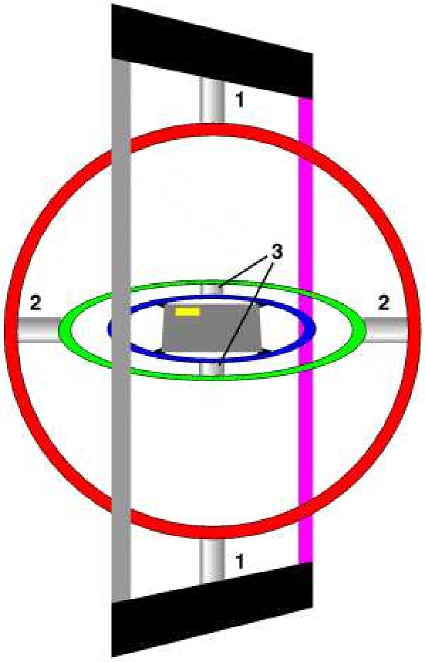


FIGURE 2

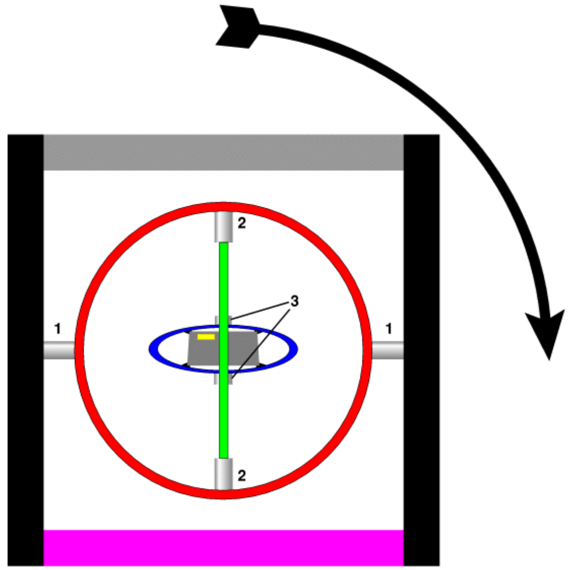


FIGURE 3

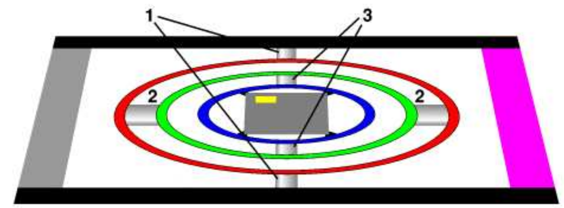


FIGURE 4

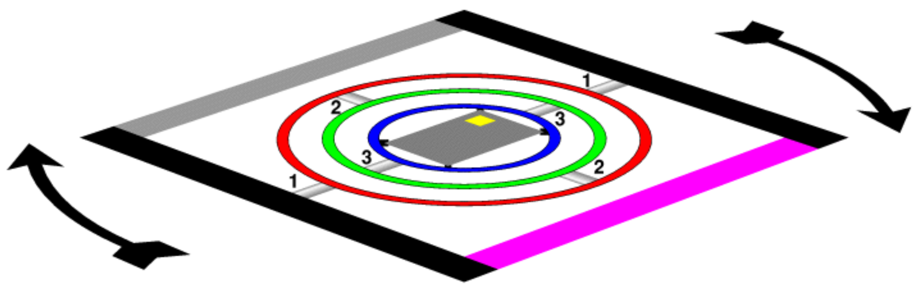


FIGURE 5

Mechanical Gimbal Lock: **Using x, y, z axes to encode orientation gives singular situations.**

Gimbal Lock — Apollo Systems

Red-painted area = Danger of real Gimbal Lock

2D Rotations

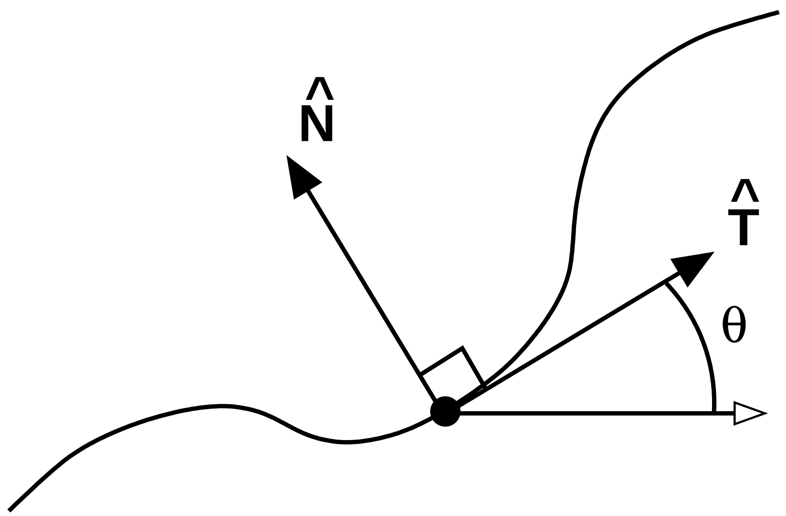
- 2D rotations $\leftrightarrow$ *complex numbers.*
- Why? $e^{i\theta}(x + iy) = (x' + iy')$

$$\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta + y \cos\theta \end{aligned}$$

- **Complex numbers** are a subspace of quaternions — so exploit 2D rotations to **introduce us to quaternions** and their geometric meaning.

Frames in 2D

The tangent and normal to 2D curve move continuously along the curve:



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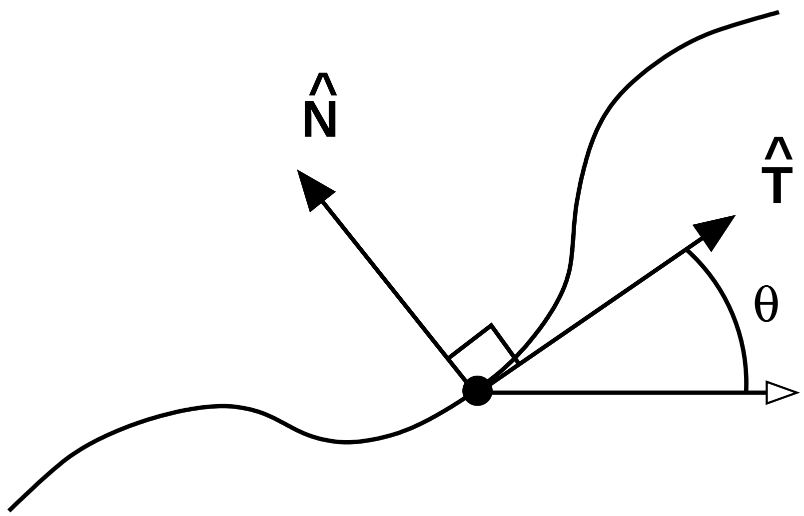

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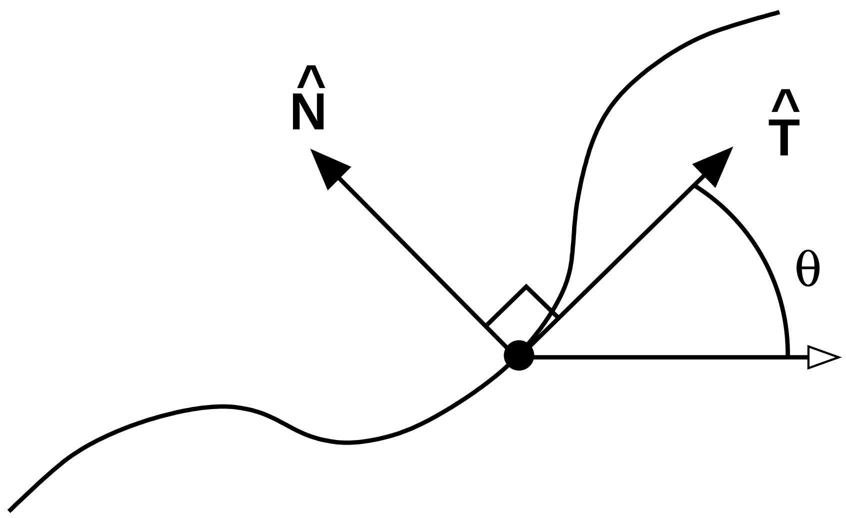


Frame Matrix in 2D

This motion is described at each point (or time) by the matrix:

$$R_2(\theta) = \begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{N}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

The Belt Trick Says:

There is a Problem...at least in 3D

How do you get $\cos\theta$ to know about 720 degrees?

The Belt Trick Says:

There is a Problem...at least in 3D

How do you get $\cos\theta$ to know about 720 degrees?

Hmmmmm. $\cos(\theta/2)$ knows about 720 degrees, right?

Half-Angle Transform:

A Fix for the Problem?

Let $a = \cos(\theta/2)$, $b = \sin(\theta/2)$,

(i.e., $\cos\theta = a^2 - b^2$, $\sin\theta = 2ab$),

and parameterize 2D rotations as:

$$R_2(a,b) \;=\; \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix} .$$

where orthonormality implies

$$(a^2 + b^2)^2 = 1$$

which reduces back to $a^2 + b^2 = 1$.

Frame Evolution in 2D

Examine the time-evolution of a 2D frame (on our way to 3D).

First use $\theta(t)$ coordinates:

$$\left[\ \hat{\mathbf{T}}\ \ \hat{\mathbf{N}}\ \right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Differentiate to find frame equations:

$$\begin{aligned} \dot{\hat{\mathbf{T}}}(t) &= +\kappa\hat{\mathbf{N}} \\ \dot{\hat{\mathbf{N}}}(t) &= -\kappa\hat{\mathbf{T}} , \end{aligned}$$

where $\kappa(t) = d\theta/dt$ is the **curvature**.

Frame Evolution in (a, b):

The basis $(\hat{\mathbf{T}}, \hat{\mathbf{N}})$ is nasty — **Four equations** with **Three constraints** from orthonormality, but just **One** true degree of freedom.

Major Simplification occurs in (a, b) coordinates!!

$$\dot{\hat{\mathbf{T}}} = 2 \begin{bmatrix} a\dot{a} - b\dot{b} \\ a\dot{b} + b\dot{a} \end{bmatrix} = 2 \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix}$$

Frame Evolution in (a, b):

But this formula for $\dot{\hat{\mathbf{T}}}$ is just $\kappa\widehat{\mathbf{N}}$, where

$$\kappa\widehat{\mathbf{N}} = \kappa \begin{bmatrix} -2ab \\ a^2 - b^2 \end{bmatrix} = \kappa \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix}$$

or

$$\kappa\widehat{\mathbf{N}} = \kappa \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

2D Quaternion Frames!

Rearranging terms, *both* $\dot{\hat{\mathbf{T}}}$ and $\dot{\hat{\mathbf{N}}}$ eqns reduce to

$$\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\kappa \\ +\kappa & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

This is the square root *of frame equations.*

2D Quaternions . . .

So *one equation* in the two *"quaternion"* variables (a, b) with the constraint $a^2 + b^2 = 1$ contains *both* the frame equations

$$\dot{\hat{\mathbf{T}}} = +\kappa \widehat{\mathbf{N}}$$

$$\dot{\hat{\mathbf{N}}} = -\kappa \widehat{\mathbf{T}}$$

$\Rightarrow$ this is much better for computer implementation, etc.

Rotation as Complex Multiplication

If we let $(a + ib) = \exp{(i\,\theta/2)}$ we see that

rotation is complex multiplication!

"Quaternion Frames" in 2D are just complex numbers, with

$$\boxed{\text{Evolution Eqns = derivative of } \exp{(i\,\theta/2)}!}$$

Rotation with no matrices!

Due to an extremely deep reason in Clifford Algebras,

$$a + ib = e^{i\theta/2}$$

represents rotations "more nicely" than the matrices $R(\theta)$.

$$(a' + ib')(a + ib) = e^{i(\theta'+\theta)/2} = A + iB$$

where if we *want* the matrix, we write:

$$R(\theta')R(\theta) = R(\theta' + \theta) = \begin{bmatrix} A^2 - B^2 & -2AB \\ 2AB & A^2 - B^2 \end{bmatrix}$$

The Algebra of 2D Rotations

The algebra corresponding to 2D rotations is easy: just complex multiplication!!

$$\begin{aligned}(a', b') * (a, b) &\cong (a' + ib')(a + ib) \\ &= a'a - b'b + i(a'b + ab') \\ &\cong (a'a - b'b,\ a'b + ab') \\ &= (A,\ B)\end{aligned}$$

2D Rotations are just **complex multiplication**, and take you around the unit circle!

Quaternion Frames

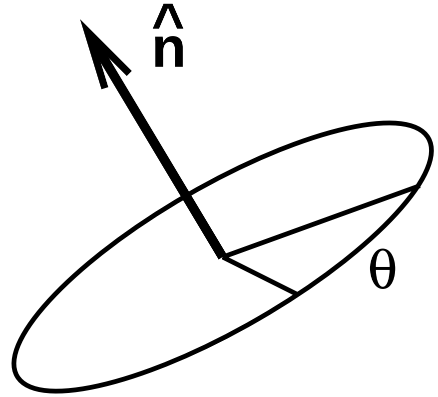
In 3D, repeat our trick: take square root of the frame, but now use *quaternions*:

- **Write down the 3D frame.**
- **Write as double-valued quadratic form.**
- **Rewrite frame evolution equations linearly in the new variables.**

The Geometry of 3D Rotations

We begin with a basic fact:

Euler theorem: *every* 3D frame can be written as a spinning by θ about a fixed axis $\hat{\mathbf{n}}$, the eigenvector of the rotation matrix:



Quaternion Frames . . .

The Matrix $R_3(\theta, \hat{\mathbf{n}})$ giving 3D rotation by θ about axis $\hat{\mathbf{n}}$ is :

$$\begin{bmatrix} c + (n_1)^2(1-c) & n_1 n_2(1-c) - s n_3 & n_3 n_1(1-c) + s n_2 \\ n_1 n_2(1-c) + s n_3 & c + (n_2)^2(1-c) & n_3 n_2(1-c) - s n_1 \\ n_1 n_3(1-c) - s n_2 & n_2 n_3(1-c) + s n_1 & c + (n_3)^2(1-c) \end{bmatrix}$$

where $c = \cos\theta$, $s = \sin\theta$, and $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$.

Can we find a 720-degree form?

Remember 2D: $a^2 + b^2 = 1$
then substitute $1 - c = (a^2 + b^2) - (a^2 - b^2) = 2b^2$
to find the remarkable expression for $\mathbf{R}(\theta, \hat{\mathbf{n}})$:

$$\begin{bmatrix} a^2 - b^2 + (n_1)^2(2b^2) & 2b^2 n_1 n_2 - 2abn_3 & 2b^2 n_3 n_1 + 2abn_2 \\ 2b^2 n_1 n_2 + 2abn_3 & a^2 - b^2 + (n_2)^2(2b^2) & 2b^2 n_2 n_3 - 2abn_1 \\ 2b^2 n_3 n_1 - 2abn_2 & 2b^2 n_2 n_3 + 2abn_1 & a^2 - b^2 + (n_3)^2(2b^2) \end{bmatrix}$$

Rotations and Quadratic Polynomials

Remember $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$ and $a^2 + b^2 = 1$;

letting

$$q_0 = a = \cos(\theta/2) \qquad \mathbf{q} = b\hat{\mathbf{n}} = \hat{\mathbf{n}}\sin(\theta/2)$$

We find a matrix $R_3(q)$

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Quaternions and Rotations . . .

HOW does $q = (q_0,\, \mathbf{q})$ represent rotations?

LOOK at

$$R_3(\theta, \hat{\mathbf{n}}) \stackrel{?}{=} R_3(q_0, q_1, q_2, q_3)$$

THEN we can verify that choosing

$$q(\theta, \hat{\mathbf{n}}) = (\cos\frac{\theta}{2},\ \hat{\mathbf{n}}\sin\frac{\theta}{2})$$

makes the R_3 equation an *IDENTITY* .

Quaternions and Rotations . . .

WHAT happens if you do **TWO** rotations?

EXAMINE the action of two rotations

$$R(q')R(q) = R(Q)$$

EXPRESS in quadratic forms in q and LOOK FOR an analog of complex multiplication:

Quaternions and Rotations . . .

RESULT: the following multiplication rule

$\boxed{q' * q = Q}$ yields **exactly** the correct 3×3 rotation matrix $R(Q)$:

$$\begin{bmatrix} Q_0 = [q' * q]_0 \\ Q_1 = [q' * q]_1 \\ Q_2 = [q' * q]_2 \\ Q_3 = [q' * q]_3 \end{bmatrix} = \begin{bmatrix} q'_0 q_0 - q'_1 q_1 - q'_2 q_2 - q'_3 q_3 \\ q'_0 q_1 + q'_1 q_0 + q'_2 q_3 - q'_3 q_2 \\ q'_0 q_2 + q'_2 q_0 + q'_3 q_1 - q'_1 q_3 \\ q'_0 q_3 + q'_3 q_0 + q'_1 q_2 - q'_2 q_1 \end{bmatrix}$$

This is Quaternion Multiplication.

Algebra of Quaternions

= 3D Rotations!

2D rotation matrices are represented

by **complex multiplication**

3D rotation matrices are represented

by **quaternion multiplication**

Algebraic 2D/3D Rotations

Therefore in 3D, the 2D complex multiplication

$$(a', b') * (a, b) = (a'a - b'b,\ a'b + ab')$$

is replaced by 4D quaternion multiplication:

$$\begin{aligned} q' * q \; = \; (& q'_0 q_0 - q'_1 q_1 - q'_2 q_2 - q'_3 q_3, \\ & q'_0 q_1 + q'_1 q_0 + q'_2 q_3 - q'_3 q_2, \\ & q'_0 q_2 + q'_2 q_0 + q'_3 q_1 - q'_1 q_3, \\ & q'_0 q_3 + q'_3 q_0 + q'_1 q_2 - q'_2 q_1) \end{aligned}$$

Algebra of Quaternions . . .

The equation is easier to remember by dividing it into a *scalar* piece q_0 and a *vector* piece $\vec{\mathbf{q}}$:

$$q' * q \;=\; (q_0' q_0 - \vec{\mathbf{q}}' \cdot \vec{\mathbf{q}},\\ q_0' \, \vec{\mathbf{q}} + q_0 \, \vec{\mathbf{q}}' + \vec{\mathbf{q}}' \times \vec{\mathbf{q}})$$

Now we can SEE quaternions!

Since $(q_0)^2 + \mathbf{q} \cdot \mathbf{q} = 1$ then

$$q_0 = \sqrt{1 - \mathbf{q} \cdot \mathbf{q}}$$

Plot just the 3D vector: $\mathbf{q} = (q_x,\, q_y,\, q_z)$

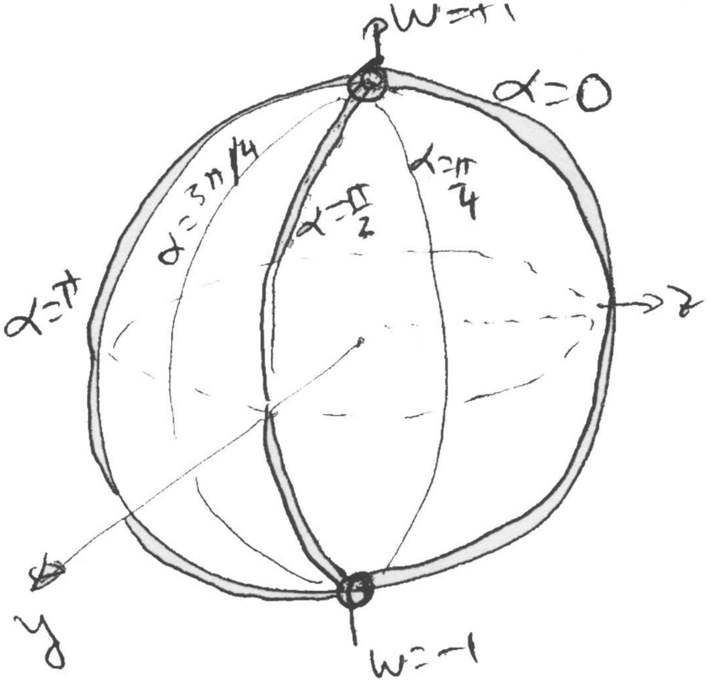
q_0 is KNOWN! We can also use any other triple: the fourth component is *dependent*.

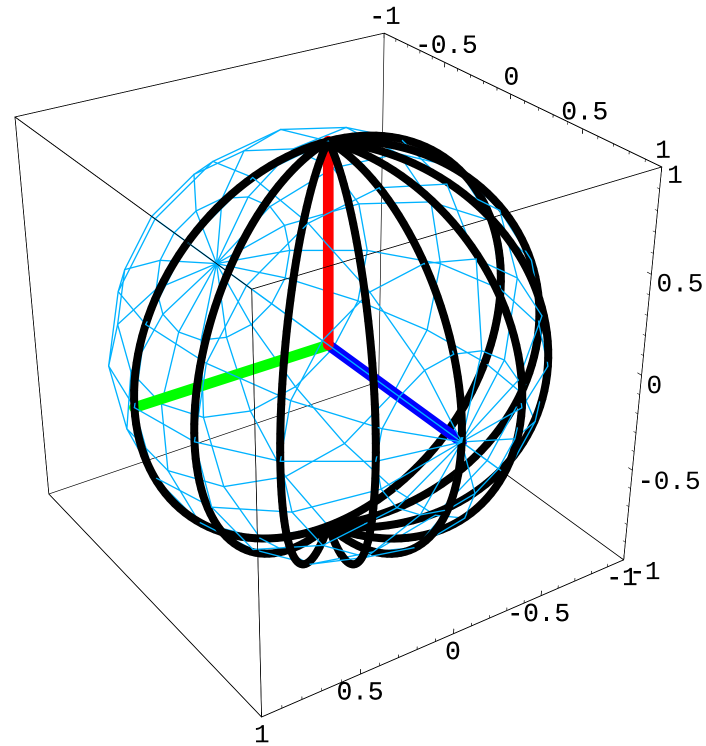
DEMO

We can now make a Quaternion Picture of each of our favorite tricks

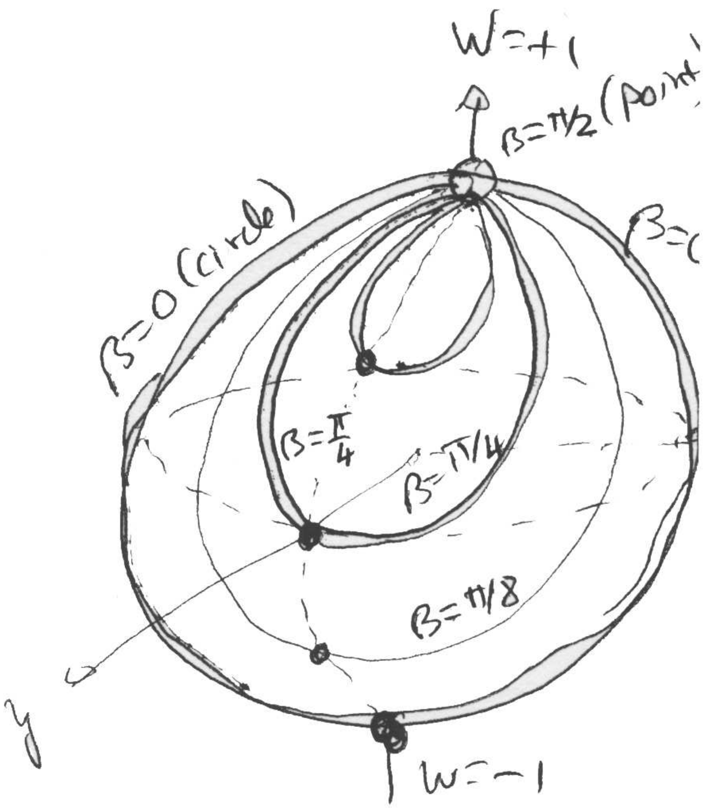
- 360° **Belt Trick in Quaternion Form.** DEMO:
- 720° **Belt Trick in Quaternion Form.**
- **Rolling Ball in Quaternion Form.** DEMO:
- **Gimbal Lock in Quaternion Form.**

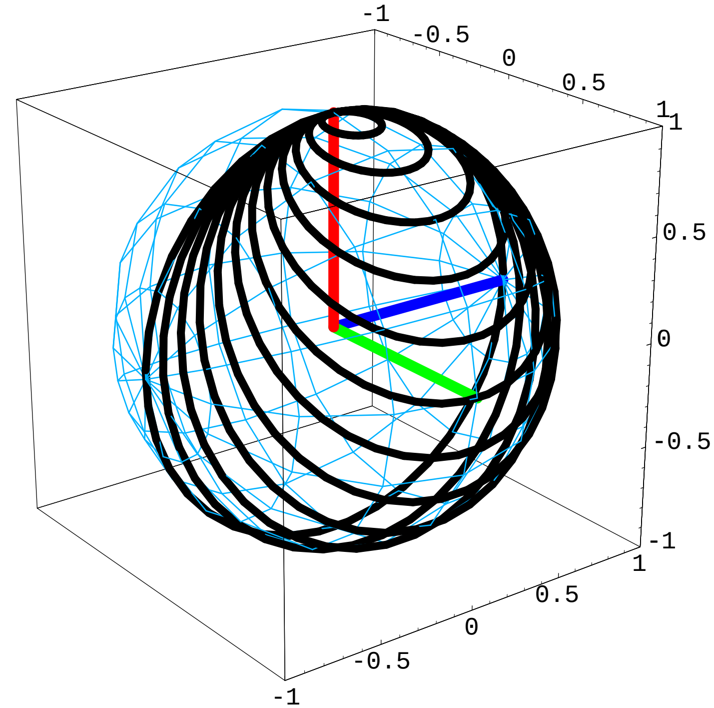
360° **Belt Trick in Quaternion Form**



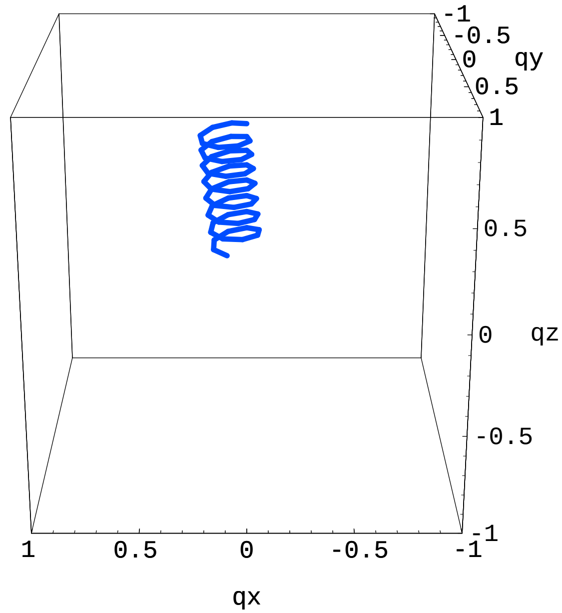


720° Belt Trick in Quaternion Form

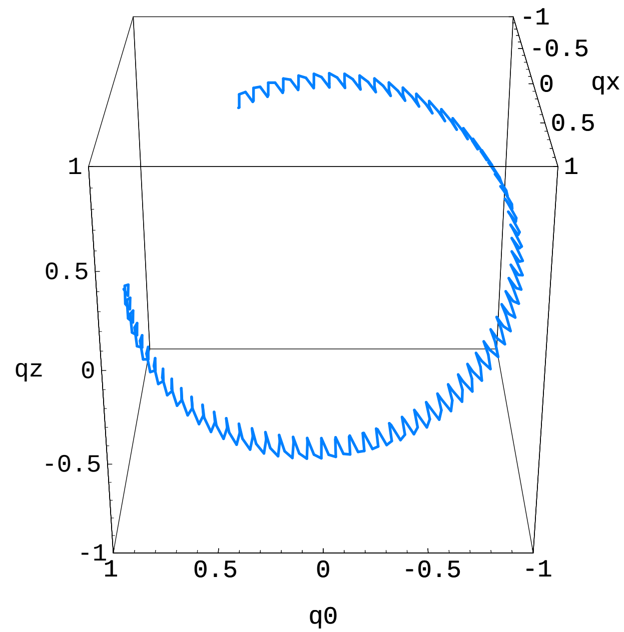




Rolling Ball in Quaternion Form

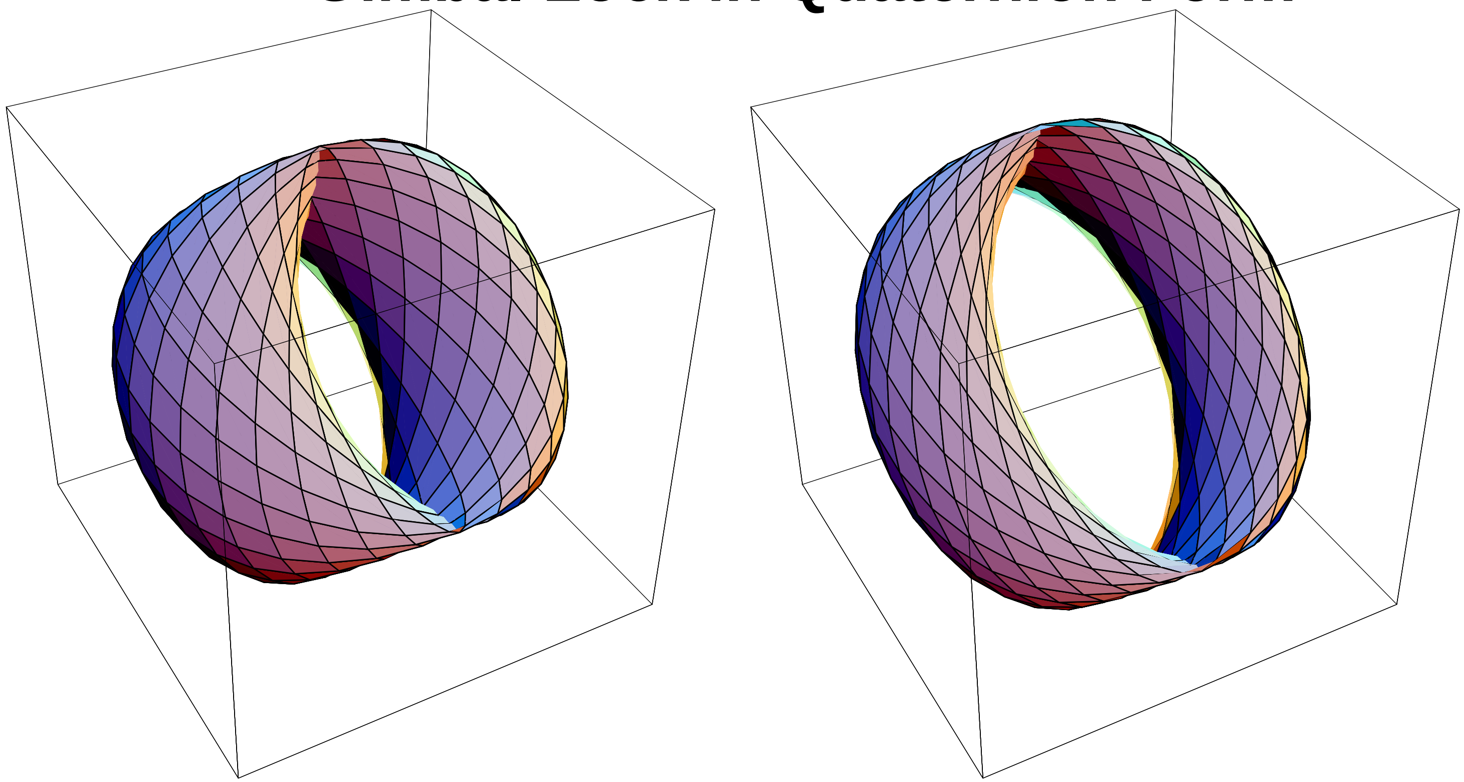


$\mathbf{q}$ vector-only plot.



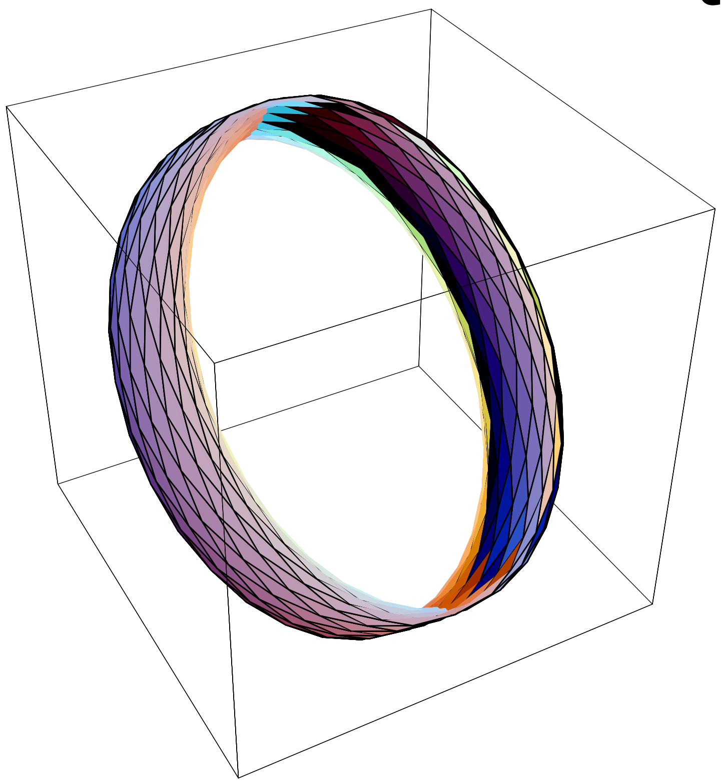
(q_0, q_x, q_z) plot

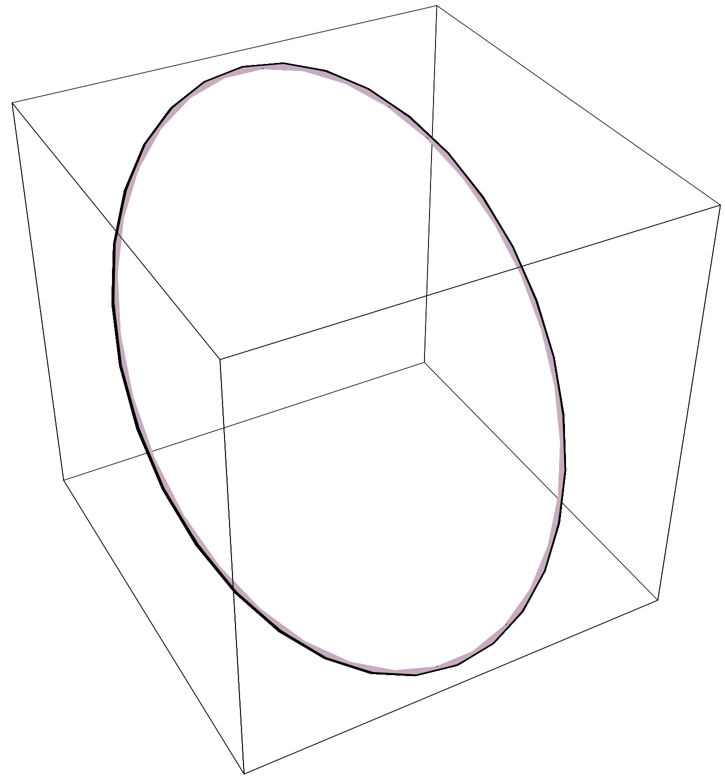
Gimbal Lock in Quaternion Form



Quaternion Plot of the *remaining* orientation degrees of freedom of $\mathbf{R}(\theta, \hat{\mathbf{x}}) \cdot \mathbf{R}(\phi, \hat{\mathbf{y}}) \cdot \mathbf{R}(\psi, \hat{\mathbf{z}})$ at $\phi = 0$ and $\phi = \pi/6$

Gimbal Lock in Quaternion Form, contd





Choosing ϕ and plotting the *remaining* orientation degrees in the rotation sequence $\mathbf{R}(\theta, \hat{\mathbf{x}}) \cdot \mathbf{R}(\phi, \hat{\mathbf{y}}) \cdot \mathbf{R}(\psi, \hat{\mathbf{z}})$, we see degrees of freedom **decrease from TWO to ONE as** $\phi \to \pi/2$

Quaternion Interpolations

- Shoemake (Siggraph '85) proposed using quaternions instead of Euler angles to get smooth frame interpolations without **Gimbal Lock**:

  BEST CHOICE: Animate objects and cameras using rotations represented on S^3 by quaternions

Interpolating on Spheres

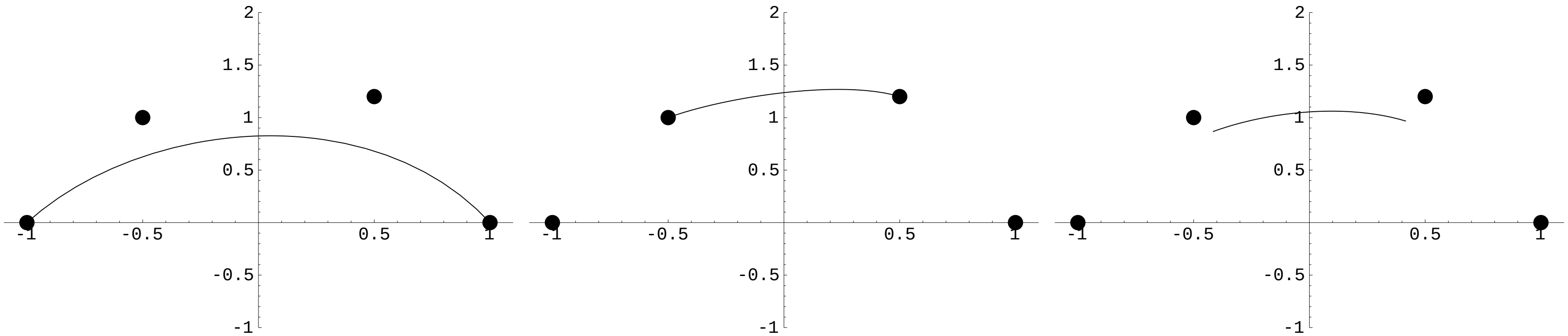
General quaternion spherical interpolation employs the “SLERP,” a constant angular velocity transition between two directions, $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$:

$$\boxed{\begin{aligned} \hat{\mathbf{q}}_{12}(t) &= \mathrm{Slerp}(\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, t) \\ &= \hat{\mathbf{q}}_1 \frac{\sin((1-t)\theta)}{\sin(\theta)} + \hat{\mathbf{q}}_2 \frac{\sin(t\theta)}{\sin(\theta)} \end{aligned}}$$

where $\cos\theta = \hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_2$.

Plane Interpolations

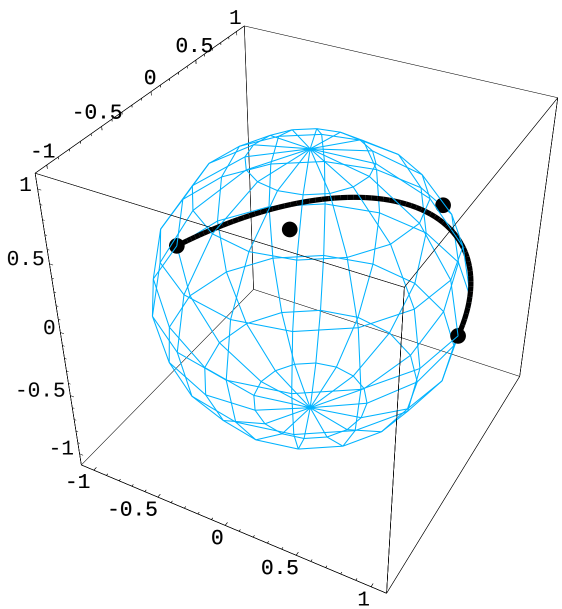
In Euclidean space, these three basic cubic splines look like this:



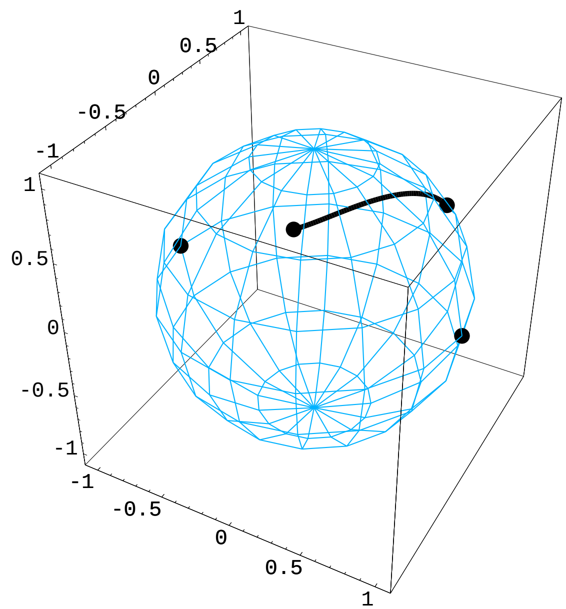
Bezier Catmull-Rom Uniform B

The differences are in the derivatives: Bezier has to start matching all over at every fourth point; Catmull-Rom matches the first derivative; and B-spline is the cadillac, matching all derivatives but *no control points*.

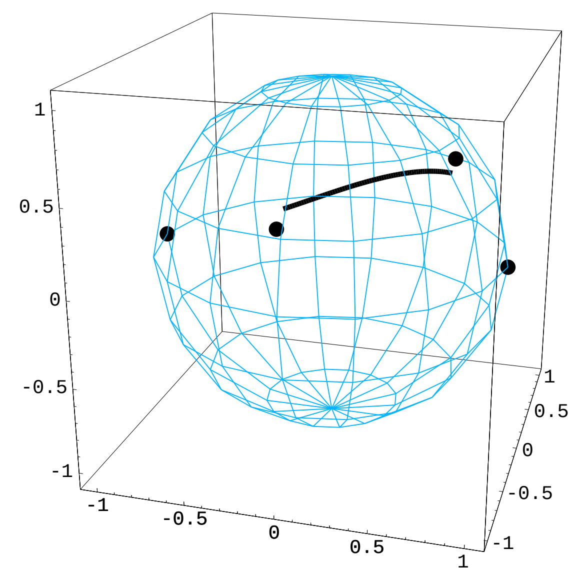
Spherical Interpolations



Bezier

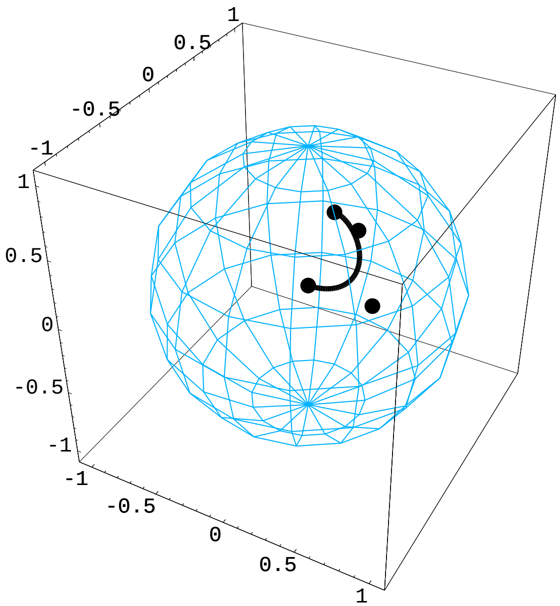


Catmull-Rom

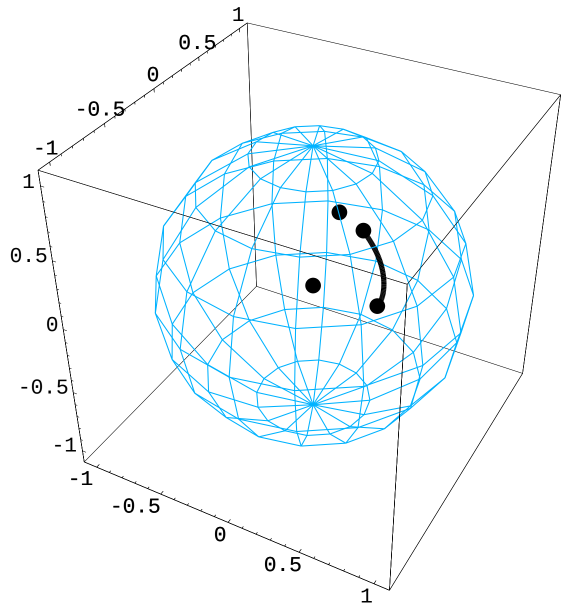


Uniform B

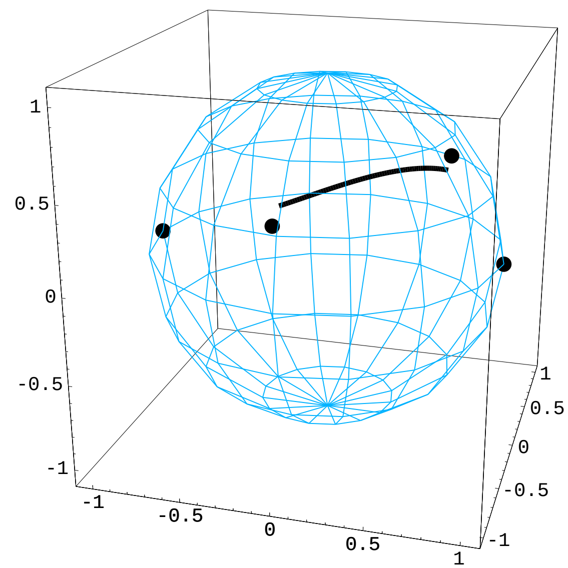
Quaternion Interpolations



Bezier



Catmull-Rom



Uniform B

Exp Form of Quaternion Rotations

In Hamilton's notation, we can generalize the 2D equation

$$a + ib = e^{i\theta/2}$$

Just set

$$\begin{aligned} q &= (q_0,\, q_1,\, q_2,\, q_3) \\ &= q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \\ &= e^{(\mathbf{I}\cdot\hat{\mathbf{n}}\theta/2)} \end{aligned}$$

with $q_0 = \cos(\theta/2)$ and $\vec{\mathbf{q}} = \hat{\mathbf{n}}\sin(\theta/2)$ and $\mathbf{I} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$, with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and $\mathbf{i} * \mathbf{j} = \mathbf{k}$ (cyclic),

Cute Quaternion Tricks!

Square Roots are cool..

A quaternion p is the **square root** of a quaternion q if

$$p * p = q \; .$$

A hint: remember that if $c = \cos\theta$, and $\gamma = \cos(\frac{\theta}{2})$, then

$$\gamma = \sqrt{\frac{1+c}{2}} = \frac{1+c}{\sqrt{2(1+c)}}$$

Cute Quaternion Tricks...

Suppose we now look at $1 + q = (1 + q_0, \mathbf{q})$. Then

$$\begin{aligned}(1+q)*(1+q) &= \Big((1+q_0)^2 - \mathbf{q}\cdot\mathbf{q},\ 2\mathbf{q}(1+q_0)\Big)\\ &= 2(1+q_0)\,q\end{aligned}$$

Dividing through by $2(1 + q_0)$, we find the $\boxed{\textbf{square root:}}$

$$p = \sqrt{q} = \frac{1+q}{\sqrt{2(1+q_0)}}$$

Tricks, contd: Lining up $\hat{\mathrm{a}}$ and $\hat{\mathrm{b}}$

A common rotation task is to line up two directions, $\hat{\mathrm{a}}$ and $\hat{\mathrm{b}}$. There is a simple **quaternion form** for this operation. Let

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta = c\,, \quad \hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{n}} \sin\theta$$

where we assume $\sin\theta > 0$. Then we can compute the rotation from $\hat{\mathrm{a}}$ to $\hat{\mathrm{b}}$ using, again, the half-angle formula:

$$\begin{aligned} R(\hat{\mathbf{a}}, \hat{\mathbf{b}}) &= (\cos(\theta/2), \hat{\mathbf{n}} \sin(\theta/2)) \\ &= \left(\sqrt{\frac{1+c}{2}}, \hat{\mathbf{a}} \times \hat{\mathbf{b}} \sqrt{\frac{1}{2(1+c)}} \right) \end{aligned}$$

where we also used $\sin\theta = 2\cos(\theta/2)\sin(\theta/2)$.

Clifford Algebras

- **All Rotations in any dimension are represented by** ***two reflections*** **using Clifford Algebra:**

  $\mathbf{A}$ and $\mathbf{B}$ define the perpendicular directions to two reflection planes, $\mathbf{A} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{B} = 1$.

- **Create Rotation Matrix $\mathbf{R}$ and solve for the Quaternion, and you amazingly get THIS:**

$$q(\mathbf{A}, \mathbf{B}) \;=\; (\mathbf{A} \cdot \mathbf{B}, \mathbf{A} \times \mathbf{B})$$

Clifford Algebra Quaternion Form . . .

Why is this a quaternion form?

$$
\begin{aligned}
q \cdot q &= (\mathbf{A} \cdot \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) \\
&= (\mathbf{A} \cdot \mathbf{A})\,(\mathbf{B} \cdot \mathbf{B}) \\
&\equiv 1
\end{aligned}
$$

If Quaternions are like the Square Roots of Rotations, then Clifford Algebras are like the Square Roots of Quaternions!

Key to Quaternion Intuition

Fundamental Intuition: We know

$$q_0 = \cos(\theta/2), \;\; \vec{\mathbf{q}} = \hat{\mathbf{n}} \sin(\theta/2)$$

We also know that *any coordinate frame* M can be written as $M = R(\theta, \hat{\mathbf{n}})$.

Therefore

$\vec{\mathrm{q}}$ **points exactly along the axis we have to rotate around to go from identity** I **to** M**, and the length of** $\vec{\mathrm{q}}$ **tells us how much to rotate.**

Summarize Quaternion Properties

- **Unit four-vector.** Take $q = (q_0, q_1, q_2, q_3) = (q_0, \vec{\mathbf{q}})$ to obey constraint $q \cdot q = 1$.

- **Multiplication rule.** The quaternion product q and p is $q * p = (q_0 p_0 - \vec{\mathbf{q}} \cdot \vec{\mathbf{p}},\ q_0 \vec{\mathbf{p}} + p_0 \vec{\mathbf{q}} + \vec{\mathbf{q}} \times \vec{\mathbf{p}})$,
  or, alternatively,

$$\begin{bmatrix} [q * p]_0 \\ [q * p]_1 \\ [q * p]_2 \\ [q * p]_3 \end{bmatrix} = \begin{bmatrix} q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2 \\ q_0 p_2 + q_2 p_0 + q_3 p_1 - q_1 p_3 \\ q_0 p_3 + q_3 p_0 + q_1 p_2 - q_2 p_1 \end{bmatrix}$$

Quaternion Summary . . .

- **Rotation Correspondence.** The unit quaternions q and $-q$ correspond to a single 3D rotation R_3:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

If

$$q = (\cos\frac{\theta}{2}, \hat{\mathbf{n}} \sin\frac{\theta}{2}) \ ,$$

with $\hat{\mathbf{n}}$ a unit 3-vector, $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$. Then $R(\theta, \hat{\mathbf{n}})$ is usual 3D rotation by θ in the plane $\perp$ to $\hat{\mathbf{n}}$.

SUMMARY

- Quaternions represent $\boxed{\text{3D frames}}$
- Quaternion multiplication represents 3D rotation
- Quaternions are $\boxed{\text{points on a hypersphere}}$
- Quaternions paths can be visualized with 3D display
- Belt Trick, Rolling Ball, and Gimbal Lock can be understood as Quaternion Paths.