

# The Undecidability of Iterated Modal Relativization

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## Abstract

In dynamic epistemic logic and other fields, it is natural to consider relativization as an operator taking sentences to sentences. When using the ideas and methods of dynamic logic, one would like to iterate operators. This leads to *iterated relativization*. We are also concerned with the transitive closure operation, due to its connection to common knowledge. We show that three fragments of the logic of iterated relativization and transitive closure are  $\Sigma_1^1$ -complete. Of these, two fragments do not include transitive closure. We also show that the question of whether a sentence in these fragments has a finite (tree) model is  $\Sigma_1^0$ -complete. These results go via reduction to problems concerning domino systems.

## 1 Introduction

If  $\varphi$  is a sentence of some language  $\mathcal{L}$  and  $A$  is an  $\mathcal{L}$ -structure, then we write  $A^\varphi$  for the submodel of  $A$  determined by  $\{a \in A : (A, a) \models \varphi\}$ , the set of points of  $A$  satisfying  $\varphi$ . This definition applies for a wide variety of languages  $\mathcal{L}$ ; we shall be interested in classical modal logic and some related languages. Specifically, we consider  $\mathcal{L}(\text{rel})$ , the *extension of modal logic by relativization*: this language has sentences  $[\varphi]\psi$  with the semantics

$$(A, a) \models [\varphi]\psi \quad \text{iff} \quad (a \in A^\varphi \text{ implies } (A^\varphi, a) \models \psi).$$

We define  $\langle\varphi\rangle$  to be dual of  $[\varphi]$ . So  $\langle\varphi\rangle\psi$  is  $\neg[\varphi]\neg\psi$ . That is,

$$(A, a) \models \langle\varphi\rangle\psi \quad \text{iff} \quad (a \in A^\varphi \text{ and } (A^\varphi, a) \models \psi),$$

and we also see that

$$(A, a) \models \langle\varphi\rangle\psi \quad \text{iff} \quad ((A, a) \models \varphi \text{ and } (A, a) \models [\varphi]\psi).$$

This language  $\mathcal{L}(\text{rel})$  was first proposed (with different names) by Plaza [15] and independently later by Gerbrandy [9, 10].  $\mathcal{L}(\text{rel})$  is important in connection with the modeling of *public announcements* in the multi-agent setting. But this paper settles technical questions and is therefore less interested in conceptual matters, so we shall not motivate this or other logical

systems. Getting back to  $\mathcal{L}(\text{rel})$ , its originators noted that  $\mathcal{L}(\text{rel})$  is equivalent in expressive power to ordinary modal logic. One way to see this is to define a translation  $t : \mathcal{L}(\text{rel}) \rightarrow \mathcal{L}$ . In stating this translation, we introduce some notation. Let  $\psi$  and  $\varphi$  be modal sentences, and suppose that  $\varphi$  is in negation normal form (i.e., all negations apply only to atomic sentences). Then we define  $\varphi^\psi$  by the following recursion:  $p^\psi = p$ ,  $(\neg p)^\psi = \neg p$ ,  $(\varphi_1 \wedge \varphi_2)^\psi = \varphi_1^\psi \wedge \varphi_2^\psi$ ,  $(\varphi_1 \vee \varphi_2)^\psi = \varphi_1^\psi \vee \varphi_2^\psi$ ,  $(\Box\varphi)^\psi = \Box(\psi \rightarrow \varphi^\psi)$ , and  $(\Diamond\varphi)^\psi = \Diamond(\psi \wedge \varphi^\psi)$ . (For  $\varphi$  not in negation normal form, we set  $\varphi^\psi = (\text{nnf } \varphi)^\psi$ , where  $\text{nnf } \varphi$  is the negation normal form of  $\varphi$ .)

**Proposition 1** *Let  $\psi$  and  $\varphi$  be modal sentences. Let  $A$  be a model, and let  $a \in A^\psi$ . Then  $(A^\psi, a) \models \varphi$  iff  $(A, a) \models \varphi^\psi$ .*

**Proof** By induction on  $\varphi$  in negation normal form. Here, for example, is the induction step for  $\Box\varphi$ , assuming the result for  $\varphi$ . Let  $a \in A^\psi$ . Assume first that  $(A^\psi, a) \models \Box\varphi$ . Then for all  $b \in A^\psi$  such that  $a \rightarrow b$ , we have by our induction hypothesis that  $(A, a) \models \varphi^\psi$ . In other words,  $(A, a) \models \Box(\psi \rightarrow \varphi^\psi)$ . That is  $(A, a) \models (\Box\varphi)^\psi$ . The converse is similar.  $\dashv$

Now we define the translation  $t$  of  $\mathcal{L}(\text{rel})$  to  $\mathcal{L}$ . The main clause is  $([\varphi]\psi)_t = \varphi_t \rightarrow (\psi_t)^{\varphi_t}$ . An induction using Proposition 1 shows that this map  $t$  preserves the semantics. And another induction shows that each  $\varphi_t$  is a purely modal sentence. Our conclusion at this point is that adding relativization to modal logic alone does not increase expressive power.

Things get more interesting when one adds further constructs. The first is the *common knowledge* (or *reflexive-transitive closure*) operator  $\Box^*$ , with the semantics

$$(A, a) \models \Box^* \varphi \quad \text{iff} \quad \text{for all } b \text{ such that } a \xrightarrow{*} b, (A, b) \models \varphi$$

Here  $\xrightarrow{*}$  is the reflexive-transitive closure of the accessibility relation of  $A$ . Call the resulting language  $\mathcal{L}(\text{rel}, \Box^*)$ . In this case, the relevant results in this direction may be found in Baltag, Moss, and Solecki [1].  $\mathcal{L}(\text{rel}, \Box^*)$  is more expressive than modal logic, and indeed more expressive than  $\mathcal{L}(\Box^*)$ ; i.e., modal logic with the transitive closure operator  $\Box^*$  added. In particular, one cannot express  $[q]\Box^*p$  in  $\mathcal{L}(\Box^*)$ . But every sentence of  $\mathcal{L}(\text{rel}, \Box^*)$  is effectively equivalent to a sentence of propositional dynamic logic (PDL). This immediately implies the finite model property and indeed the decidability of  $\mathcal{L}(\text{rel}, \Box^*)$ . Furthermore, there are sound and complete logical systems for this notion. Even more, it is possible to extend this positive result by generalizing the notion of relativization to many other types of “epistemic actions” on models. It would take us too far afield to get into this matter here, but one should see [2].

In this paper, we go one step further. We consider the *iterated relativization operator*  $[\varphi^*]$ . The semantics is given by

$$(A, a) \models [\varphi^*]\psi \quad \text{iff} \quad (A, a) \models [\varphi]^n \psi \text{ for all } n.$$

So we also have a dual operation  $\langle \varphi^* \rangle$ , and then

$$(A, a) \models \langle \varphi^* \rangle \psi \quad \text{iff} \quad (A, a) \models \langle \varphi \rangle^n \psi \text{ for some } n.$$

It is also convenient to note that

$$(A, a) \models [\varphi^*](\varphi \wedge \psi) \quad \text{iff} \quad (A, a) \models \langle \varphi \rangle^n \psi \text{ for all } n.$$

We define the logics  $\mathcal{L}(\text{rel}, \text{rel}^*)$  and  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  in the obvious ways.

**Examples** Throughout this paper, we write  $D$  for  $\diamond\text{True}$ , so  $[D^*]$  means  $[(\diamond\text{True})^*]$ . Semantically, the operation of relativizing by  $\diamond\text{True}$  removes those points of a model which have no children. We use the letter  $D$  because this operation reminds us of the Cantor-Bendixson derivative of a set of real numbers, wherein one removes the isolated points. Indeed, we write  $A'$  for  $A^D$  and define  $A^{(n)}$  by  $A^{(0)} = A$  and  $A^{(n+1)} = (A^{(n)})'$ .

Our first example of the iteration of the derivative operation is  $\langle D^* \rangle \Box \text{False}$ . By induction on  $n$ , we see that  $(A, a) \models \langle D \rangle^n \Box \text{False}$  iff the longest path in  $A$  beginning at  $a$  has length exactly  $n$ . It follows that  $(A, a) \models \langle D^* \rangle \Box \text{False}$  iff there is some  $n$  such that all paths in  $A$  starting from  $a$  are of length at most  $n$ .

$[D^*] \diamond \text{True}$  is then the dual of  $\langle D^* \rangle \Box \text{False}$ . It holds at a point  $a$  if there is no bound on the lengths of paths from  $a$ .

It was observed in [1] that  $\mathcal{L}(\text{rel}, \text{rel}^*)$  does not have the finite model property because  $[D^*] \diamond \text{True}$  is satisfiable but only by an infinite model. Nevertheless, the second author conjectured that the satisfiability problem for this logic was still decidable. This conjecture was refuted by the first author. The results of this paper show this in several ways. Specifically, we show that the satisfiability problem for the following fragments of  $\mathcal{L}(\text{rel}, \text{rel}^*)$  and  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  are  $\Sigma_1^1$ -complete:

1. The fragment generated by  $[D^*]$ ,  $\Box^*$ ,  $\Box$ ,  $\wedge$ , and  $\neg$ .
2. The fragment generated by two iterated relativization operators  $[D_x^*]$  and  $[D_y^*]$ , in addition to  $\Box$ ,  $\wedge$ ,  $\neg$ , and atomic sentences. Here  $D_x$  and  $D_y$  are two fixed modal sentences.
3. The fragment generated by  $[D^*]$ , arbitrary modal relativizations,  $\Box$ ,  $\wedge$ ,  $\neg$ , and atomic sentences.

One difference between these results is that in the first and third, we iterate only one very simple relativization  $D$  but we also add either the transitive closure operator  $\Box^*$  or else complex modal relativizations. The second instead calls on the iteration of two particular (purely modal) sentences. Also, the first fragment does not use atomic sentences.

We also prove that the problems of deciding whether a sentence in  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  has a finite model, or a finite tree model, are  $\Sigma_1^0$ -complete.

## 1.1 Domino systems

All of our  $\Sigma_1^1$ -hardness results go via reduction to the tiling problem for recurring domino systems. So we recall the basic definitions. The original paper on this is Harel [11], and the book by Blackburn, de Rijke, and Venema [4] has applications to modal logic.

**Definition** A *domino system* is a tuple  $\mathcal{D} = (\text{Dominoes}, H, V)$ , where  $\text{Dominoes}$  is a finite set, and  $H, V \subseteq \text{Dominoes} \times \text{Dominoes}$ .

The *first quadrant* is the set  $Q = \mathbb{N} \times \mathbb{N}$ . A *tiling of  $Q$  by  $\mathcal{D}$*  is a function  $t : Q \rightarrow \text{Dominoes}$ . The tiling  $t$  is *proper* if for all  $n, m \in \mathbb{N}$ ,

1.  $H(t(n, m), t(n+1, m))$ .
2.  $V(t(n, m), t(n, m+1))$ .

A *recurring domino system* is a pair  $(\mathcal{D}, d_0)$  with  $d_0 \in \text{Dominoes}$ ; a *proper tiling* of  $(\mathcal{D}, d_0)$  is a tiling of  $Q$  by  $\mathcal{D}$  in which  $t(n, 0) = d_0$  for infinitely many  $n$ .

We shall use the result of Harel [11] on the problem of deciding whether a recurring domino system  $(\mathcal{D}, d_0)$  has a proper tiling: this problem is  $\Sigma_1^1$ -complete.

In Section 7, we prove a result by reduction to the periodicity problem for domino systems. We recall the relevant definitions later.

## 1.2 Translating $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$ into the Modal Iteration Calculus

In [6], Dawar, Grädel, and Kreutzer introduced a *Modal Iteration Calculus MIC*. Among other things, they showed that the satisfiability problem for *MIC* is undecidable. It was suggested by van Benthem [16] that adding iterated relativization to modal logic gives a fragment of *MIC*. We want to discuss this result, since *MIC* is the smallest previously-studied logical system containing  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  that we are aware of.<sup>1</sup>

*MIC* is defined by adding two things to the syntax of modal logic: set variables  $X_1, X_2, \dots$ , and a formula constructing operator

$$\mathbf{ifp}(X_j : X_1 \leftarrow \varphi_1, \dots, X_k \leftarrow \varphi_k). \quad (1)$$

Here the  $\varphi$ 's are again formulas and  $1 \leq j \leq k$ . We understand  $\mathbf{ifp}$  to be a variable binding operator, and  $X_1, \dots, X_k$  are bound in (1). We define the semantics of all formulas of *MIC*. Assume that the free set variables of  $\varphi$  are among  $Y_1, \dots, Y_n$  and that  $B_1, \dots, B_n \subseteq A$  and  $a \in A$ . We define  $(A, B_1, \dots, B_n, a) \models \varphi$  by recursion on  $\varphi$ . For example,  $(A, B_1, \dots, B_n, a) \models Y_j$  iff  $a \in B_j$ . The main clause is for  $\mathbf{ifp}$ -formulas as in (1). Let the free set variables of each  $\varphi_i$  be included in the list  $Y_1, \dots, Y_n, X_1, \dots, X_k$ , and let  $B_1, \dots, B_n \subseteq A$ . Define *iterates*  $S_i^\alpha \subseteq A$  for  $\alpha$  an ordinal and  $1 \leq i \leq k$  by recursion on  $\alpha$ : let  $S_i^0 = \emptyset$ ,

$$S_i^{\alpha+1} = S_i^\alpha \cup \{a \in A : (A, B_1, \dots, B_n, S_1^\alpha, \dots, S_k^\alpha, a) \models \varphi_i\},$$

and for  $\lambda$  a limit ordinal,  $S_i^\lambda = \bigcup_{\beta < \lambda} S_i^\beta$ . For each  $i$ , the sequence  $S_i^\alpha$  is an increasing sequence of subsets of  $A$ , and we set  $S_i^*$  to be its eventual value. Then

$$(A, B_1, \dots, B_n, a) \models \mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i}) \quad \text{iff} \quad a \in S_j^*.$$

We now discuss the relation of the language  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  to *MIC*. As it happens, van Benthem in [16] modified the semantics of sentences of the form  $[\varphi^*]\psi$  to use iteration over *all ordinals* rather than only over the natural numbers. To get an exact match, we need to do a bit more.

Suppose that  $\varphi(X)$  is a formula with just  $X$  free. Consider the formula  $\mathbf{ifp}(X : X \leftarrow \varphi)$ . Let  $U, V$ , and  $W$  be new variables. Define

$$\mathbf{ifp}_\omega(X : X \leftarrow \varphi) = \mathbf{ifp}(V : U \leftarrow V, V \leftarrow (\neg W \wedge \varphi(\emptyset)) \vee (W \wedge \varphi(V) \wedge \neg \varphi(U)), W \leftarrow \text{True}). \quad (2)$$

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<sup>1</sup>The results of this section are not needed in later sections of the paper.

Fix a model  $A$ , and consider the interpretation of (2) in  $A$ . We write  $S^\alpha$  for the iterates of the original system and  $U^\alpha$ ,  $V^\alpha$  and  $W^\alpha$  for the iterates of the new system. Clearly  $U^1 = \emptyset$ ,  $V^1 = \varphi(\emptyset) = S^1$ ,  $W^0 = \emptyset$  and for  $n \geq 1$ ,  $W^n = A$ . An induction on the natural number  $n$  shows that  $U^{n+1} = V^n = S^n$  and  $V^{n+1} = S^{n+1}$ . We have already checked this for  $n = 0$ . Assume for  $n$  that  $U^{n+1} = V^n = S^n$  and  $V^{n+1} = S^{n+1}$ . Then we see that  $U^{n+2} = V^{n+1} = S^{n+1}$  (easily), and also that

$$\begin{aligned} V^{n+2} &= V^{n+1} \cup (\varphi(S^{n+1}) \setminus \varphi(S^n)) \\ &= S^{n+1} \cup (\varphi(S^{n+1}) \setminus \varphi(S^n)) \\ &= S^{n+2} \end{aligned}$$

It follows that  $U^\omega = V^\omega = S^\omega$ . And then for  $\alpha \geq \omega$  we see that  $U^\alpha = V^\alpha = S^\omega$ . The inductive step is that  $V^{\alpha+1} = V^\alpha \cup (\varphi(S^\omega) \setminus \varphi(S^\omega)) = S^\omega \cup \emptyset = S^\omega$ .

Our conclusion here is that for all formulas  $\varphi(X)$  in the language, there is another formula  $\mathbf{ifp}_\omega(X : X \leftarrow \varphi)$  (as in (2)) whose interpretation in any model  $A$  is the  $\omega$ -th inflationary iterate  $S^\omega$  in  $A$ .

Furthermore, we note that  $MIC$  is closed under relativization in the following sense. If  $\varphi$  is a formula of  $MIC$  and  $\psi$  is a sentence of it, we define  $\varphi^\psi$  by the same recursion as earlier, except that we also add  $X^\psi = X \wedge \psi$ , and also

$$\mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i})^\psi = \mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i^\psi})$$

From this, we define a translation  $t$  of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  into the sentences of  $MIC$ . The main clauses are

$$\begin{aligned} (\diamond^* \varphi)_t &= \mathbf{ifp}(X : X \leftarrow \varphi_t \vee \diamond X) \\ ([\varphi] \psi)_t &= \mathbf{ifp}(Y : X \leftarrow \varphi_t, Y \leftarrow Z \wedge \psi_t^X, Z \leftarrow \text{True}) \\ (\langle \psi^* \rangle \varphi)_t &= \mathbf{ifp}_\omega(X : X \leftarrow \varphi_t^{-Y}, Y \leftarrow \neg(\psi_t^{-Y})) \end{aligned}$$

We check that this works for the sentences  $\langle \psi^* \rangle \varphi$ , assuming that it works for  $\psi$  and  $\varphi$ . We write  $X^n$  for the  $n$ -th iteration of  $X$ , and similarly for  $Y$ . Fix a model  $A$ , and define subsets and submodels  $A_n$  by  $A_0 = A$  and  $A_{n+1} = (A_n)^\psi$ . We first check by induction that  $Y^n = -A_n$ . This is clear for  $n = 0$ . Assuming that  $Y^n = -A_n$ , we have

$$\begin{aligned} Y^{n+1} &= -( \psi^{A_n} ) \cup Y^n \\ &= -A_{n+1} \cup (-A_n) \\ &= -A_{n+1} \end{aligned}$$

And then we also see that  $X^n = \varphi^A \cup \varphi^{A_1} \cup \dots \cup \varphi^{A_n}$ . So we are after  $X^\omega = \bigcup_{n < \omega} X^n$ . Since we use  $\mathbf{ifp}_\omega$ , this is given by our formula above.

**Implications** As a result of this translation and the  $\Sigma_1^1$ -hardness results to come, we have an improvement of Theorem 3.5 of [6]. That result exhibits an encoding of first-order arithmetic into the satisfiability problem for  $MIC$ . We also get the  $\Sigma_1^0$ -completeness of the finite satisfiability problem, and this appears to be new. Of perhaps more importance is that we have shown that some small fragments of  $MIC$  are undecidable. So the search for expressive fragments of  $MIC$  that go beyond the modal  $\mu$ -calculus will have to involve logics formed from different principles than the ones we study here.

## 2 The satisfiability problem for $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$ is in $\Sigma_1^1$

We sketch a proof that the set  $\{\chi \in \mathcal{L}(\text{rel}, \text{rel}^*, \Box^*) : \chi \text{ is satisfiable}\}$  is  $\Sigma_1^1$ . This is in contrast for *MIC*, where the estimate of [6] gives  $\Sigma_2^1$ . What accounts for the difference is precisely that our semantics of the  $[\varphi^*]\psi$  construct involves iteration over *numbers* rather than arbitrary ordinals. Were one to modify the semantics (as van Benthem does in [16]), then the  $\Sigma_1^1$  upper bound is presumably false.

It will be useful to take **True** to be a primitive symbol of  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$ . It is also worth remarking that the syntax of  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  allows sentences of the form  $[\varphi^*]\psi$ . We often use abbreviations  $[\varphi]^m\psi$ , but these are exactly that: abbreviations. For example,  $[p]^3q$  is an abbreviation for  $[p][p][p]q$ .

We define a map *pre* from  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)^*$ , the set of sequences from  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$ , to  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$ :  $\text{pre}(\lambda) = \text{True}$ , where  $\lambda$  is the empty sequence,  $\text{pre}(\varphi) = \varphi$ ; and for  $n \geq 2$ ,  $\text{pre}(\varphi_1, \dots, \varphi_n) = \varphi_1 \wedge [\varphi_1]\text{pre}(\varphi_2, \dots, \varphi_n)$ . (*pre* stands for “precondition”. The name comes from [2], where a generalization of this function plays an important role.)

**Lemma 2** *Let  $A$  be any model, and define relations  $R \subseteq A \times A$ ,  $P \subseteq \omega \times A$ , and  $X \subseteq \mathcal{L}(\text{rel}, \text{rel}^*, \Box^*) \times A$  as follows:  $R$  is the accessibility relation of  $A$ ,  $P(k, x)$  iff  $(A, x) \models p_k$ , and  $X(\varphi, x)$  iff  $(A, x) \models \varphi$ . Then  $X(\text{True}, x)$  holds for all  $x$ . And each instance of the following biconditionals also holds:*

$$\begin{aligned}
X([\varphi_1] \cdots [\varphi_n] p_k, x) &\leftrightarrow X(\text{pre}(\varphi_1, \dots, \varphi_n), x) \rightarrow P(k, x) \\
X([\varphi_1] \cdots [\varphi_n] \neg\psi, x) &\leftrightarrow X(\text{pre}(\varphi_1, \dots, \varphi_n), x) \rightarrow \neg X([\varphi_1] \cdots [\varphi_n] \psi, x) \\
X([\varphi_1] \cdots [\varphi_n] (\psi_1 \wedge \psi_2), x) &\leftrightarrow X([\varphi_1] \cdots [\varphi_n] \psi_1, x) \wedge X([\varphi_1] \cdots [\varphi_n] \psi_2, x) \\
X([\varphi_1] \cdots [\varphi_n] \Box\psi, x) &\leftrightarrow X(\text{pre}(\varphi_1, \dots, \varphi_n), x) \rightarrow (\forall y)(R(x, y) \rightarrow X([\varphi_1] \cdots [\varphi_n] \psi, y)) \\
X([\varphi_1] \cdots [\varphi_n] \Box^m\psi, x) &\leftrightarrow (\forall m) X([\varphi_1] \cdots [\varphi_n] \Box^m\psi, x) \\
X([\varphi_1] \cdots [\varphi_n] [\psi^*]\chi, x) &\leftrightarrow (\forall m) X([\varphi_1] \cdots [\varphi_n] [\psi]^m\chi, x)
\end{aligned} \tag{3}$$

Moreover, each sentence of  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  other than **True** is an instance of some (unique) sentence occurring on the left-hand side of one of these biconditionals.

**Proof** All of the equivalences are special cases of results from [2].

The “moreover” assertion is checked by induction on  $\varphi$ . If  $\varphi$  is of the form  $p_k$ ,  $\neg\psi$ ,  $\psi_1 \wedge \psi_2$ ,  $\Box\psi$ ,  $\Box^m\psi$ , or  $[\psi^*]\chi$ , then we may take  $n = 0$ . If  $\varphi$  is of the form  $[\psi]\chi$ , then by induction hypothesis,  $\chi$  is an instance of the left side of one of the biconditionals in (3). And then so is  $[\psi]\chi$ .  $\dashv$

**Lemma 3** *There is a wellfounded relation  $<$  on  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  such that if  $\varphi_L$  occurs on the left-hand side of one of the biconditionals in (3), and  $\varphi_R$  occurs on the right-hand side of the same biconditional, then  $\varphi_R < \varphi_L$ .*

**Remark** Here is an example of what we mean in this lemma, based on the fourth biconditional above. For all  $\varphi_1, \dots, \varphi_n, \psi$ , we have  $\text{pre}(\varphi_1, \dots, \varphi_n) < [\varphi_1] \cdots [\varphi_n] \Box\psi$ , and we also have  $[\varphi_1] \cdots [\varphi_n] \psi < [\varphi_1] \cdots [\varphi_n] \Box\psi$ .

Concerning the last biconditional, we mean that for all  $m$ ,

$$[\varphi_1] \cdots [\varphi_n] [\psi]^m\chi < [\varphi_1] \cdots [\varphi_n] [\psi^*]\chi.$$

**Proof** We obtain  $<$  as a lexicographic partial order (LPO) of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$ . For background on LPO, see the surveys by Dershowitz [7] and Plaisted [14]. We regard  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  as an algebra of terms, using the constructors  $\text{True}, p_1, \dots, p_k, \dots, \neg, \wedge, \square, \square^*, \text{rel}$ , and  $\text{rel}^*$ . The latter are two-place function symbols:  $\text{rel}(\varphi, \psi)$  is an alternative for  $[\varphi]\psi$ , and  $\text{rel}^*(\varphi, \psi)$  is an alternative for  $[\varphi^*]\psi$ . In this proof, we shall let  $f$  and  $g$  range over these symbols. We define the ordering  $<$  on the function symbols of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  to be the smallest transitive relation containing

$$\text{True} < p_k, \wedge, \neg, \square < \square^* < \text{rel} < \text{rel}^*.$$

This wellfounded relation generates an LPO on  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$ ; as usual we denote this ordering by  $<$  as well. Concretely, this is the smallest relation such that

**(LPO1)** If  $(t_1, \dots, t_n) < (s_1, \dots, s_n)$  in the lexicographic ordering on  $n$ -tuples, and if  $t_j < f(s_1, \dots, s_n)$  for  $1 \leq j \leq n$ , then  $f(t_1, \dots, t_n) < f(s_1, \dots, s_n)$ .

**(LPO2)** If  $t \leq s_i$  for some  $i$ , then  $t < f(s_1, \dots, s_n)$ .

**(LPO3)** If  $g < f$  and  $t_i < f(s_1, \dots, s_n)$  for all  $i \leq m$ , then  $g(t_1, \dots, t_m) < f(s_1, \dots, s_n)$ .

It is a general result on LPO that  $<$  is wellfounded. It also has the subterm property: if  $\varphi$  is a strict subsentence of  $\psi$ , then  $\varphi < \psi$ .

We check by induction on  $n \geq 1$  that

$$\text{pre}(\varphi_1, \dots, \varphi_n) < [\varphi_1] \cdots [\varphi_n] \psi.$$

This is where we use the assumption that  $\wedge < \text{rel}$ . Further inductions show that  $[\psi]^m \chi < [\psi^*] \chi$  for all  $m$ , and that  $\square^m \psi < \square^* \psi$  for all  $m$ . (LPO3) is used in these, as are the assumptions that  $\square < \square^*$  and  $\text{rel} < \text{rel}^*$ . These preliminary remarks establish the base case ( $n = 0$ ) of an induction on  $n$  that if  $\varphi_L$  is the left-hand side of one of the biconditionals in (3), and  $\varphi_R$  is on the right side of the same biconditional, then  $\varphi_R < \varphi_L$ . The induction step follows easily from (LPO1), (LPO2) and the subterm property.  $\dashv$

In the statement and proof of our result below, we assume a “nice” coding of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  by a subset of  $\omega$ . We need to know that several functions are recursive in the codes. These include  $\varphi_1, \dots, \varphi_n \mapsto \text{pre}(\varphi_1, \dots, \varphi_n)$ ;  $n, \varphi, \psi \mapsto [\varphi]^n \psi$ ; and  $n, \varphi \mapsto \square^n \varphi$ .

**Theorem 4**  $\{\chi \in \mathcal{L}(\text{rel}, \text{rel}^*, \square^*) : \chi \text{ is satisfiable}\}$  is  $\Sigma_1^1$ .

**Proof** We claim that a sentence  $\chi$  is satisfiable if there are sets  $R \subseteq \omega \times \omega$ ,  $P \subseteq \omega \times \omega$ , and  $X \subseteq \mathcal{L}(\text{rel}, \text{rel}^*, \square^*) \times \omega$  such that  $X(\text{True}, y)$  holds for all  $y$ , and all instances of the biconditionals of (3) hold, and that there is some  $x$  such that  $X(\chi, x)$ .

In one direction, we appeal to a result stated in [6] that says that if  $\chi$  is satisfiable, then it has a countable model  $A$ . (In [6] this was stated for *MIC*, based on an extension of a result originally shown by Flum [8]: the logic *LFP* has the Löwenheim-Skolem property. Since we know that  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  is a sublogic of *MIC*, we now have this property for it.) So we may assume that the universe of  $A$  is  $\omega$ , and now the rest follows from Lemma 2.

In the other direction, fix a sentence  $\chi$ . Assume that we have  $R$ ,  $P$ , and  $X$ . Let  $A$  be the model with universe  $\omega$  whose structure is given by  $R$  and  $P$  in the obvious way. Let  $<$  be a wellfounded relation as in Lemma 3. We argue by induction on  $<$  that for all  $\varphi$  in the field of

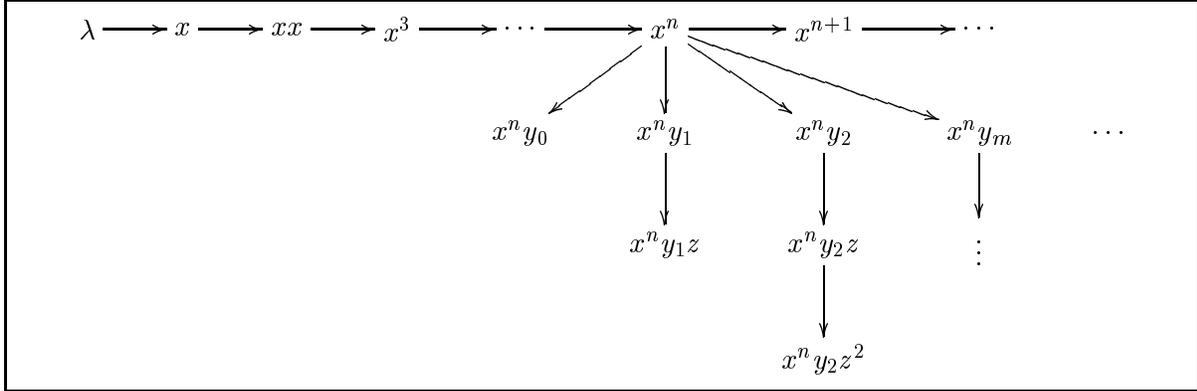


Figure 1: The frame  $F$  used in work on the fragment for  $[D^*]$ ,  $\square^*$ ,  $\square$ , and atomic sentences.

$<$ ,  $X(\varphi, x)$  iff  $(A, x) \models \varphi$ . The induction is an easy consequence of Lemma 2. We also use the last assertion in Lemma 2 to know that all sentences belong to the field of  $<$ . In particular, the sentence  $\chi$  with which we began belongs. And from this, the claim easily follows.

At this point, we have shown our claim. We conclude by noting that our condition on  $R$ ,  $P$  and  $X$  in the first paragraph of this proof is arithmetic: it involves universal quantification over sequences from  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  as well as application of some functions which we assume to be recursive. It follows that our equivalent formulation of satisfiability is  $\Sigma_1^1$ .  $\dashv$

### 3 $[D^*]$ , $\square^*$ , $\square$ , and atomic sentences

In this section, we prove that satisfiability is  $\Sigma_1^1$ -complete for the language with  $[D^*]$ ,  $\square^*$ ,  $\square$  and *atomic sentences*. We strengthen this in Section 4 to eliminate the atomic sentences. That is, we shall prove the following result:

**Theorem 5** *To every recurring domino system  $(\mathcal{D}, d_0)$ , we can effectively associate a sentence  $\varphi_{\mathcal{D}, d_0}$  of the language of  $[D^*]$ ,  $\square^*$ ,  $\square$ , True, and the boolean connectives such that the following are equivalent:*

1. *There is a proper tiling of  $Q$  by  $(\mathcal{D}, d_0)$ .*
2.  *$\varphi_{\mathcal{D}, d_0}$  is satisfiable.*

**Corollary 6** *The satisfiability problem for the fragment of Theorem 5 is  $\Sigma_1^1$ -complete.*

Fix a recurring domino system  $(\mathcal{D}, d_0)$ . We take a language with atomic sentences corresponding to the (finitely many) dominoes. Concretely, let  $d$  correspond to  $d$ .

**The intended frame for the first quadrant** Let  $x, y_0, y_1, \dots, y_m, \dots$ , and  $z$  be different symbols. We construct a frame  $F$  from a subset of  $\{x, z, y_0, \dots, y_m, \dots\}^*$ , the set of words on our symbols. The set of worlds of  $F$  is

$$\{x^n : 0 \leq n\} \cup \{x^n y_m z^p : 0 \leq n \text{ and } p \leq m\}.$$

stalk	$[D^*] \diamond \text{True}$
$\chi_d$	$d \wedge \square \text{False}$
$\varphi_{\mathcal{D}}$	stalk $\wedge \square^*(\text{stalk} \rightarrow \diamond \text{stalk})$ $\wedge \square^*[D^*](\text{stalk} \rightarrow \diamond \bigvee_d \chi_d)$ $\wedge \square^*[D^*](\text{stalk} \rightarrow \neg \bigvee_{\neg H(d,d')} (\diamond \chi_d \wedge \diamond \chi_{d'}))$ $\wedge \square^*[D^*](\text{stalk} \rightarrow \neg \bigvee_{\neg V(d,d')} (\diamond \chi_d \wedge \diamond \langle D \rangle \chi_{d'}))$
recurring( $d_0$ )	$\square^*(\text{stalk} \rightarrow \diamond^*(\text{stalk} \wedge \diamond \chi_{d_0}))$
$\varphi_{\mathcal{D}, d_0}$	$\varphi_{\mathcal{D}} \wedge \text{recurring}(d_0)$

Figure 2: Abbreviations in the fragment for  $[D^*]$ ,  $\square^*$ ,  $\square$ , and atomic sentences.

We use standard notation here; for example,  $x^5 y_3 z^0$  here really is the word  $xxxxxy_3$ . Note that  $F$  contains the empty word  $\lambda$ . The accessibility relation is given by  $x^n \rightarrow x^{n+1}$ ,  $x^n \rightarrow x^n y_m$ , and  $x^n y_m z^p \rightarrow x^n y_m z^{p+1}$ . A picture of  $F$  may be found in Figure 1.

The points of the form  $x^n$  are called the *stalk* of  $F$ , since if one rotated our picture  $90^\circ$  it would be a single stalk with one branch of each finite length coming off of the each point of the stalk. Note that  $(F, x^n y_m z^p) \models \square \text{False}$  iff  $p = m$ . And the derivative  $F'$  of  $F$  is isomorphic as a frame to  $F$ , via the map  $x^n \mapsto x^n$  and  $x^n y_{m+1} z^{p+1} \mapsto x^n y_m z^p$ .

**Tilings give models** Our intention is that  $x^n y_m$  is a surrogate for the point  $(n, m)$  of the first quadrant. To get a model from a frame we need only specify the semantics of our atomic sentences. Let  $t : Q \rightarrow \mathcal{D}$  be a tiling of the first quadrant. We construct a model  $F_t$  from  $t$  (and the underlying frame  $F$  described above) by declaring the atomic sentence  $d$  to be true at  $x^n y_m$  iff  $t(n, m) = d$ . No other atomic sentences are true anywhere else.

**The sentence  $\varphi_{\mathcal{D}}$**  In Figure 2 we list several sentences used in our this section. We begin with stalk, shorthand for  $[D^*] \diamond \text{True}$ . Note that  $\models \text{stalk} \rightarrow [D^*] \text{stalk}$ .

The sentence  $\chi_d$  will be our sentence saying of a node that it codes the domino  $d$ . But we must use derivatives to associate squares in the quadrant to the points of  $F$ .

We now consider the sentence  $\varphi_{\mathcal{D}}$ . In the third clause, we mean to take disjunctions over all pairs  $(d, d')$  such that  $\neg H(d, d')$ . Similarly for the last clause. That last clause may also be written without  $\langle D \rangle$  as

$$\square^*[D^*] \left( \text{stalk} \rightarrow \neg \bigvee_{\neg V(d,d')} (\diamond \chi_d \wedge \diamond (d' \wedge \diamond \text{True} \wedge \square \square \text{False})) \right).$$

**The intended models work** We next check that  $(F_t, \lambda) \models \varphi_{\mathcal{D}}$ , where  $\lambda$  again is the empty word. Recall that the stalk in  $F_t$  and in all its derivatives  $F_t^{(m)}$  is the set of points of the form  $x^n$  for some  $n$ . This implies the first condition at the empty word  $\lambda$ . For the second, an induction on  $m$  shows that  $x^n y_k z^p \in F_t^{(m)}$  iff  $k - p \geq m$ , and also that for  $k \geq m$ ,

$$(F_t^{(m)}, x^n) \models \diamond^{k-m} \text{True} \wedge \neg \diamond^{k-m+1} \text{True}.$$

(when  $k = m$ , we intend that  $\diamond^0 \text{True} = \text{True}$ ). As a result,

$$(F_t^{(m)}, x^n) \models \diamond \chi_d \quad \text{iff} \quad d = t(n, m). \quad (4)$$

This implies the second clause of  $\varphi_{\mathcal{D}}$ . And more crucially, the properness of the tiling  $t$  implies the last two clauses in  $\varphi_{\mathcal{D}}$ . (Indeed, the last two clauses could even be strengthened by dropping the mention of stalk. The point is that the only points with children satisfying any  $\chi_d$  sentence are the points on the stalk. But the formulation as in Figure 2 will be needed in Section 4.)

**Any model of  $\varphi_{\mathcal{D}}$  gives a proper tiling** The significant direction is to show that models of  $\varphi_{\mathcal{D}}$  gives proper tilings. There is a slightly stronger result that we use in the next section.

**Lemma 7** *Let stalk be any sentence so that  $\models \text{stalk} \rightarrow [D^*]\text{stalk}$ . Let  $\chi_d$  be any sentences for  $d \in D$ . Let  $\varphi_{\mathcal{D}}$  be as in Figure 2, using stalk and  $\chi_d$ . Let  $(A, a_0)$  be an arbitrary model of  $\varphi_{\mathcal{D}}$ . There is a sequence in  $A$*

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n \rightarrow \cdots$$

with  $a_n \models \text{stalk}$  for all  $n$ . Moreover,

1. There is a function  $t : Q \rightarrow \mathcal{D}$  with the property that for  $(n, m) \in Q$ ,  $(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)}$ .
2. Each such function  $t$  is a proper tiling of  $Q$  by  $\mathcal{D}$ .

**Remark** This lemma says that every model of  $\varphi_{\mathcal{D}}$  gives a proper tiling in all possible ways. It does not say that all models of  $\varphi_{\mathcal{D}}$  are related in any way to the intended models, or that all models of  $\varphi_{\mathcal{D}}$  give tilings in a unique or canonical way. For satisfiability, we need only know that proper tilings exist. This is the content of the present lemma.

**Proof** The sequence  $a_0 \rightarrow a_1 \rightarrow \cdots$  exists by the first two clauses ( $\text{stalk} \wedge \square^*(\text{stalk} \rightarrow \diamond \text{stalk})$ ) of  $\varphi_{\mathcal{D}}$ . The condition on the sentence stalk shows that  $a_n \models [D^*][D^*]\diamond \text{True}$  as well; i.e.,  $a_n \models [D^*]\text{stalk}$ . Then the second clause of  $\varphi_{\mathcal{D}}$  insures that for each  $n$  and  $m$  that there will be some  $d \in D$  so that  $(A^{(m)}, a_n) \models \diamond \chi_d$ . That is, some tiling  $t$  exists which satisfies the condition  $(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)}$ .

We turn to the second part. First, consider  $t(n, m)$  and  $t(n+1, m)$ . By definition of  $t$ ,  $(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)}$  and  $(A^{(m)}, a_{n+1}) \models \diamond \chi_{t(n+1, m)}$ . Since  $a_n \rightarrow a_{n+1}$ , we have

$$(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)} \wedge \diamond \diamond \chi_{t(n+1, m)}.$$

And since  $(A, a_0) \models \varphi_{\mathcal{D}}$ , we must have  $H(t(n, m), t(n+1, m))$ .

Second, consider  $t(n, m)$  and  $t(n, m+1)$ . We have  $(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)}$  and  $(A^{(m+1)}, a_n) \models \diamond \chi_{t(n, m+1)}$ . Let  $a_n \rightarrow b$  be such that  $(A^{(m+1)}, b) \models \chi_{t(n, m+1)}$ . Then  $(A^{(m)}, b) \models \langle D \rangle \chi_{t(n, m+1)}$ , so

$$(A^{(m)}, a_n) \models \diamond \chi_{t(n, m)} \wedge \diamond \langle D \rangle \chi_{t(n, m+1)}.$$

As above, we have  $V(t(n, m), t(n, m+1))$ . ⊣

**Recurring domino systems** If the original tiling  $t$  has  $d_0$  infinitely often on the  $x$ -axis, then the intended model  $(F_t, \lambda)$  satisfies  $\text{recurring}(d_0)$  from Figure 2. Conversely, let  $\varphi_{\mathcal{D}, d_0} = \varphi_{\mathcal{D}} \wedge \text{recurring}(d_0)$ . If  $(A, a_0) \models \varphi_{\mathcal{D}, d_0}$ , we may choose the path  $a_0 \rightarrow a_1 \rightarrow \dots$  so that for infinitely many  $i$ ,  $a_i \models \diamond \chi_{d_0}$ . Then we may arrange that the tiling that we get from this path has  $d_0$  infinitely often on the  $x$ -axis.

**Summary** Beginning with a tiling  $t$ , we constructed a sentence  $\varphi_{\mathcal{D}, d_0}$  with the property that models of  $\varphi_{\mathcal{D}, d_0}$  give proper tilings of  $(\mathcal{D}, d_0)$ . Conversely, every proper tiling of  $(\mathcal{D}, d_0)$  gives a model of  $\varphi_{\mathcal{D}, d_0}$ . This would complete the proof of Theorem 5 stated at the beginning of this section, except that we would like to strengthen the result to avoid the atomic sentences  $d$  corresponding to the dominoes.

## 4 Eliminating atomic sentences

We eliminate the atomic sentences by making the models more complicated and doing extra work in the coding. The overall strategy is to *redefine*  $\chi_d$  to be a certain sentence built only from  $[D^*]$ ,  $\square^*$ ,  $\square$ , and the boolean connectives. We only need to show that proper tilings from a domino system  $\mathcal{D}$  give models of  $\varphi_{\mathcal{D}}$ , or rather the version of  $\varphi_{\mathcal{D}}$  obtained by the redefinition. We only need to find *some* model of  $\varphi_{\mathcal{D}}$ ; this was the easy part in the previous section. Then Lemma 7 tells us that *any* model of  $\varphi_{\mathcal{D}}$  gives a proper tiling in each of its paths through the stalk and in each sequence of choices along the path.

Again, we fix a domino system  $\mathcal{D}$  for the remainder of this section. It will be convenient to take the dominoes to be a set of the form  $\{2, 3, \dots, K\}$ . That is,  $d \geq 2$  for  $d \in \text{Dominoes}$ . The reason for this will become clear as we develop our coding.

**Models from proper tilings** Let  $t : Q \rightarrow \mathcal{D}$  be a proper tiling of  $Q$  by  $\mathcal{D}$ . We construct a frame  $G_t$  as follows. We again begin with infinitely many different symbols  $x, y_0, y_1, \dots, y_m, \dots$  and  $z$ . The set of worlds of  $G_t$  is the following set of words:

$$\{x^n : 0 \leq n\} \cup \{x^n y_m^q z^p : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m), 0 \leq p \leq m + 1\}$$

The accessibility relation is given by  $u \rightarrow v$  iff both belong to  $G_t$  and if  $v$  is a one-letter extension of  $u$ .

**Some examples of the coding** We take  $n = 5, m = 2$ . Suppose that  $t(5, 2) = 4$ . Then the model would contain the points in Figure 3 as an induced substructure.

There are other arrows from  $\lambda, x, \dots, x^5$ , but for all of the other points shown, there are no other arrows besides what is in the figure. The derivative operation removes  $x^5 y_2 z^3, \dots, x^5 y_2^4 z^3$ . The second derivative removes  $x^5 y_2 z^2, \dots, x^5 y_2^4 z^2$ . Recalling that  $n = 5, m = 2$ , and  $t(5, 2) = 4$ , we have

$$(G_t^{(m)}, x^n y_m) \models \diamond \square \text{False} \wedge \diamond^{t(n, m)} \square \text{False} \wedge \neg \diamond^{t(n, m) + 1} \square \text{False}$$

This is the key point for our coding. Taking a third derivative here leaves only the top row. Then the next four derivatives remove in turn  $x^5 y_2^4, \dots, x^5 y_2$ . One can check that again for  $n = 5, m = 2$ , and  $t(5, 2) = 4$ , if  $r \neq m$ ,

$$(G_t^{(r)}, x^n y_m) \models \neg(\diamond \square \text{False} \wedge \diamond^{t(n, m)} \square \text{False})$$

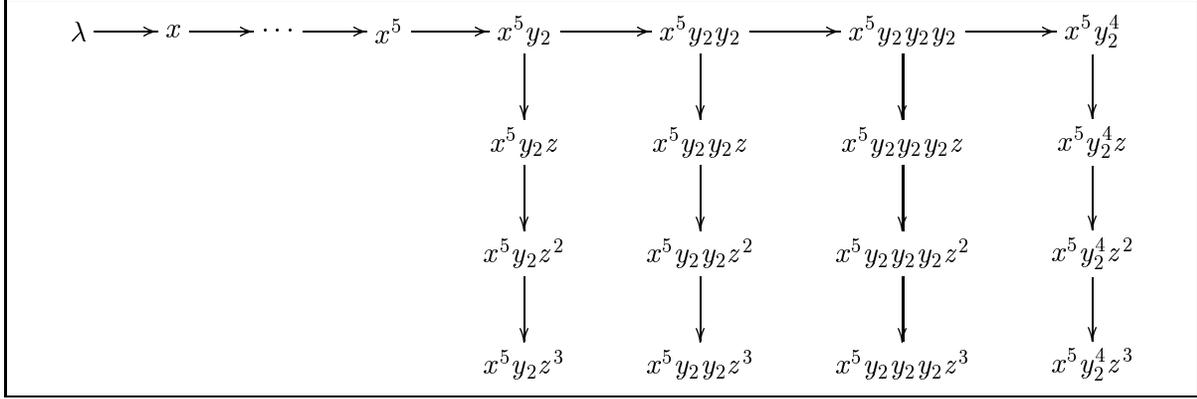


Figure 3: Part of  $G_t$  with  $n = 5$ ,  $m = 2$ , and  $t(5, 2) = 4$ .

As we shall see, this holds for all  $n$  and  $m$ , using the assumption that  $t(n, m) \geq 2$ .

**The sentences  $\chi_d$  and  $\varphi_{\mathcal{D}}$**  Recall that  $D = \{2, \dots, K\}$ . For  $d \in D$ , let

$$\chi_d = \diamond \square \text{False} \wedge \diamond^d \square \text{False} \wedge \neg \diamond^{d+1} \square \text{False}.$$

We again take  $\text{stalk} = [D^*] \diamond \text{True}$ . Observe that these sentences are defined independently of the intended models. Then we construct  $\varphi_{\mathcal{D}}$  from these sentences exactly as in Figure 2. It remains to show that  $(F_t, \lambda) \models \varphi_{\mathcal{D}}$ .

**Derivatives** Recall that

$$F_t = \{x^n : n \geq 0\} \cup \{x^n y_m^q z^p : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m), 0 \leq p \leq m + 1\}.$$

By induction on  $r \geq 0$  we see that  $F_t^{(r)}$  is

$$\begin{aligned} & \{x^n : n \geq 0\} \cup \{x^n y_m^q z^{p-r} : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m), r \leq p \leq m + 1\} \\ & \cup \{x^n y_m^q : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m) + m + 1 - r, m + 1 < r\}. \end{aligned}$$

It follows that  $u \models \text{stalk}$  iff  $u$  is of the form  $x^n$  for some  $n$ . This immediately gives the first clause of  $\varphi_{\mathcal{D}}$ . For all  $n$  and  $m$ ,  $F_t^{(m)}$  contains a submodel

$$\begin{array}{ccccc} x^n & \longrightarrow & x^n y_m & \longrightarrow & \cdots & \longrightarrow & x^n y_m^{t(n,m)} \\ & & \downarrow & & & & \downarrow \\ & & x^n y_m z & & & & x^n y_m^{t(n,m)} y \end{array}$$

Except for  $x^n$ , none of the points above have any other neighbors besides the ones shown. As a result,  $(F_t^{(m)}, x^n y_m) \models \chi_{t(n,m)}$ . This for all  $n$  and  $m$  shows that  $(F_t, \lambda)$  satisfies the condition  $\square^*[D^*](\text{stalk} \rightarrow \diamond \bigvee_d \chi_d)$ .

For the same  $n$  and  $m$ , the only  $d$  such that  $(F_t^{(m)}, x^n y_m) \models \chi_d$  is  $t(n, m)$ . This follows easily from the definition of  $\chi_d$ .

We claim in addition that if  $k \neq m$ , then for  $(F_t^{(m)}, x^n y_k)$  satisfies no sentence  $\chi_d$ . (Actually,  $x^n y_k$  only belongs to  $F_t^{(m)}$  when  $m \leq t(n, k) + k$ .) When  $k > m$ ,  $(F_t^{(m)}, x^n y_k) \models \neg \diamond \square \text{False}$ . And when  $k < m \leq t(n, k) + k$ , the relevant submodel of  $F_t^{(m)}$  is

$$x^n \longrightarrow x^n y_k \longrightarrow \cdots \longrightarrow x^n y_k^{t(n,k)+k+1-m}$$

The only way to have  $(F_t^{(m)}, x^n y_k) \models \diamond \square \text{False}$  is to have  $t(n, k) + k + 1 - m = 1$ . Then  $(F_t^{(m)}, x^n y_k) \models \neg \diamond^2 \text{True}$ . Hence for all  $d \geq 2$ ,  $(F_t^{(m)}, x^n y_k) \models \neg \chi_d$ .

Now we see that the same equation as (4) holds:

$$(F_t^{(m)}, x^n) \models \diamond \chi_d \quad \text{iff} \quad d = t(n, m). \quad (5)$$

We check the last clause of  $\varphi_{\mathcal{D}}$  holds; the third clause is similar. The only points satisfying stalk in any derivative are the  $x^n$  points. Suppose toward a contradiction that  $d$  and  $d'$  are such that  $\neg V(d, d')$  and yet  $(F_t^{(m)}, x^n) \models \diamond \chi_d \wedge \langle D \rangle \diamond \chi_{d'}$ . Then  $(F_t^{(m+1)}, x^n) \models \diamond \chi_{d'}$ . So by (5),  $d = t(n, m)$  and  $d' = t(n, m + 1)$ . But this contradicts the properness of the tiling  $t$ .

## 5 Two iterated modal derivatives, modal logic, but no $\square^*$

In this section, we prove the following result:

**Theorem 8** *There are two modal sentences  $D_x$  and  $D_y$  such that to every recurring domino system  $(\mathcal{D}, d_0)$ , we can effectively associate a sentence  $\varphi_{\mathcal{D}, d_0}$  built from  $[D_x^*]$ ,  $[D_y^*]$ ,  $\square$ ,  $\text{True}$ , atomic sentences and the boolean connectives, such that the following are equivalent:*

1. *There is a proper tiling of  $Q$  by  $(\mathcal{D}, d_0)$ .*
2.  *$\varphi_{\mathcal{D}, d_0}$  is satisfiable.*

We again take atomic sentences  $d$  corresponding to the (finitely many) dominoes. We also take new atomic sentences north and east. From all these we form the sentences listed in Figure 4.

We define sentences  $D_x$  and  $D_y$  to be  $x \rightarrow \diamond \text{True}$  and  $y \rightarrow \diamond \text{True}$ , respectively. For any model  $A$ , let  $D_x(A) = A^{D_x}$  and  $D_y(A) = A^{D_y}$ . Intuitively,  $D_x(A)$  is  $A$  after deleting the set of  $x$ -points of  $A$  which are endpoints.

**Intended models for the squares in  $Q$**  Our intended model for the square  $(i, j)$  is  $W_{i,j}$  as shown below:

$$-(i+1) \longleftarrow -i \longleftarrow \cdots \longleftarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow j \longrightarrow j+1$$

with  $0 \models \text{north} \wedge \text{east}$ ;  $1, \dots, j+1 \models y$ ; and  $-1, \dots, -i, -(i+1) \models x$ .

**Observation**  $\langle W_{i,j}, 0 \rangle \models \text{square}_1 \wedge \text{square}_2$ . This is trivial for  $\text{square}_1$ . Note that  $D_x(W_{i+1,j}) = W_{i,j}$  and  $D_y(W_{i,j+1}) = W_{i,j}$ . These imply that that

$$\langle W_{i,j}, 0 \rangle \models \square(x \rightarrow \langle D_x \rangle^i \square \text{False}) \wedge \square(y \rightarrow \langle D_y \rangle^j \square \text{False}).$$

This implies  $\text{square}_2$ .

$x$	$\text{east} \wedge \neg \text{north}$
$y$	$\text{north} \wedge \neg \text{east}$
$D_x$	$x \rightarrow \diamond \text{True}$
$D_y$	$y \rightarrow \diamond \text{True}$
$\text{square}_1$	$\text{east} \wedge \text{north} \wedge \square(x \vee y)$
$\text{square}_2$	$\square(x \rightarrow \langle D_x^* \rangle \square \text{False}) \wedge \square(y \rightarrow \langle D_y^* \rangle \square \text{False})$
$\chi_d$	$d \wedge \diamond(x \wedge \square \text{False}) \wedge \diamond(y \wedge \square \text{False})$
$\text{tiling}_1$	$\neg \text{east} \wedge \neg \text{north} \wedge \square(\text{square}_1 \wedge \text{square}_2)$
$\text{tiling}_2$	$[D_x^*][D_y^*] \bigvee_d \diamond \chi_d$
$\text{proper}_1$	$[D_x^*][D_y^*] \neg \bigvee_{\neg H(d,d')} (\diamond \chi_d \wedge \langle D_x \rangle \diamond \chi_{d'})$
$\text{proper}_2$	$[D_x^*][D_y^*] \neg \bigvee_{\neg V(d,d')} (\diamond \chi_d \wedge \langle D_y \rangle \diamond \chi_{d'})$
$\text{recurring}(d_0)$	$[D_x^*] \langle D_x^* \rangle \diamond \chi_{d_0}$
$\varphi_{\mathcal{D}, d_0}$	$\text{tiling}_1 \wedge \text{tiling}_2 \wedge \text{proper}_1 \wedge \text{proper}_2 \wedge \text{recurring}(d_0)$

Figure 4: Sentences used in the fragment with  $[D_x^*]$ ,  $[D_y^*]$  and modal logic.

**Intended models for the tilings** Let  $t : Q \rightarrow D$ . We encode  $t$  as the model  $T = T(t)$  whose set of worlds is

$$\{*\} + \sum_{N \times N} W_{i,j}$$

That is, disjoint copies of all of the models  $W_{i,j}$  from above together with a new point  $*$ . We write  $0^{i,j}$  for the 0 of  $W_{i,j}$ . The accessibility relation has  $* \rightarrow 0^{i,j}$  for all  $i, j$ . The copies  $W_{i,j}$  are just as before. The atomic sentences are the same as before except now we must take care of the sentences corresponding to the dominoes. We specify that  $0^{i,j} \models d$  iff  $t(i, j) = d$ . The top point  $*$  satisfies nothing.

**The intended models satisfy  $\varphi_{\mathcal{D}, d_0}$**  We check that for all  $d_0$ -recurring tilings  $t$ ,  $(T(t), *) \models \varphi_{\mathcal{D}, d_0}$ . The main point is that

$$D_x^n (D_y^m (T(t), *)) = (T(\lambda r s. t(r+n, s+m)), *).$$

This implies all of our properties.

### Any model of $\varphi_{\mathcal{D}, d_0}$ gives a proper recurrent tiling

**Lemma 9** *Suppose that  $(W, w) \models \langle D_x^* \rangle \square \text{False}$ . Then every point  $u$  of  $W$  reachable from  $w$  via a path of length  $\geq 1$  satisfies  $x$ .*

**Proof** Suppose not. Let  $u$  be of minimal distance from  $w$  such that for some  $v$ ,  $u \models x$ ,  $u \rightarrow v$ , and  $v \models \neg x$ . We have a sequence  $w = w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_{n+1} = u$ . By minimality,  $w_1, \dots, w_{n-1}$  all satisfy  $x$ . But  $D_x$  maintains  $w_1, \dots, w_{n+1} = u$ , and  $v$ . Indeed, for all  $n$ ,  $D_x^n$  maintains all of these. A fortiori,  $(W, w) \models [D_x^*] \diamond \text{True}$ . This is a contradiction.  $\dashv$

**Lemma 10** *Let  $\varphi_{\mathcal{D},d_0}$  be as in Figure 4. Let  $(A, a_0) \models \varphi_{\mathcal{D},d_0}$ . Then*

1. *There is a function  $t : Q \rightarrow \mathcal{D}$  with the property that for  $(n, m) \in Q$ ,  $(A, a_0) \models \langle D_x \rangle^n \langle D_y \rangle^m \diamond \chi_{t(n,m)}$ .*
2. *Each such function  $t$  is a proper tiling of  $Q$  by  $\mathcal{D}$ .*
3. *There is some  $t$  such that  $d_0$  occurs infinitely often on the  $x$ -axis.*

**Proof** We may assume that each point of  $A$  is reachable from  $a_0$ . By tiling<sub>1</sub>,  $a_0$  does not satisfy  $x$  or  $y$ . Therefore  $a_0 \in D_x^n(D_y^m(A))$ , for all  $n$  and  $m$ . Now the existence of  $t$  is immediate from tiling<sub>2</sub>. To check that  $t$  is proper, consider  $t(n, m)$  and  $t(n+1, m)$ . As we know,

$$\begin{aligned} (A, a_0) &\models \langle D_x \rangle^n \langle D_y \rangle^m \diamond \chi_{t(n,m)} \\ (A, a_0) &\models \langle D_x \rangle^{n+1} \langle D_y \rangle^m \diamond \chi_{t(n+1,m)} \end{aligned} \tag{6}$$

The heart of the matter is that the two derivatives commute on  $A$ . To prove this, we explicitly determine  $D_x^p D_y^q(A, a_0)$  and  $D_y^q D_x^p(A, a_0)$ . Note that  $A$  consists of  $a_0$ , (satisfying  $\neg \text{east} \wedge \neg \text{north}$ ), its children (all satisfying  $\text{east} \wedge \text{north}$ ), and the descendants of its children. All of them satisfy  $x \vee y$ . By Lemma 9 (or the version of it with  $y$  replacing  $x$  throughout), if some point  $b \in A$  satisfies  $x$  (or  $y$ ) then so do all the children of  $b$ . It follows from this that  $A$  is partitioned into three sets:

$$\{a_0\} \cup \{b \in A : a_0 \rightarrow b\} \cup \{b \in A : b \models x\} \cup \{b \in A : b \models y\}.$$

Again, the points in the last two groups have all their children in the same group. It follows that

$$\begin{aligned} D_x^p D_y^q(A, a_0) &= \{a_0\} \cup \{b \in A : a_0 \rightarrow b\} \\ &\quad \cup \{b \in A : b \models x \wedge \diamond^p \text{True}\} \\ &\quad \cup \{b \in A : b \models y \wedge \diamond^q \text{True}\} \end{aligned}$$

And this set is also exactly  $D_y^q D_x^p(A, a_0)$ . We apply this to the second equation of (6). First, we take  $p = n+1$  and  $q = m$ . Then we take  $p = n$  and  $q = m$ , and read the equation backwards. The upshot is that

$$(A, a_0) \models \langle D_x \rangle^n \langle D_y \rangle^m \langle D_x \rangle \diamond \chi_{t(n+1,m)}.$$

Recall that  $(A, a_0) \models \text{proper}_1$ . This implies  $H(t(n, m), t(n+1, m))$ .

Similar work shows  $V(t(n, m), t(n, m+1))$  for all  $n$  and  $m$ . Indeed, this work is simpler because one does not have to know that the derivative operations commute.

Finally, we check that there is a recurring tiling. The sentence  $\text{recurring}(d_0)$  implies that there is some tiling  $t$  such that for infinitely many  $n$ ,  $t(n, 0) = d_0$ .  $\dashv$

## 6 $[D^*]$ , modal relativization, and $\square$ ; but no $\square^*$

We next get undecidability for the fragment with  $[D^*]$ , modal announcements, and the usual modal apparatus. Crucially, the fragment does not include  $\square^*$ .

**Theorem 11** *There are fixed model sentences  $\varphi_1, \dots, \varphi_3$  such that we can effectively associate to every recurring domino system  $(\mathcal{D}, d_0)$  a sentence  $\varphi_{\mathcal{D},d_0}$  of the language of  $[D^*]$ ,  $[\varphi_1], \dots, [\varphi_3]$ ,  $\square$ ,  $\text{True}$ , atomic sentences and the boolean connectives such that the following are equivalent:*

1. *There is a proper tiling of  $Q$  by  $(\mathcal{D}, d_0)$ .*
2.  *$\varphi_{\mathcal{D}, d_0}$  is satisfiable.*

## 6.1 Frames and models

Fix a recurring domino system  $(\mathcal{D}, d_0)$ . We take a language with atomic sentences corresponding to the (finitely many) dominoes. Concretely, let  $d$  correspond to  $d$ . We also take new symbols *root*, *column*, *a*, *b*, *red*, *blue*, and *yellow*. The role of *red*, *blue*, and *yellow* will be to fix the order of the columns, which would otherwise be lost when trying to interpret the encoding tiling. We will require that *red* columns are followed by *blue*, *blue* by *yellow*, and *yellow* by *red*.

**The intended frame for the first quadrant** This time, we construct a frame  $F$  by taking the set of symbols  $\{x_0, \dots, x_i, \dots, y, z_0, \dots, z_m, \dots\}$  and from these the set of worlds

$$\begin{aligned} \{\lambda\} \cup \{x_i : 0 \leq i\} &\cup \{x_i y^p : 0 \leq i, 1 \leq p \leq i + 1\} \\ &\cup \{x_i z_m^p : 0 \leq i, 0 \leq m, 1 \leq p \leq i + m + 1\} \end{aligned}$$

The accessibility relation is given by  $\lambda \rightarrow x_i$ ,  $x_i y^p \rightarrow x_i y^{p+1}$ , and  $x_n z_m^p \rightarrow x_n z_m^{p+1}$ . The picture is shown in Figure 5.

Now fix a proper recurring tiling  $t$  of  $Q$  by  $\mathcal{D}$ . We get a model  $F_t$  as follows:

1.  $\lambda \models \text{root}$ .
2.  $x_n \models \text{column}$ .
3.  $x_n \models \text{red}$  iff  $n \equiv 0 \pmod{3}$ ,  
 $x_n \models \text{blue}$  iff  $n \equiv 1 \pmod{3}$ , and  
 $x_n \models \text{yellow}$  iff  $n \equiv 2 \pmod{3}$ .
4.  $x_n y^p \models a$  for  $1 \leq p \leq n + 1$ .
5.  $x_n z_m^p \models b$ , for  $1 \leq p \leq n + m + 1$ .
6.  $x_n z_m \models t(n, m)$ .

**The sentence  $\varphi_{\mathcal{D}, d_0}$**  We consider the sentences in Figure 6.

**The intended models work** First, note that *structure* is true at all worlds of  $F$ . Thus relativizing by it does no work. By induction on  $n \geq 0$ ,  $F_t^{(n)}$  is

$$\begin{aligned} \{\lambda\} \cup \{x_i : 0 \leq i\} &\cup \{x_i y^p : 0 \leq i, 1 \leq p \leq i + 1 - n\} \\ &\cup \{x_i z_j^p : 0 \leq i, 1 \leq p \leq i + j + 1 - n\} \end{aligned}$$

In  $F_t^{(n)}$ , the only point satisfying  $\text{column} \wedge \diamond a \wedge \neg \diamond \diamond a$  is  $x_n$ . It follows that the part of  $F_t^{(n)(\text{activecol})}$  reachable from  $\lambda$  is

$$\{\lambda, x_n, x_n y\} \cup \{x_n z_j^p : 0 \leq j, 1 \leq p \leq j + 1\}. \quad (7)$$

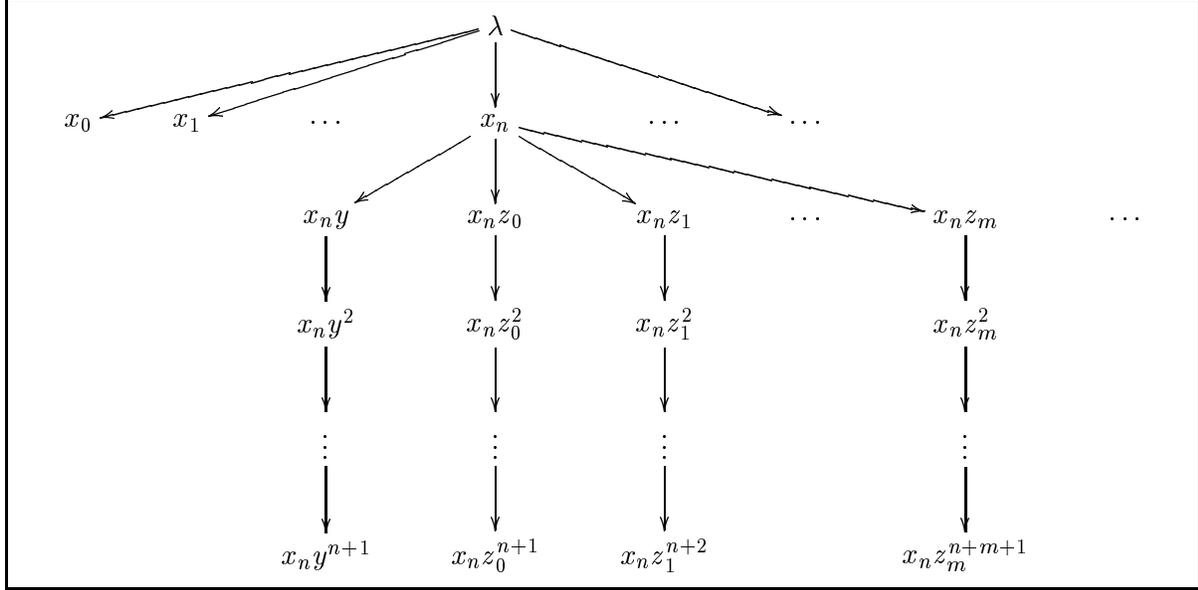


Figure 5: The intended model for the fragment with  $[D^*]$  and modal relativizations.

structure	“exactly one of $\{\text{root}, \text{column}, a, b\}$ holds” $\wedge(\text{root} \rightarrow \Box \text{column}) \wedge (\text{column} \rightarrow \Box(a \vee b)) \wedge (a \rightarrow \Box a) \wedge (b \rightarrow \Box b)$ $\wedge(\text{column} \rightarrow \text{“exactly one of } \{\text{red}, \text{blue}, \text{yellow}\}\text{”})$
activecol	$\text{column} \rightarrow (\Diamond a \wedge \neg \Diamond \Diamond a)$
nextcol	$\text{column} \rightarrow (\Diamond \Diamond a \wedge \neg \Diamond \Diamond \Diamond a)$
twocol	$\text{activecol} \vee \text{nextcol}$
$\chi_d$	$b \wedge d \wedge \Box \text{False}$
$\alpha$	$\langle \text{structure} \rangle [D^*] (D \wedge \langle \text{activecol} \rangle [D^*] \bigvee_d \Diamond \Diamond \chi_d)$
$\beta$	$[\text{structure}] [D^*] [\text{activecol}] [D^*] \neg \bigvee_{\neg V(d, d')} (\Diamond \Diamond \chi_d \wedge \langle D \rangle \Diamond \Diamond \chi_{d'})$
$\gamma_1$	$[\text{structure}] \Box (\text{activecol} \rightarrow \text{red})$
$\gamma_2$	$[\text{structure}] [D^*]$ $(\Diamond (\text{activecol} \wedge \text{red}) \rightarrow \Box (\text{nextcol} \rightarrow \text{blue}))$ $\wedge (\Diamond (\text{activecol} \wedge \text{blue}) \rightarrow \Box (\text{nextcol} \rightarrow \text{yellow}))$ $\wedge (\Diamond (\text{activecol} \wedge \text{yellow}) \rightarrow \Box (\text{nextcol} \rightarrow \text{red}))$
$\delta$	$[\text{structure}] [D^*] [\text{twocol}] [D^*] \neg \bigvee_{\neg H(d, d')}$ $(\Diamond (\text{red} \wedge \Diamond \chi_d) \wedge \langle D \rangle \Diamond (\text{blue} \wedge \Diamond \chi_{d'}))$ $\vee (\Diamond (\text{blue} \wedge \Diamond \chi_d) \wedge \langle D \rangle \Diamond (\text{yellow} \wedge \Diamond \chi_{d'}))$ $\vee (\Diamond (\text{yellow} \wedge \Diamond \chi_d) \wedge \langle D \rangle \Diamond (\text{red} \wedge \Diamond \chi_{d'}))$
recurring( $d_0$ )	$[\text{structure}] \langle \text{activecol} \rangle [D^*] \langle D^* \rangle \Diamond \Diamond \chi_{d_0}$
$\varphi_{\mathcal{D}, d_0}$	$\text{root} \wedge \alpha \wedge \beta \wedge \gamma_1 \wedge \gamma_2 \wedge \delta \wedge \text{recurring}(d_0)$

Figure 6: Sentences used in the fragment with  $[D^*]$  and modal relativizations.

For all  $n \geq 0$  and  $m \geq 1$ , the part of  $F_t^{(n)(\text{activecol})(m)}$  accessible from  $\lambda$  is

$$\{\lambda, x_n\} \cup \{x_n z_j^p : m \leq j, 1 \leq p \leq j + 1 - m\}.$$

The only point of  $F_t^{(n)(\text{activecol})(m)}$  which satisfies any  $\chi_d$  sentence is  $x_n z_m$ . This easily implies that  $(F_t^{(n)}, \lambda) \models \alpha$ . Appealing to the properness of the tiling  $t$ , we see also that  $(F_t^{(n)}, \lambda) \models \beta$ .

Moreover, the part of  $F_t^{(n)(\text{nextcol})}$  reachable from  $\lambda$  is

$$\{\lambda, x_{n+1}, x_{n+1}y, x_{n+1}y^2\} \cup \{x_{n+1}z_j^p : 0 \leq j, 1 \leq p \leq j + 2\}. \quad (8)$$

From (7) and (8) and the definition of  $F_t$ , we easily see that  $(F_t^{(n)}, \lambda) \models \gamma_1 \wedge \gamma_2$ . Continuing our discussion, we see that for all  $m$ , the part of  $F_t^{(n)(\text{twocol})(m)}$  reachable from  $\lambda$  is

$$\begin{aligned} \{\lambda, x_n, x_{n+1}\} &\cup \{x_n z_j^p : m \leq j, 1 \leq p \leq j + 1 - m\} \\ &\cup \{x_{n+1} z_j^p : m + 1 \leq j, 1 \leq p \leq j + 2 - m\}. \end{aligned} \quad (9)$$

(For  $0 \leq m \leq 2$  we also have  $x_n y$ ,  $x_{n+1} y$  and  $x_{n+1} y^2$ . But these are not relevant to our discussion, and we shall ignore them.)

We now come to the most critical part of the verification, the part about  $\delta$ . We may assume, without loss of generality, that  $x_n \models \text{red}$ . Therefore  $x_{n+1} \models \text{blue}$ . Also note that  $(F_t^{(n)(\text{twocol})(m+1)}, x_{n+1}) \models \diamond \chi_{t(n+1, m)}$ , so that  $(F_t^{(n)(\text{twocol})(m)}, \lambda) \models \langle D \rangle \diamond \diamond \chi_{t(n+1, m)}$ . Moreover, the only sentence of one of the forms listed in the statement of  $\delta$  which is satisfied by  $(F_t^{(n)(\text{twocol})(m)}, \lambda)$  is

$$\diamond(\text{red} \wedge \diamond \chi_{t(n, m)}) \wedge \langle D \rangle \diamond(\text{blue} \wedge \diamond \chi_{t(n+1, m)}).$$

And for this sentence, we do have  $H(t(n, m), t(n+1, m))$ . This concludes the verification of  $\delta$ .

At this point, we can explain the need for red, blue and yellow. Suppose we drop the colors from the statement of  $\delta$ . Then  $(F_t^{(n)(\text{twocol})(m)}, \lambda)$  might satisfy a sentence of the form

$$\diamond \diamond \chi_{t(n, m)} \wedge \langle D \rangle \diamond \diamond \chi_{t(n+1, m)}$$

where  $H(t(n, m), t(n+1, m))$  is false. The problem is that in (9), we have no way to know which nodes code squares in the  $n$ -th column of the desired model, and which code squares in the  $(n+1)$ -st column. That is, once the derivatives have eliminated the  $a$ -points, we have no way to tell right from left in  $F_t^{(n)(\text{twocol})(m)}$ .

Returning to the final point concerning the intended models, the fact that  $t(n, 0) = d_0$  for infinitely many  $n$  implies that  $(F_t^{(n)}, \lambda) \models \text{recurring}(d_0)$ .

**Any model of  $\varphi_{\mathcal{D}, d_0}$  gives a proper tiling** Let  $(A, w) \models \varphi_{\mathcal{D}, d_0}$ . We may assume that every element of  $A$  is reachable from  $w$  in finitely many transitions along  $\rightarrow$ . Indeed, throughout this proof, in *all* models, we assume that every point is reachable from  $w$ .

The sentence structure is *universal*; that is, it may be written in terms of atomic sentences and their negations using  $\wedge$ ,  $\vee$ , and  $\square$  (but not  $\diamond$ ). Any such universal sentence  $\rho$  has the property that  $[\rho] \square^* \rho$  is valid; that is, for all  $(X, x)$ , if  $(X, x) \models \rho$ , then after relativizing with  $\rho$ , we see that  $\rho$  holds at all points reachable from  $x$ . Therefore,  $(A, w)^{\text{structure}} \models \square^* \text{structure}$ .

Of course, this point could be verified directly without using the more general fact concerning positivity.

At this point, we know something about the structure of  $(A, w)^{\text{structure}}$ . First,  $w \models \text{root}$ , and  $w$  is the only point with this property. The children of  $w$  satisfy **column** and they are the only points to do so. They also satisfy some color sentence. The rest of the model consists of **a**-points and **b**-points; the children of each of these types is again of the same type.

For the rest of this argument, we save on notation by replacing  $(A, w)$  by  $(A, w)^{\text{structure}}$ ; thus we assume that our remarks in the previous paragraph apply to the original  $(A, w)$ .

As a consequence of  $\alpha$ , for each  $n$ ,  $(A^{(n)}, w) \models \langle \text{activecol} \rangle \diamond \text{True}$ . So  $(A^{(n)}, w) \models \diamond \text{activecol}$  also. Let

$$\begin{aligned} C_n &= \{x : w \rightarrow x, (A^{(n)}, x) \models \text{activecol}\}, \\ N_n &= \{x : w \rightarrow x, (A^{(n)}, x) \models \text{nextcol}\}. \end{aligned}$$

Each  $C_n$  is nonempty. Note that the following are equivalent:

1.  $(A^{(n)}, x) \models \text{nextcol} = (\text{column} \rightarrow (\diamond \diamond \mathbf{a} \wedge \neg \diamond \diamond \mathbf{a}))$
2.  $(A^{(n+1)}, x) \models \text{activecol} = (\text{column} \rightarrow (\diamond \mathbf{a} \wedge \neg \diamond \diamond \mathbf{a}))$

In other words,  $N_n = C_{n+1}$ . By  $\gamma_1$ , for each  $x \in C_0$  we have  $x \models \text{red}$ . By an induction using  $\gamma_2$ , we see that each  $x \in C_n$  satisfies the same color and that the colors cycle through red, blue, yellow, red,  $\dots$ , as desired.

Now we define a tiling  $t$  from  $(A, w)$ . We know from  $\alpha$  that

$$(A^{(n)(\text{activecol})(m)}, w) \models \diamond \diamond \chi_d$$

for some domino  $d$ . We choose one such  $d$  and define  $t(n, m) = d$ . The main point of the construction is to make sure that  $t$  is a proper tiling. The fact that  $V(t(n, m), t(n, m + 1))$ , for all  $n$  and  $m$ , comes from  $\beta$ . The hard work comes in checking that for all  $n$  and  $m$ ,  $H(t(n, m), t(n + 1, m))$ .

$A^{(n+1)(\text{activecol})}$  consists of  $w$ ,  $C_{n+1}$ , the **a**-children in  $A^{(n+1)}$  of the elements of  $C_{n+1}$  (these last are end nodes of  $A^{(n+1)}$ , since  $\diamond \mathbf{a} \wedge \neg \diamond \diamond \mathbf{a}$  holds on  $C_{n+1}$ ), and all of the **b**-descendants in  $A^{(n+1)}$  of the elements of  $C_{n+1}$  (these are exactly the **b**-nodes in the original  $A$  which are descendants of some element of  $C_{n+1}$  and which satisfy  $\diamond^{n+1} \text{True}$ ).

$A^{(n)(\text{nextcol})}$  consists of  $w$ , the set  $N_n$ , the **a**-descendants in  $A^{(n)}$  of the elements of  $N_n$  (but recall that elements of  $N_n$  all satisfy  $\diamond^2 \mathbf{a} \wedge \neg \diamond^3 \mathbf{a}$  in  $A^{(n)}$ ), and all of the **b**-descendants in  $A^{(n)}$  of the elements of  $N_n$  (these are exactly the **b**-nodes in the original  $A$  which are descendants of some element of  $N_n$  and which satisfy  $\diamond^n \text{True}$ ).

It follows from these observations and from the fact that  $N_n = C_{n+1}$  that

$$A^{(n)(\text{nextcol})(1)} = A^{(n+1)(\text{activecol})}.$$

So by induction on  $m$ ,

$$A^{(n+1)(\text{activecol})(m)} = A^{(n)(\text{nextcol})(m+1)} \subseteq A^{(n)(\text{twocol})(m+1)}. \quad (10)$$

To conclude, we fix  $n$  and  $m$  and check that  $H(t(n, m), t(n + 1, m))$ . Let  $x \in C_n$ ,  $y \in A^{(n)(\text{activecol})(m)}$ ,  $u \in C_{n+1}$ , and  $v \in A^{(n+1)(\text{activecol})(m)}$  be such that

1.  $w \rightarrow x \rightarrow y$
2.  $(A^{(n)(\text{activecol})}(m), y) \models \chi_{t(n,m)}$
3.  $w \rightarrow u \rightarrow v$
4.  $(A^{(n+1)(\text{activecol})}(m), v) \models \chi_{t(n+1,m)}$

The last point here tells us that  $(A^{(n)(\text{nextcol})}(m), v) \models \langle D \rangle \chi_{t(n+1,m)}$ . Without loss of generality, take  $x \models \text{red}$ . So  $u \models \text{blue}$ . By (10) and the points above,

$$(A^{(n)(\text{twocol})}(m), w) \models \diamond(\text{red} \wedge \diamond \chi_{t(n,m)}) \wedge \langle D \rangle \diamond(\text{blue} \wedge \diamond \chi_{t(n+1,m)}).$$

We see from  $\delta$  that  $H(t(n, m), t(n+1, m))$ , as desired.

The recurrence condition is easy to check.

This concludes the proof of Theorem 11.

## 7 Undecidability of satisfiability on finite (tree) models

In this section, we show that the problem of determining whether a sentence  $\psi$  of our language  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  has a *finite* model is  $\Sigma_1^0$ -complete. The same work shows that the problem of determining whether  $\psi$  has a finite *tree* model is also  $\Sigma_1^0$ -complete. Note that the relation  $A \models \varphi$  is decidable for sentences  $\varphi \in \mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  and finite models  $A$ . Therefore, the set of  $\varphi$  which have a finite model is  $\Sigma_1^0$ . The proof that this problem is  $\Sigma_1^0$ -hard goes by reduction from the problem of deciding whether a domino system has a *periodic* tiling of the first quadrant.

**Definition** A *rectangle* is a subset of the first quadrant of the form

$$R = \{0, \dots, r\} \times \{0, \dots, s\}. \quad (11)$$

Let  $\mathcal{D} = (\text{Dominoes}, H, V)$ , be a domino system. A *repeatable rectangle* (for  $\mathcal{D}$ ) is a pair  $(R, t)$ , where  $R$  is a rectangle, and  $t : R \rightarrow \text{Dominoes}$  satisfies the following conditions:

1.  $H(t(n, m), t(n+1, m))$  for  $0 \leq n < r$  and  $0 \leq m \leq s$ .
2.  $V(t(n, m), t(n, m+1))$  for  $0 \leq n \leq r$  and  $0 \leq m < s$ .
3.  $H(t(r, m), t(0, m))$  for  $0 \leq m \leq s$ .
4.  $V(t(n, s), t(n, 0))$  for  $0 \leq n \leq r$ .

A repeatable rectangle is just a witness to the existence of a periodic tiling of the plane or first quadrant.

**Proposition 12** *The question of whether a domino system has a repeatable rectangle is  $\Sigma_1^0$ -complete.*

This result is originally due to Berger [3]. It appears as Theorem 3.1.7 of [5] with a proof in Cyril Allauzen and Bruno Durand's appendix of [5]. Another reference on this matter is Lin [12]. (Incidentally, Lin's paper is in English but seems not to be known to later workers on tiling.)

**Theorem 13** *For every domino system  $\mathcal{D}$  we can effectively find a sentence  $\varphi_{\mathcal{D}}$  of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  such that the following are equivalent:*

1.  $\mathcal{D}$  has a repeatable rectangle.
2.  $\varphi_{\mathcal{D}}$  is satisfied on a finite tree.
3.  $\varphi_{\mathcal{D}}$  is satisfied on some (finite or infinite) model.

Moreover, such a  $\varphi_{\mathcal{D}}$  can be found in the fragment of  $\mathcal{L}(\text{rel}, \text{rel}^*, \square^*)$  considered in each of Sections 3, 5 and 6.

We shall prove this result in this section for just one of our fragments, the one of Section 3. We shall not attempt to work without atomic sentences, and indeed this time around we need more atomic sentences than before. The intended models are basically the obvious finite versions of the stalk models which we have seen. But the sentences that encode the models are substantially more complicated.

Incidentally, when one changes to a different fragment, many details in the proof of Theorem 13 change. We believe that it would be possible to encode an undecidable problem into all of our fragments in such a way as to make it easier to go in a “fragment-independent” way from the  $\Sigma_1^1$ -completeness results on general satisfiability to the  $\Sigma_1^0$ -completeness results for finite satisfiability. However, to do this, one would need to encode a new tiling problem created just for this purpose. We opt for quoting a known tiling problem (in Proposition 12), and so we only give the details of the finite satisfiability result in one fragment.

**The intended finite frames corresponding to rectangles** We use the notation from Section 3. Recall that associated to the first quadrant  $Q$  we have a frame  $F$ . For the rectangle  $R$  as in (11), we let  $F_R$  be the subframe of  $F$  determined by

$$\{x^n : 0 \leq n \leq r\} \cup \{x^n y_m z^p : 0 \leq n \leq r; 0 \leq m \leq s; \text{ and } 0 \leq p \leq m\}.$$

These are just the points of the original frame that figure into the coding of the points in  $R$ . We keep the accessibility relation  $\rightarrow$  exactly as before.

**Repeatable rectangles give models** Let  $(R, t)$  be a repeatable rectangle, so  $t : R \rightarrow \text{Dominoes}$ . As in our earlier work, we take atomic sentences  $d$  for  $d \in \text{Dominoes}$ . This time we take stalk to be an atomic sentence, not an abbreviation. We also need atomic sentences  $\text{hmax}$  and  $\text{vmax}$  that are true of points coding squares that are “rightmost” and “uppermost”. The purposes of these are perhaps best gleaned from the intended models.

We construct a model  $F_{(R,t)}$  from  $t$  (and the underlying frame  $F_R$  described above) by declaring

$$x^n \models \text{stalk},$$

$$\text{if } t(n, m) = d, \text{ then } x^n y_m \models d,$$

$$x^r y_m \models \text{hmax},$$

$$x^n y_s \models \text{vmax},$$

for all  $0 \leq n \leq r$  and  $0 \leq m \leq s$ . No other atomic sentences are true at any other points.

stalk	this is now an atomic sentence
structure	$\bigwedge_{d \neq d'} (d \rightarrow \neg d') \wedge (\diamond^* \text{hmax} \rightarrow \text{stalk xor hmax})$
$\chi_d$	$d \wedge \square \text{False}$
active	$\bigvee_d \chi_d$
$\varphi_{\mathcal{D}}$	$\text{stalk} \wedge \square^*(\text{stalk} \rightarrow \diamond \text{active}) \wedge \square^* \text{structure}$ $\wedge \diamond^* \text{hmax}$ $\wedge \langle D^* \rangle \diamond (\text{active} \wedge \text{vmax})$ $\wedge [D^*] (\diamond^*(\text{stalk} \wedge \diamond (\text{active} \wedge \text{vmax})) \rightarrow \square^*(\text{stalk} \rightarrow \square (\text{active} \rightarrow \text{vmax})))$ $\wedge [D^*] (\diamond^*(\text{stalk} \wedge \diamond (\text{active} \wedge \neg \text{vmax})) \rightarrow \square^*(\text{stalk} \rightarrow \langle D \rangle \diamond \text{active}))$ $\wedge \square^*(\diamond \text{hmax} \rightarrow \square \text{hmax})$ $\wedge \square^*[D^*] \neg \bigvee_{\neg H(d,d')} (\diamond \chi_d \wedge \diamond \chi_{d'})$ $\wedge \square^*[D^*] \neg \bigvee_{\neg V(d,d')} (\diamond \chi_d \wedge \diamond \langle D \rangle \chi_{d'})$ $\wedge [D^*] \bigwedge_d (\diamond \chi_d \rightarrow \square^*(\text{active} \wedge \text{hmax} \rightarrow \bigvee_{d':H(d,d')} d'))$ $\wedge \square^* \bigwedge_d (\diamond \chi_d \rightarrow [D^*] (\diamond \text{vmax} \rightarrow \square (\text{active} \rightarrow \bigvee_{d':V(d',d)} d')))$

Figure 7: Sentences in the finite model result for the fragment  $[D^*]$ ,  $\square^*$ ,  $\square$ , and atomic sentences.

**The sentence  $\varphi_{\mathcal{D}}$**  See Figure 7. We might note that there are natural sentences which are true in the intended models but which we do not take as conjuncts of  $\varphi_{\mathcal{D}}$ . Among these are  $\text{stalk} \leftrightarrow \neg \bigvee_d d$  and  $\text{hmax} \vee \text{vmax} \rightarrow \neg \text{stalk}$ . The reasons for not incorporating these into  $\varphi_{\mathcal{D}}$  are: (a) the proof goes through without them; and (b), the argument would not be substantially shorter if we did add the extra clauses.

**The intended models work** We verify some of the clauses of  $\varphi_{\mathcal{D}}$ . As in our earlier work, we first check that

$$(F_{R,t}^{(m)}, x^n) \models \diamond \chi_d \quad \text{iff} \quad d = t(n, m). \quad (12)$$

These are the only points that satisfy  $\diamond \chi_d$ . Moreover, the only points of  $F_{R,t}^{(m)}$  satisfying active are those of the form  $x^n y_m$ .

We remind the reader that we write  $\lambda$  for  $x^0$ . So with  $m = 0$ , we have  $(F_{R,t}, \lambda) \models \square^*(\text{stalk} \rightarrow \diamond \text{active})$  via the points  $x^n y_0$ .

Next, we check all of the clauses of  $\varphi_{\mathcal{D}}$  mentioning hmax. The points where hmax holds are those of the form  $x^r y_m$ . And the only path from  $\lambda$  to a point of this form is  $\lambda \rightarrow x \rightarrow \dots \rightarrow x^r \rightarrow x^r y_m$ . This implies that  $(A, \lambda) \models \square^*(\diamond^* \text{hmax} \rightarrow \text{stalk xor hmax})$ .

Taking  $n = 0$  in (12), we see that for each  $m$ ,  $(F_{R,t}^{(m)}, \lambda) \models \diamond \chi_{t(0,m)}$ . And the only point of  $F_{R,t}^{(m)}$  satisfying active  $\wedge$  hmax is  $x^r y_m$ . Let  $d' = t(r, m)$ . Then  $x^r y_m \models d'$ . And by the assumption that  $R$  is a repeatable rectangle, we have  $H(t(r, m), t(0, m))$ . This discussion shows that  $(F_{R,t}, \lambda)$  satisfies the last sentence involving hmax.

Finally, we check the clauses mentioning vmax. We have  $(F_{R,t}, \lambda) \models \langle D \rangle^* \diamond (\text{active} \wedge \text{vmax})$  because  $\lambda \rightarrow x^0 y_s$  and  $(F_{R,t}^{(s)}, x^0 y_s) \models \text{active} \wedge \text{vmax}$ . The two long conditions on vmax are actually easy to check in the intended models. The first says informally that as we take derivatives, if any stalk point has a child which is active and satisfies vmax, then all stalk points have such a child. The second says that if any stage a stalk point has a child which is active but does not

satisfy  $\text{vmax}$  (so the stage is below  $s$ ), then at this stage all stalk points have an active child in the *next* derivative. We omit the argument for the last  $\text{vmax}$  condition.

**Any model of  $\varphi_{\mathcal{D}}$  gives a repeatable rectangle** We are checking  $(3) \implies (1)$  in Theorem 13. Let  $\varphi_{\mathcal{D}}$  be as in Figure 7. Let  $(A, a_0)$  be an arbitrary model of  $\varphi_{\mathcal{D}}$ . We note that for all  $k$ ,  $(A^{(k)}, a) \models \text{structure}$ .

**Lemma 14** *Then there are numbers  $r$  and  $s$  and points  $a_n$  and  $b_{n,m}$  for  $0 \leq n \leq r$  and  $0 \leq m \leq s$  such that*

1.  $a_0$  is the given point that satisfies  $\varphi_{\mathcal{D}}$  in  $A$ .
2.  $a_0 \rightarrow \dots \rightarrow a_n \rightarrow \dots \rightarrow a_r$ .
3.  $a_r \models \diamond \text{hmax}$ .
4.  $a_n \models \text{stalk}$ .
5.  $a_n \rightarrow b_{n,m}$ .
6.  $(A^{(m)}, b_{n,m}) \models \text{active}$ .
7.  $b_{n,s} \models \text{vmax}$ .
8.  $b_{r,m} \models \text{hmax}$ .

**Proof** Let  $r$  be least such that  $(A, a_0) \models \diamond^{r+1} \text{hmax}$ . From  $a_0$  and  $r$ , we get the  $a$ -points so that parts (1)–(3) hold. We need to check in (4) that each  $a_n \models \text{stalk}$ . Certainly  $a_n \models \diamond^* \text{hmax}$ . So by structure,  $a_n \models \text{stalk xor hmax}$ . By minimality of  $r$ , no  $a_n$  can satisfy  $\text{hmax}$ . Let  $s$  be least so that  $(A, a_0) \models \langle D \rangle^s \diamond (\text{active} \wedge \text{vmax})$ . It is possible that  $r = 0$  or  $s = 0$ .

**Claim** For  $0 \leq n \leq r$  and  $0 \leq m \leq s$ ,  $(A, a_n) \models \langle D \rangle^m \diamond \text{active}$ . For  $m < s$ ,  $(A, a_n) \models \langle D \rangle^m \square (\text{active} \rightarrow \neg \text{vmax})$ .

**Proof** By induction on  $m$ . For  $m = 0$ , one of the clauses of  $\varphi_{\mathcal{D}}$  is  $\square^*(\text{stalk} \rightarrow \diamond \text{active})$ . We already know that  $a_n \models \text{stalk}$ , and so  $(A, a_n) \models \diamond \text{active}$ . Suppose in addition that  $0 < s$ . We claim that for all  $n$ ,  $(A, a_n) \models \square (\text{active} \rightarrow \neg \text{vmax})$ . For suppose not. Then  $(A, a_n) \models \diamond (\text{active} \wedge \text{vmax})$ . One of the clauses in  $\varphi_{\mathcal{D}}$  is

$$[D^*](\diamond^*(\text{stalk} \wedge \diamond(\text{active} \wedge \text{vmax})) \rightarrow \square^*(\text{stalk} \rightarrow \square(\text{active} \rightarrow \text{vmax}))).$$

Also,  $(A, a_n) \models \text{stalk} \wedge \diamond(\text{active} \wedge \text{vmax})$ . So  $(A, a_0) \models (\text{stalk} \rightarrow \square(\text{active} \rightarrow \text{vmax}))$ . This in turn implies that  $(A, a_0) \models \diamond(\text{active} \wedge \text{vmax})$ . Looking back to the definition of  $s$ , we see that  $s = 0$ . This is a contradiction.

Next, assume our claim for  $m$ . By this induction hypothesis,  $(A^{(m)}, a_n) \models \diamond(\text{active} \wedge \neg \text{vmax})$ . As we know,  $(A^{(m)}, a_n) \models \text{stalk}$ . Another clause in  $\varphi_{\mathcal{D}}$  is

$$[D^*](\diamond^*(\text{stalk} \wedge \diamond(\text{active} \wedge \neg \text{vmax})) \rightarrow \square^*(\text{stalk} \rightarrow \langle D \rangle \diamond \text{active})).$$

And so we see that  $(A^{(m)}, a_n) \models \langle D \rangle \diamond \text{active}$ . That is,  $(A, a_n) \models \langle D \rangle^{m+1} \diamond \text{active}$ . And exactly as above, if  $m + 1 < s$ , then we have  $(A, a_n) \models \langle D \rangle^{m+1} \square (\text{active} \rightarrow \neg \text{vmax})$ .  $\dashv$

For  $0 \leq n \leq r$  and  $0 \leq m \leq s$ , let  $b_{n,m}$  be such that  $a_n \rightarrow b_{n,m}$ ,  $(A^{(m)}, b_{n,m}) \models \text{active}$ , and in addition with  $(A^{(s)}, b_{n,s}) \models \text{vmax}$ . Most of the parts of our lemma are immediate. We verify in the last part that  $b_{r,m} \models \text{hmax}$ . For this, recall that  $a_r \models \diamond \text{hmax}$ . One of the clauses in  $\varphi_{\mathcal{D}}$  is that  $\Box^*(\diamond \text{hmax} \rightarrow \Box \text{hmax})$ . So  $a_r \models \Box \text{hmax}$ . Since  $a_r \rightarrow b_{r,m}$ , we are done.  $\dashv$

We continue with the proof of Theorem 13. Fix  $n$ ,  $m$  and any points  $a_n$  and  $b_{n,m}$  as in Lemma 14. Let  $R$  be the rectangle  $\{(n, m) : 0 \leq n \leq r \text{ and } 0 \leq m \leq s\}$ . Define  $t$  on  $R$  by:

$$t(n, m) = \text{the unique } d \text{ such that } (A^{(m)}, b_{n,m}) \models d. \quad (13)$$

So  $(A^{(m)}, b_{n,m}) \models \chi_{t(n,m)}$ .

**Lemma 15**  $(R, t)$  is a repeatable rectangle for  $\mathcal{D}$ .

**Proof** There are four conditions. The first two have to do with  $t$  working correctly “inside”  $R$ . The arguments here are the same as in Lemma 7, so we omit them. Instead we check the periodicity conditions, which we repeat below:

3.  $H(t(r, m), t(0, m))$  for  $0 \leq m \leq s$ .
4.  $V(t(n, s), t(n, 0))$  for  $0 \leq n \leq r$ .

Here is an argument for (3). Consider  $b_{0,m}$  and  $b_{r,m}$ . As we know,  $(A^{(m)}, b_{0,m}) \models \chi_{t(0,m)}$ , and  $(A^{(m)}, b_{r,m}) \models \chi_{t(r,m)}$ . Using  $b_{0,m}$ , we see that

$$(A^{(m)}, a_0) \models \Box^*(\text{active} \wedge \text{hmax} \rightarrow \bigvee_{d': H(d', t(0,m))} d').$$

As we know from parts (6) and (8) of Lemma 14,  $(A^{(m)}, b_{r,m}) \models \text{active} \wedge \text{hmax}$ . So there is some  $d'$  such that  $H(d', t(0, m))$  and  $b_{r,m} \models d'$ . By (13),  $d' = t(r, m)$ . This means that  $H(t(r, m), t(0, m))$ , as desired.

Finally, we check the periodicity condition (4). Consider  $b_{n,0}$  and  $b_{n,s}$ . This time we have  $(A, a_n) \models \diamond(\text{active} \wedge \chi_{t(n,0)})$ . By our last periodicity clause in  $\chi_{\mathcal{D}}$ , we have

$$(A^{(s)}, a_n) \models \diamond \text{vmax} \rightarrow \Box(\text{active} \rightarrow \bigvee_{d': V(d', t(n,0))} d').$$

Now  $(A^{(s)}, a_n) \models \diamond(\text{vmax} \wedge \text{active})$  via  $b_{n,s}$ . So there is some  $d'$  such that  $V(d', d)$  and  $b_{n,s} \models d'$ . Again by (13),  $d' = t(n, s)$ . Therefore  $V(t(n, s), t(n, 0))$ .  $\dashv$

This completes the proof of Theorem 13. And from the theorem and Proposition 12, we infer the  $\Sigma_1^0$ -completeness of the question of whether a sentence of  $\mathcal{L}(\text{rel}, \text{rel}^*, \Box^*)$  has a finite (tree) model. A fortiori, the same holds for *MIC*.

**Open problems** We only removed the atomic sentences from the fragment of Section 2. So it is open to re-work the remaining results without atomic sentences.

We conclude with another problem raised by our work, a problem which we find more interesting. We do not know whether the satisfiability problem for the fragment determined by  $[D^*]$ ,  $\Box$ ,  $\wedge$ ,  $\neg$ , and **True** is decidable. If it were  $\Sigma_1^1$ -complete, then the result would subsume the parallel results for the fragments in this paper. And if it were decidable (with atomic sentences), then it would be “maximal” in the sense that adding any of the following features would destroy decidability: the transitive closure operation  $\Box^*$ , another iterated derivative, or relativization by modal sentences.

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