Quaternion Frenet Frames: Making Optimal Tubes and Ribbons from Curves

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◊ Introduction ◊

Our purpose here is to show how the quaternion formalism can be applied with great success not only to the interpolation between coordinate frames, but also to a remarkably elegant description of the evolving coordinate-frame geometry of curves. Specific applications of these techniques include the generation of optimal renderable ribbons and tubes corresponding to smooth mathematical curves appearing in computer graphics or scientific visualization applications.

The correspondence between the orientation of a 3D object represented by a 3×3 orthonormal matrix in the group $SO(3)$ and unit quaternions has long been known to physicists (see, e.g., (Misner et al. 1973)) and mathematicians (see, e.g., (Helgason 1962, Cartan 1981)), and was brought to the attention of the computer graphics community by (Shoemake 1985). Unit quaternions are isomorphic to the topological 3-sphere S^3, which is also the topological space of the Lie group $SU(2)$, the simply connected two-fold cover of the group $SO(3)$ describing rotations in ordinary 3D Euclidean space. The key motivation for representing 3D rotations in terms of quaternions is the fact that the geodesics in S^3 correspond to beautiful interpolations between coordinate frames in 3D space that cannot be directly achieved with other rotation representations such as Euler angles. Below, we show how to extend this general structure to moving coordinate frames of space curves.

◊ Framed Curves ◊

The differential geometry of curves (Gray 1993,Flanders 1963,Eisenhart 1960) traditionally begins with a vector $\vec{\mathbf{x}}(s)$ that describes the curve parametrically as a function of s that is at least thrice-differentiable. Then the tangent vector $\vec{\mathbf{T}}(s)$ is well-defined at every point $\vec{\mathbf{x}}(s)$ and we may choose two additional orthogonal vectors in the plane perpendicular to $\vec{\mathbf{T}}(s)$ to form a complete local orientation frame. Provided the curvature of $\vec{\mathbf{x}}(s)$ vanishes nowhere, we can choose this local coordinate system to be the *Frenet frame* (also known as the Frenet-Serret frame) consisting of the tangent $\vec{\mathbf{T}}(s)$, the binormal $\vec{\mathbf{B}}(s)$, and the principal normal $\vec{\mathbf{N}}(s)$, which are given in terms of the curve itself by these expressions:

$$\begin{aligned} \vec{\mathbf{T}}(s) &= \frac{\vec{\mathbf{x}}'(s)}{\|\vec{\mathbf{x}}'(s)\|} \\ \vec{\mathbf{B}}(s) &= \frac{\vec{\mathbf{x}}'(s) \times \vec{\mathbf{x}}''(s)}{\|\vec{\mathbf{x}}'(s) \times \vec{\mathbf{x}}''(s)\|} \\ \vec{\mathbf{N}}(s) &= \vec{\mathbf{B}}(s) \times \vec{\mathbf{T}}(s)\ . \end{aligned} \tag{1}$$

We illustrate this standard frame configuration in Figure 1. Differentiating the Frenet frames yields the classic Frenet equations:

$$\begin{bmatrix} \vec{\mathbf{T}}'(s) \\ \vec{\mathbf{N}}'(s) \\ \vec{\mathbf{B}}'(s) \end{bmatrix} = v(s) \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{T}}(s) \\ \vec{\mathbf{N}}(s) \\ \vec{\mathbf{B}}(s) \end{bmatrix} . \tag{2}$$

Here $v(s) = \|\vec{\mathbf{x}}'(s)\|$ is the scalar magnitude of the curve derivative, and the intrinsic geometry of the curve is embodied in the curvature $\kappa(s)$ and the torsion $\tau(s)$, which may be written in terms of the curve itself as

$$\begin{aligned} \kappa(s) &= \frac{\|\vec{\mathbf{x}}'(s) \times \vec{\mathbf{x}}''(s)\|}{\|\vec{\mathbf{x}}'(s))\|^3} \\ \tau(s) &= \frac{(\vec{\mathbf{x}}'(s) \times \vec{\mathbf{x}}''(s)) \cdot \vec{\mathbf{x}}'''(s)}{\|\vec{\mathbf{x}}'(s) \times \vec{\mathbf{x}}''(s)\|^2}\ . \end{aligned} \tag{3}$$

Ribbons and tubes. Ribbons and tubes centered on the curve may easily be generated using the continuous values of the normal plane coordinate system defined by $(\vec{\mathbf{N}}(s), \vec{\mathbf{B}}(s))$. Detailed methods for accomplishing this are spelled out in (Gray 1993). Closed curves are guaranteed by construction to have matching frames at the closure point, so this is an ideal method for creating ribbons and tubes to represent "thickened" curves in computer graphics applications.

Inverse versions of this procedure are often of interest as well:

- *Curvature driven*: If we specify *a priori* the velocity (non-vanishing), the curvature (non-vanishing), and the torsion of a curve, then we may start the curve with an initial Frenet frame and integrate the Frenet equations (2) to get the values of the entire frame triad at every point. Integrating the tangent vector (2) then yields the curve $\vec{\mathbf{x}}(s)$ itself, along with any desired ribbon or tube centered on the curve, e.g., by sweeping lines along the curve in the $(\vec{\mathbf{N}}, \vec{\mathbf{B}})$ plane. (See (Gray 1993) for examples.)

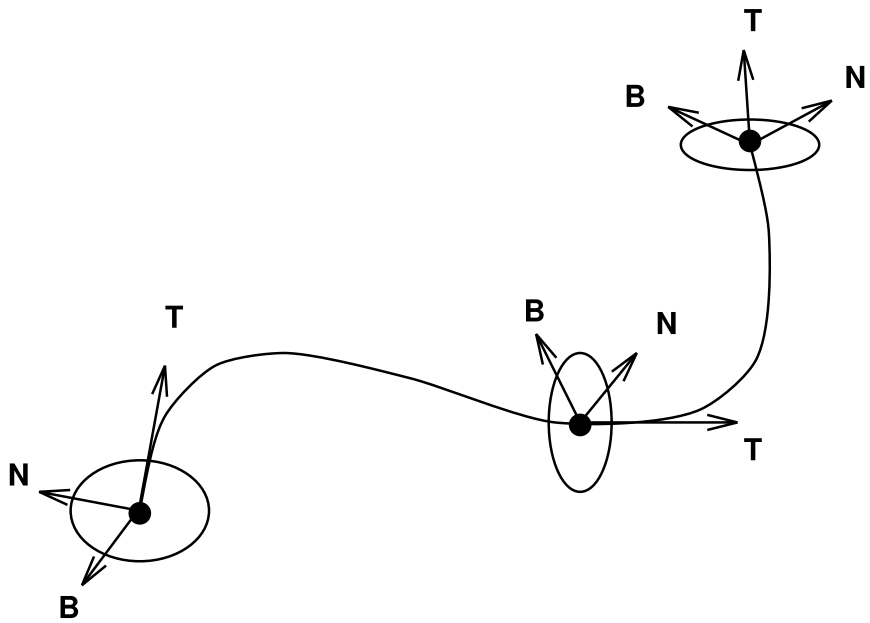

Figure 1. The triad of orthogonal axes forming the Frenet frame for a curve with non-vanishing curvature.

- *Frame-driven.* Alternatively, we may already have an expression for the frame triad $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$ as a function of s. In this case, one *derives* the curvature and torsion using (2) and (4) and integrates $\vec{\mathbf{T}}(s)$ directly to get $\vec{\mathrm{x}}(s)$ (and a corresponding ribbon or tube, if desired).

Vanishing Curvature and Parallel Transport Frames

But what if the curvature vanishes because $\vec{\mathrm{x}}''(s) = 0$ at some set of points? The Frenet frame before and after the zero-curvature set can be entirely different, so there may be no way to define a unique continuous Frenet frame over the whole curve, and any heuristics one might use to mend the situation are arbitrary.

This difficulty led (Bishop 1975) to introduce an alternative framing based on parallel transport rather than local curve derivatives. The basic concept is to observe that while $\vec{\mathbf{T}}(s)$ is unique, we may choose any convenient basis $(\vec{\mathbf{N}}_1(s), \vec{\mathbf{N}}_2(s))$ in the plane perpendicular to $\vec{\mathbf{T}}(s)$ at each point. If the derivatives of $(\vec{\mathbf{N}}_1(s), \vec{\mathbf{N}}_2(s))$ depend only on $\vec{\mathbf{T}}(s)$ and not each other, we can make $\vec{\mathbf{N}}_1(s)$ and $\vec{\mathbf{N}}_2(s)$ vary smoothly throughout the

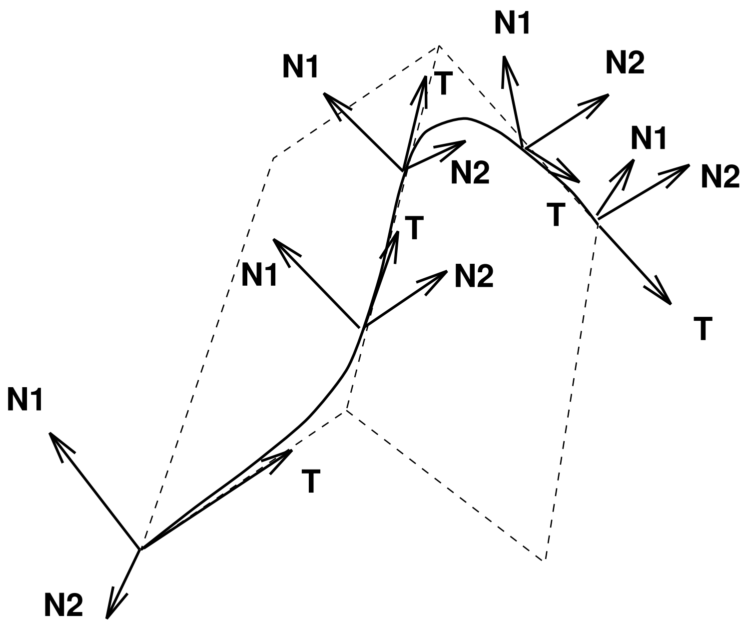

Figure 2. The parallel-transport curve framing of (Bishop 1975). The Frenet frame would be discontinuous along the "roof peak" where the curvature vanishes.

path regardless of the curvature. We therefore choose the alternative frame equations

$$\begin{bmatrix} \vec{\mathbf{T}}'(s) \\ \vec{\mathbf{N}}_1'(s) \\ \vec{\mathbf{N}}_2'(s) \end{bmatrix} = v(s) \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{T}}(s) \\ \vec{\mathbf{N}}_1(s) \\ \vec{\mathbf{N}}_2(s) \end{bmatrix} , \tag{4}$$

illustrated in Figure 2. One can show that

$$\begin{aligned} \kappa(s) &= \left((k_1)^2 + (k_2)^2\right)^{1/2} \\ \theta(s) &= \arctan\left(\frac{k_2}{k_1}\right) \\ \tau(s) &= \theta'(s) , \end{aligned}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)\, ds$.

Just as for the Frenet frame, one can begin with a curve $\vec{\mathbf{x}}(s)$ and an initial frame, or a pair of functions $(k_1(s), k_2(s))$ and an initial frame, or a frame over the entire curve, and then integrate where needed to compute the missing variables.

Closed Ribbons and Tubes. One minor drawback of the parallel transport frame for creating ribbons and tubes is that the frames at the beginning and end of a closed curve do not necessarily match up as they must for a Frenet frame; this phenomenon is easily corrected by measuring the relative rotation of the beginning and ending frames, and distributing the total rotation deficit evenly around the entire curve.

◊ Quaternion Frames ◊

Next, we sketch the correspondence between the unit quaternions and the orthonormal coordinate frames; this will take us to our main result, which is a reformulation of the Frenet and parallel-transport frames in terms of quaternions only.

Theory of Quaternion Frames. A quaternion frame is a unit-length four-vector $q = (q_0, q_1, q_2, q_3) = (q_0, \vec{\mathbf{q}})$ that corresponds to exactly one 3D coordinate frame and is characterized by the following properties:

- **Unit Norm.** The components of a unit quaternion obey the constraint,

$$(q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2 = 1 \tag{5}$$

  and therefore lie on S^3, the three-sphere.
- **Multiplication rule.** Two quaternions q and p obey the following multiplication rule, which is isomorphic to multiplication in the group SU(2), which is the double covering of the ordinary 3D rotation group SO(3):

$$q \cdot p = \left\{ \begin{array}{c} [q \cdot p]_0 \\ [q \cdot p]_1 \\ [q \cdot p]_2 \\ [q \cdot p]_3 \end{array} \right\} = \left\{ \begin{array}{c} q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ q_0 p_1 + p_0 q_1 + q_2 p_3 - q_3 p_2 \\ q_0 p_2 + p_0 q_2 + q_3 p_1 - q_1 p_3 \\ q_0 p_3 + p_0 q_3 + q_1 p_2 - q_2 p_1 \end{array} \right\} . \tag{6}$$

- **Inverse.** The inverse quaternion is defined as $\overline{q} = q^{-1} = (q_0, -\vec{\mathbf{q}})$, so that $\overline{q}q = q\overline{q} = (1, \vec{\mathbf{0}})$.
- **Mapping to 3D rotations.** Every possible 3D rotation R (a 3×3 orthogonal matrix) can be constructed from either of two related quaternions, $q = (q_0, q_1, q_2, q_3)$ or $-q = (-q_0, -q_1, -q_2, -q_3)$, using the transformation law:

$$[q \cdot \vec{\mathbf{V}} \cdot \overline{q}]_i = \sum_{j=1}^{3} R_{ij} V_j$$

where, with $v = (0, \vec{\mathbf{V}})$ a pure 3-vector, we can compute R_{ij} directly from Eq. (6) to be the quadratic formula

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_1 q_3 + 2q_0 q_2 \\ 2q_1 q_2 + 2q_0 q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_2 q_3 + 2q_0 q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} . \tag{7}$$

We can quickly check that all rows of this matrix expressed in this form are orthogonal by construction, and that the squared length of any one row or column reduces to $((q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2)^2$, which is unity if the constraint (5) holds.

Rotation Correspondence. When we substitute $q = (\cos\frac{\theta}{2}, \hat{\mathbf{n}} \sin\frac{\theta}{2})$ into Eq. (7), where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ is a unit 3-vector lying on the 2-sphere S^2, $R(\theta, \hat{\mathbf{n}})$ becomes the standard matrix for a rotation by θ in the plane perpendicular to $\hat{\mathbf{n}}$; the quadratic form ensures that two distinct unit quaternions in S^3, q and $-q$ correspond to the *same* $SO(3)$ rotation.

◊ Quaternion Moving Frames. ◊

The quadratic form (7) for a general orthonormal $SO(3)$ frame suggests that the Frenet and parallel transport frames and their evolution equations might be expressible directly in terms of a *linear equation* in the quaternion variables. If we identify the columns of (7) as $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$, respectively, we find that differentiation yields

$$\begin{aligned} d\vec{\mathbf{T}} &= 2[A] \cdot [dq] \\ d\vec{\mathbf{N}} &= 2[B] \cdot [dq] \\ d\vec{\mathbf{B}} &= 2[C] \cdot [dq] \end{aligned}$$

where

$$\begin{aligned} [A] &= \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \\ [B] &= \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix} \\ [C] &= \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix} . \end{aligned} \tag{8}$$

The Frenet equations themselves must then take the form

$$\begin{aligned} [A]\cdot[q'] = \vec{\mathbf{T}}' &= v\kappa\vec{\mathbf{N}} \\ [B]\cdot[q'] = \vec{\mathbf{N}}' &= -v\kappa\vec{\mathbf{T}} + v\tau\vec{\mathbf{B}} \\ [C]\cdot[q'] = \vec{\mathbf{B}}' &= -v\tau\vec{\mathbf{N}}\ . \end{aligned}$$

By simply writing out the right-hand sides of these equations and grouping terms, we derive the following fundamental expression, the *quaternion Frenet frame equation*:

$$[q'(s)] = \begin{bmatrix} q'_0 \\ q'_1 \\ q'_2 \\ q'_3 \end{bmatrix} = \frac{1}{2}v \begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ \kappa & 0 & -\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} . \tag{9}$$

This equation has the following key properties:

- The matrix on the right hand side is antisymmetric, so that $q(s)\cdot q'(s) = 0$ by construction. Thus all unit quaternions remain unit quaternions as they evolve by this equation.
- The number of equations has been reduced from nine coupled equations with six orthonormality constraints on a non-simply-connected space to four coupled equations incorporating a single constraint that keeps the solution vector confined to the simply-connected 3-sphere.

Equation (9) allows us to do the same sort of thing we did with Eq. (2):

- *Curvature Specification.* Given $v(s)$, $\kappa(s)$, and $\tau(s)$, where only $\tau(s)$ may vanish, Eq. (9) can be integrated directly to give $q(s)$, which in turn uniquely generates $(\vec{\mathbf{T}}(s), \vec{\mathbf{N}}(s), \vec{\mathbf{B}}(s))$ via Eq. (7).
- *Frame Specification.* If only the 4-vector field $q(s)$ corresponding to a smooth moving frame is specified, a simple differentiation gives us the curvature and torsion for the curve provided we specify $v(s)$.

Similarly, a parallel-transport frame system with $(\vec{\mathbf{N}}_1(s), \vec{\mathbf{T}}(s), \vec{\mathbf{N}}_2(s))$ (in that order) corresponding to columns of Eq. (7) can be shown easily to be completely equivalent to the following the *parallel-transport quaternion frame equation*:

$$[q'(s)] = \begin{bmatrix} q'_0 \\ q'_1 \\ q'_2 \\ q'_3 \end{bmatrix} = \frac{1}{2}v \begin{bmatrix} 0 & -k_2 & 0 & k_1 \\ k_2 & 0 & -k_1 & 0 \\ 0 & k_1 & 0 & k_2 \\ -k_1 & 0 & -k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} . \tag{10}$$

where

$$[B]\cdot[q'] = \vec{\mathbf{T}}' = vk_1\vec{\mathbf{N}}_1 + vk_2\vec{\mathbf{N}}_2$$

$$\begin{aligned}[A] \cdot [q'] = \vec{\mathbf{N}}_1' &= -vk_1 \vec{\mathbf{T}} \\ [C] \cdot [q'] = \vec{\mathbf{N}}_2' &= -vk_2 \vec{\mathbf{T}} \ .\end{aligned}$$

Spinors. We append one parenthetic observation to fill in a gap in the story of quaternion frames. The question to ask is a simple one: if orthogonal matrices act linearly on vectors, and quaternions are like square roots of orthogonal matrices, on what do quaternions *themselves* act linearly? The answer is that quaternions act linearly to generate rotations of *spinors*, which are the subject of an incredibly vast literature in mathematics and physics (see, e.g, (Misner et al. 1973,Cartan 1981)). This is not obviously important for computer graphics, but is interesting as general background knowledge if one is concerned with where quaternion-like geometric descriptions actually fit in the "big mathematical picture." We will perhaps pursue this subject another time.

◊ Conclusion ◊

In order to generate acceptable renderable structures corresponding to curves in a computer graphics scene, we must often thicken the curve to produce a belt, ribbon, or tube. The Frenet-frame and parallel-transport-frame equations provide the mathematical machinery to accomplish that; however, the nine-component equations that result are unwieldy and subject to accumulating errors in the maintenance of the six constraints necessary to reduce the actual number of parameters of the frame to the three Euler angles. The quaternion frame approach greatly improves this situation by reducing the problem to four quaternion frame variables with the single constraint that the quaternions lie on the unit three-sphere. Possible extensions to consider would include a similar treatment of the differential geometry of surfaces.

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◊ Remark ◊

The author has consulted numerous mathematical references and mathematicians, and has thus far been unable to discover an existing formulation of Frenet frames equivalent to the one presented here. Most likely, identical observations were made in the late nineteenth century but have been lost to contemporary mathematical practice. One might surmise that when the compelling formalism for treating moving coordinate frames using the exterior algebra of differential forms (see, e.g., (Flanders 1963)) replaced the tensor formalism for coordinate frames of curves used in classical treatments of differential geometry like (Eisenhart 1960), there was little motivation to seek or utilize other expressions.

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