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REALIZING DATA STRUCTURES AS LATTICES

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Abstract

We prove that if T is any endofunctor on the category of complete lattices which satisfies a weak continuity condition, then there is a canonical solution to the isomorphism $L \cong T(L)$.

The idea of defining data types recursively dates back at least to [4](*), e.g., "A list is either an atom or a pair of lists." In general, we want to find an object X such that $X = T(X)$, in this case, $X = A \cup X \times X$. Scott pointed out [8,9 & 10] that certain transformations T, such as $T(X) = X^X$, had no solutions in the category of sets, but there was a solution to the (weakened) equation $X \cong T(X)$ in the category of complete lattices. Scott also provided solutions for a number of interesting T's. Reynolds [7] pointed out that Scott's constructions could be unified. In this note, we prove that the unified construction is essentially categorical in nature: if T is an endofunctor on the category of complete lattices, and T satisfies a certain continuity condition, then the equation $X \cong T(X)$ has a solution L_∞ which is canonical, i.e., if $M \cong T(M)$, then there exists a canonical morphism $L_\infty \to M$. This continues the program suggested by Scott [10].

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*Of course, defining sets by induction is a much older idea, in general. Here we mean the application of this idea to computer programming.

1. Definitions

We presume familiarity with the following standard concepts: category, morphism, functor, and colimit. There are a number of elementary expositions of this material, notably [5] and [3, appendix].

On the other hand, there are several distinct notions of the category of complete lattices. A complete lattice is a partially ordered set $(L,\leq)$ with the property that every subset S of L has a least upper bound, denoted $\bigsqcup S$. A subset D of L is called directed iff every finite subset of D has an upper bound in D. (Thus every directed set is nonempty). If $(L,\leq)$ and $(M,\leq')$ are complete lattices, with least upper bounds denoted $\bigsqcup$ and $\bigsqcup'$, we say a function $f\colon L \to M$ is continuous iff for any directed $D \subseteq L$,

$$f(\bigsqcup D) = \bigsqcup{}'\{f(x) \mid x \varepsilon D\}$$

The set of all continuous maps from L to M forms a complete lattice under the ordering

$$f \leq g \quad \text{iff} \quad (\forall x \varepsilon L)[f(x) \leq' g(x)]$$

This lattice is denoted $[L \to M]$. We will use the terms "lattice" and "complete lattice" interchangeably.

Let $\underline{CL}$ denote the category of complete lattices, with morphisms the continous maps. Let $\underline{CLP}$ be the category whose objects are those of $\underline{CL}$ and with morphism sets given by $\underline{CLP}(L,M) = \underline{CL}(L,M) \times \underline{CL}(M,L)$. If $\phi = \langle f,g\rangle \;\varepsilon\; \underline{CLP}(L,M)$, let $\phi^{+} = \langle g,f\rangle \;\varepsilon\; CLP(M,L)$ with $\langle f,g\rangle\langle f',g'\rangle = \langle ff',g'g\rangle$. Let $\underline{CLR}$ (the category of retractions) be the subcategory of $\underline{CLP}$ of morphisms $\phi \in \underline{CLP}(L,M)$ such that $\phi\phi^+ \le 1_M$ and $\phi^+\phi = 1_L$, where the ordering is the natural one on $\underline{CLP}(L,M) = [L\to M] \times [M\to L]$. We say a functor $T: \underline{CLP} \to \underline{CLP}$ is <u>continuous on morphism sets</u> iff $\bigsqcup T(\phi_i) = T(\bigsqcup \phi_i)$ for any directed set of morphisms. Let $\underline{\omega}$ be the free category generated by the graph $\{\omega, \{(n,n+1)|n\varepsilon\omega\}\}$. Note that any functor $F: \underline{\omega} \to \underline{C}$ is uniquely specified by its values on the objects and its values on the (unique) arrows $n \to n+1$.

<u>Proposition 1</u> (i) $(\phi\cdot\psi)^+ = \psi^+\cdot\phi^+$

(ii) $\langle f,g\rangle \in \underline{CLP}(L,M)$ is a retraction iff $fg \le 1_M$ and $gf = 1_L$

(iii) $\phi \in \underline{CLR}(L,M)$ is an isomorphism iff $\phi\phi^+ = 1_M$

(iv) If $T:\underline{CLP} \to \underline{CLP}$ is an endofunctor continuous on the morphism sets, then T preserves retractions.

(v) If $\langle f,g\rangle$ and $\langle f',g\rangle$ are retractions, then $f = f'$

(vi) $\{1\}$ is an initial object of CLR

<u>Proof</u>. All trivial except (v): If $\langle f,g\rangle \in CLR(L,M)$, then $f(x) = \bigsqcap\{y\varepsilon M | g(y) = x\}$. (vi) follows from (v) and the fact that $\{1\}$ is a final object in $\underline{CL}$.

If there is a retraction $\langle f,g\rangle \in \underline{CLR}(L,M)$, then L is embeddable in M via f, and every $x \in M$ has a unique best approximation in the image of L, given by $fg(x)$. So if $\underline{CLR}(L,M)$ is nonempty, we may think of L as a sub-datatype of M.

2. Lattice-Theoretic Lemmata

Throughout the following let F be a functor $\underline{\omega} \to \underline{CLR}$ given by $F(n) = L_n$ and $F(n \to n+1) = \theta_n = \langle f_n, g_n \rangle$. Let L_∞ denote the complete lattice whose underlying set is given by

$$\{(x_0, x_1, \ldots) \mid x_i \in L_i \ \&\ x_i = g_i(x_{i+1})\}$$

under the ordering $x \le y$ iff $(\forall i)[x_i \le y_i]$. If $n \le m$, we write $\langle f_{nm}, g_{mn} \rangle$ for the image under F of the unique morphism in $\underline{\omega}(n,m)$.

Lemma 1. L_∞ is a complete lattice.

Proof. Let $S \subseteq L_\infty$. Let $S_k = \{x_k \mid x \in S\} \subseteq L_k$. Then for each k, S_k has a least upper bound $\bigsqcup S_k \in L_k$. Let $y_k = \bigsqcup_{n \ge k} g_{nk}(\bigsqcup S_n)$. Setting $n = k$, we see $y_k \ge \bigsqcup S_k$, and y is a member of L_∞ by the construction of the g_{nk}. If $x \in S$, then for each k, $x_k \le \bigsqcup S_k \le y_k$, so $x \le y$. Hence y is an upper bound for S in L_∞. Let z be another upper bound for S in L_∞. Then for every n, $\bigsqcup S_n \le z_n$. Now $z \in L_\infty$, so for every $n \ge k$, $z_k = g_{nk}(z_n)$. So $z_k = \bigsqcup_{n \ge k} g_{nk}(z_n) \ge \bigsqcup_{n \ge k} g_{nk}(\bigsqcup S_n) = y_k$. So $y \le z$, and $y = \bigsqcup S$. $\square$

Define a morphism $\phi_n = \langle f_{n\infty}, g_{\infty n} \rangle \in \underline{CLP}(L_n, L_\infty)$ by

$$g_{\infty n}((x_0, x_1, \ldots)) = x_n$$

$$(f_{n\infty}(y))_k = \begin{cases} g_{nk}(y) & \text{if } k \le n \\ f_{nk}(y) & \text{if } k > n \end{cases}$$

Thus

$g_{\infty k} f_{n\infty} = g_{nk}$ if $k \leq n$ and f_{nk} if $k \geq n$, and $g_{\infty n} f_{n\infty} = 1_{L_n}$.

<u>Lemma 2</u>. $\phi_n \in \underline{CLR}(L_n, L_\infty)$.

<u>Proof</u>. We must first show that the range of $f_{n\infty}$ is in fact L_∞ and not just ΠL_n , that is, that $(f_{n\infty}(y))_k = g_k(f_{n\infty}(y))_{k+1}$. If $k < n$, then $g_k(f_{n\infty}(y))_{k+1} = g_k(g_{n,k+1}(y)) = g_{nk}(y) = (f_{n\infty}(y))_k$. If $k \geq n$, then $g_k(f_{n\infty}(y))_{k+1} = g_k f_{n,k+1}(y) = g_{k+1,k} f_{n,k+1}(y) = g_{k+1,k} f_{k,k+1} f_{n,k}(y) = f_{nk}(y) = (f_{n\infty}(y))_k$. We may now show that ϕ_n is a retraction: $\phi_n \phi_n^+ = \langle f_{n\infty}, g_{\infty n}\rangle \cdot \langle g_{\infty n}, f_{n\infty}\rangle = \langle f_{n\infty} g_{\infty n}, f_{n\infty} g_{\infty n}\rangle$. $f_{n\infty} g_{\infty n}(x_0, x_1, \ldots) = f_{n\infty}(x_n) = \langle g_{n0}(x_n), g_{n1}(x_n), \ldots, x_n, f_{n,n+1}(x_n), \ldots f_{nk}(x_n) \ldots\rangle$

For $k \leq n$, $g_{nk}(x_n) = x_k$; for $k > n$, $f_{nk}(x_n) = f_{nk} g_{kn}(x_k) \leq x_k$, since $\langle f_{nk}, g_{kn}\rangle \in \underline{CLR}(L_n, L_k)$. So $\phi_n \phi_n^+ \leq 1_{L_\infty}$. $\phi_n^+ \phi_n = \langle g_{\infty n} f_{n\infty}, g_{\infty n} f_{n\infty}\rangle$. So $g_{\infty n} f_{n\infty}(y) = (f_{n\infty}(y))_n = g_{nn}(y) = y$. Hence $\phi_n^+ \phi_n = 1_{L_n}$. So ϕ_n is a retraction. □

<u>Lemma 3</u>. $\bigsqcup_n f_{n\infty} g_{\infty n} = 1_{L_\infty}$.

<u>Proof</u>. For each n , $f_{n\infty} g_{\infty n} \leq 1_{L_\infty}$, so $\bigsqcup_n f_{n\infty} g_{\infty n} \leq 1_{L_\infty}$. Furthermore, if $x = (x_0, x_1, \ldots) \in L_\infty$, then $x_n = g_{\infty n}(x) = g_{\infty n} f_{n\infty} g_{\infty n}(x) = (f_{n\infty} g_{\infty n}(x))_n \leq (\bigsqcup_n f_{n\infty} g_{\infty n}(x))_n$. So $x \leq \bigsqcup f_{n\infty} g_{\infty n}(x)$, and thus $1_{L_\infty} \leq \quad f_{n\infty} g_{\infty n}(x)$. □

<u>Corollary</u>. $\bigsqcup \phi_n \phi_n^+ = 1_{L_\infty}$

Lemma 4. The morphisms $\phi_n\phi_n^+ \in \underline{CLR}(L_\infty, L_\infty)$ form a directed set.

Proof. It will suffice to show that for each n and k, $g_{\infty k}f_{n\infty}g_n \le g_{\infty k}f_{n+1,\infty}$, for then we would have $f_{n\infty}g_n \le f_{n+1,\infty}$, and consequently $f_{n\infty}g_{\infty n} = f_{n\infty}g_n g_{\infty,n+1} \le f_{n+1,\infty}g_{\infty,n+1}$. This, in turn, guarantees that $\phi_n\phi_n^+ \le \phi_{n+1}\phi_{n+1}^+$, from which the lemma follows.

If $k \le n$, then $g_{\infty k}f_{n\infty}g_n = g_{nk}g_n = g_{n+1,k} = g_{\infty k}f_{n+1,\infty}$. If $k \ge n+1$, then $g_{\infty k}f_{n\infty}g_n = f_{nk}g_n = f_{n+1,k}f_n g_n \le f_{n+1,k} = g_{\infty k}f_{n+1,\infty}$ $\square$.

3. Existence of limits

<u>Theorem 1</u>. Let $F: \underline{\omega} \to \underline{CLR}$, L_∞, and ϕ_n be as before. Then L_∞ is the colimit of F, and the ϕ_n form the limiting cone.

<u>Proof</u>. We must prove (A) that the ϕ_n form a cone, i.e. for all $n,m \in \underline{\omega}$ and $\mu \in \underline{\omega}(n,m)$

$$\begin{tikzcd} L_n \arrow[rr, "F(\mu)"] \arrow[dr, "\phi_n"'] & & L_m \arrow[dl, "\phi_m"] \\ & L_\infty & \end{tikzcd}$$

and (B) if ϕ' is a cone from F to some M, then there is a unique $\Psi \in \underline{CLR}(L_\infty, M)$ such that for all $n \in \underline{\omega}$

$$\begin{tikzcd} L_n \arrow[r, "\phi_n"] \arrow[dr, "\phi'_n"'] & L_\infty \arrow[d, "\Psi"] \\ & M \end{tikzcd}$$

To show (A), we need only show that for each n

$$\begin{tikzcd} L_n \arrow[r, "f_n"] \arrow[dr, "f_{n\infty}"'] & L_{n+1} \arrow[d, "f_{n+1,\infty}"] \\ & L_\infty \end{tikzcd} \qquad \text{(i)}$$

and

$$\begin{tikzcd} L_n & L_{n+1} \arrow[l, "g_n"'] \\ & L_\infty \arrow[ul, "g_{\infty n}"] \arrow[u, "g_{\infty,n+1}"'] \end{tikzcd} \qquad \text{(ii)}$$

To show (i): Let $y \in L_n$. If $k \le n$, $(f_{n+1,\infty}(f_n(y)))_k = g_{n+1,k}(f_n(y)) = g_{nk} \cdot g_n \cdot f_n(y) = g_{nk}(y) = (f_{n\infty}(y))_k$. If $k > n$, $(f_{n+1,\infty}(f_n(y)))_k = f_{n+1,k} \cdot f_n(y) = f_{nk}(y) = (f_{n\infty}(y))_k$. So for all k, $(f_{n+1,\infty}(f_n(y)))_k = (f_{n\infty}(y))_k$, and $f_{n+1,\infty} \cdot f_n = f_{n\infty}$.

To show (ii), if $x \in L_\infty$, $g_n(g_{\infty,n+1}(x)) = g_n(x_{n+1}) = x_n = g_{\infty n}(x)$.

To show (B), let ϕ' be a cone from F to M, with ϕ'_n given by $L_n \underset{g_{Mn}}{\overset{f_{nM}}{\rightleftarrows}} M$. Let $\Psi = \bigsqcup \phi'_n \phi^+_n$, that is $\Psi = \langle h,j \rangle$, where

$$h = \bigsqcup f_{nM} g_{\infty n}$$

$$j = \bigsqcup f_{n\infty} g_{Mn}$$

We must first show that Ψ is a retraction:

$$\begin{aligned}
j \cdot h &= (\bigsqcup_n f_{n\infty} g_{Mn}) \cdot (\bigsqcup_n f_{nM} g_{\infty n}) \\
&= \bigsqcup_n f_{n\infty} g_{Mn} f_{nM} g_{\infty n} && \text{(by continuity of composition)} \\
&= \bigsqcup_n f_{n\infty} g_{\infty n} && \text{(since } \langle f_{nM}, g_{Mn} \rangle \text{ is a retraction)} \\
&= 1_{L_\infty} && \text{(by Lemma 3)}
\end{aligned}$$

$$\begin{aligned}
h \cdot j &= (\bigsqcup_n f_{nM} g_{\infty n}) \cdot (\bigsqcup_n f_{n\infty} g_{Mn}) \\
&= \bigsqcup_n f_{nM} g_{\infty n} f_{n\infty} g_{Mn} \\
&\le \bigsqcup_n f_{nM} g_{Mn} \\
&\le 1_{L_n}
\end{aligned}$$

To show that Ψ is the required mediating arrow, we must show for each n

$$\begin{tikzcd} L_n \arrow[r, "f_{n\infty}"] \arrow[dr, "f_{nM}"'] & L_\infty \arrow[d, "h"] \\ & M \end{tikzcd} \quad \text{(iii)} \qquad\qquad \begin{tikzcd} L_n & L_\infty \arrow[l, "g_{\infty n}"'] \\ & M \arrow[u, "j"'] \arrow[ul, "g_{Mn}"] \end{tikzcd} \quad \text{(iv)}$$

To show (iii): If $y \in L_n$,

$$h(f_{n\infty}(y)) = \bigsqcup_p f_{pM} g_{\infty p} f_{n\infty}(y)$$

$$= \bigsqcup (\{f_{pM} f_{np}(y) \mid p \geq n\} \cup \{f_{pM} g_{np}(y) \mid p<n\})$$

$$= \bigsqcup (\{f_{nM}(y) \mid p>n\} \cup \{f_{pM} g_{np}(y) \mid p<n\})$$

The last equality comes from the fact that the f_{nM} form a cone. So (iii) will be established if only we can show that for $p < n$, $f_{pM} g_{np} < f_{nM}$. But $f_{pM} g_{np} = f_{nM} f_{pn} g_{np} \leq f_{nM}$. Here the equality comes from the cone property and the inequality from the definition of a retraction. This completes the proof of (iii).

To show (iv):

$$g_{\infty n} \cdot j(m) = g_{\infty n}(\bigsqcup_p f_{p\infty} g_{Mp}(m))$$

$$= \bigsqcup_p g_{\infty n} f_{p\infty} g_{Mp}(m)$$

$$= \bigsqcup_p (f_p \cdot g_{Mp}(m))_n$$

$$= \bigsqcup (\{g_{pn} g_{M_p}(m) \mid p<n\} \cup \{f_{pn} g_{Mp}(m) \mid p>n\}$$

$$= \bigsqcup (\{g_{Mn}(m)\} \cup \{f_{pn} g_{Mp}(m) \mid p>n\})$$

Again, we need only verify that for $p > n$, $f_{pn} g_{Mp} \leq g_{Mn}$. If $p \geq n$, since the g's form a cone, we have $f_{pn} g_{np} = f_{pn} g_{np} g_{Mn} \leq g_{Mn}$. This completes (iv).

Last, we must verify that Ψ is unique. It will suffice to show j is unique. So let $\Psi' = \langle h', j' \rangle$ be another mediating arrow. Then $g_{\infty n} j' = g_{Mn}$. So $j' = (\bigsqcup f_{n\infty} g_{\infty n}) j' = \bigsqcup f_{n\infty} g_{\infty n} j' =$ $\bigsqcup f_{n\infty} g_{Mn} = j$. $\square$

<u>Theorem 2</u>. Let $T: \underline{CLP} \to \underline{CLP}$ be a functor continuous on the morphism sets with $T(\phi^+) = (T(\phi))^+$. Let L_0 be a complete lattice, and $\theta_0 \in \underline{CLR}(L_0, T(L_0))$. Define a functor $F: \underline{\omega} \to \underline{CLR}$ as follows:

$$F(0) = L_0$$

$$F(k+1) = T(F(k)) \qquad k \geq 0$$

$$F(0 \to 1) = \theta_0$$

$$F(k+1 \to k+2) = T(F(k \to k+1)) \qquad k \geq 0$$

Let L_n denote $F(n)$, $\theta_n = \langle f_n, g_n \rangle$ denote $F(n \to n+1)$, and let L_∞ be as before. Then $L_\infty \cong T(L_\infty)$.

<u>Proof</u>. By Proposition 1, the range of F does lie in $\underline{CLR}$. Let $\phi_n \in \underline{CLR}(L_n, L_\infty)$ be the components of the limiting cone. Define retractions $\phi_n': L_n \to T(L_\infty)$ by

$$\phi_n' = \begin{cases} T(\phi_{n-1}) & \text{if } n > 0 \\ T(\phi_0) \cdot \theta_0 & \text{if } n = 0 \end{cases}$$

We claim that the ϕ_n' form a cone from F to $T(L_\infty)$, that is for each n :

$$\begin{tikzcd} L_n \arrow[rr, "\theta_n"] \arrow[dr, "\phi_n'"'] & & L_{n+1} \arrow[dl, "\phi_{n+1}'"] \\ & T(L_\infty) & \end{tikzcd}$$

For $n = 0$, $\phi_0' = T(\phi_0) \cdot \theta_0 = \phi_1' \cdot \theta_0$.

For $n > 0$, we have $\phi'_n = T(\phi_{n-1}) = T(\phi_n \cdot \theta_{n-1}) = T(\phi_n) \cdot T(\theta_{n-1}) = \phi'_{n+1} \cdot \theta_n$. So, by Theorem 1, we have a retraction $\psi : L_\infty \to T(L_\infty)$. By Proposition 1, we need only show that $\psi\psi^+ = 1_{T(L_\infty)}$.

$$
\begin{aligned}
\psi\psi^+ &= (\bigsqcup \phi'_n \phi^+_n)(\bigsqcup \phi'_n \phi^+_n)^+ \\
&= (\bigsqcup \phi'_n \phi^+_n) \cdot (\bigsqcup \phi^{++}_n \cdot (\phi'_n)^+) \\
&= (\bigsqcup \phi'_n \phi^+_n) \cdot (\bigsqcup \phi_n (\phi'_n)^+) \\
&= \bigsqcup \phi'_n \phi^+_n \phi_n (\phi'_n)^+ \\
&= \bigsqcup \phi'_n (\phi'_n)^+ \\
&= \bigsqcup T(\phi_{n-1})(T(\phi_{n-1}))^+ \\
&= \bigsqcup T(\phi_{n-1}) T(\phi^+_{n-1}) \\
&= \bigsqcup T(\phi_{n-1} \phi^+_{n-1}) \\
&= T(\bigsqcup \phi_{n-1} \phi^+_{n-1}) \\
&= T(1_{L_\infty}) \\
&= 1_{T(L_\infty)} \quad \square
\end{aligned}
$$

4. Uniqueness Results

Theorem 3 With the hypothesis of the previous theorem, let $L_0 = \{1\}$ and let θ_0 be the unique retraction $\{1\} \to T(\{1\})$. Let M be any "partial solution" to $\chi \cong T(\chi)$, that is, there is a $\xi \in \underline{CLR}(T(M), M)$.

Then there is a retraction $\Psi \in \underline{\mathrm{CLR}}(L_\infty , M)$. Furthermore, there is a unique Ψ with the following property: let α be the unique member of $\underline{\mathrm{CLR}}(\{1\} , M)$. Define $\xi_k \in \underline{\mathrm{CLR}}(T^k(M) , M)$ by $\xi_0 = 1_M$; $\xi_{k+1} = \xi \cdot T(\xi_k)$. Then for each n , $\Psi \cdot \phi_n = \xi_n \cdot T^n(\alpha)$.

<u>Proof</u>. All this follows merely from the definition of colimit and the fact that the $\xi_n \cdot T^n(\alpha)$ form a cone from F to M . To verify the cone property, we need only show that for each $n \geq 0$

$$\begin{tikzcd}
T^n(\{1\}) \arrow[rr, "T^n(\theta_0)"] \arrow[dr, "\xi_n \cdot T^n(\alpha)"'] & & T^{n+1}(\{1\}) \arrow[dl, "\xi_{n+1} \cdot T^{n+1}(\alpha)"] \\
& M &
\end{tikzcd}$$

i.e., $\xi_n \cdot T^n(\alpha) = \xi_{n+1} \cdot T^{n+1}(\alpha) \cdot T^n(\theta_0)$.

For $n = 0$, we have $\xi_0 \cdot T^{(0)}(\alpha) = \alpha = \xi_1 \cdot T(\alpha) \cdot \theta_0$ by uniqueness of α . Assume the required identity holds for n . Then

$$\xi_{n+2} \cdot T^{n+2}(\alpha) \cdot T^{n+1}(\theta_0) = \xi \cdot T(\xi_{n+1}) \cdot T^{n+2}(\alpha) \cdot T^{n+1}(\theta_0)$$

$$= \xi \cdot T(\xi_{n+1} \cdot T^{n+1}(\alpha) \cdot T^n(\theta_0))$$

$$= \xi \cdot T(\xi_n \cdot T^n(\alpha)) \qquad \text{(by IH)}$$

$$= \xi \cdot T(\xi_n) \cdot T^{n+1}(\alpha)$$

$$= \xi_{n+1} \cdot T^{n+1}(\alpha) \quad \square$$

Theorem 4. (Main Result). Let $T: \underline{CL} \to \underline{CL}$ be any endofunctor continuous on the morphism sets. Then there exists a solution L_∞ to the equation $\chi \cong T(\chi)$ which is canonical in the sense that if $M \cong T(M)$, then there is a retraction $\Psi \in \underline{CLR}(L_\infty, M)$.

Proof. T extends to $T' : \underline{CLP} \to \underline{CLP}$ via $T'(\langle f,g \rangle) = \langle Tf, Tg \rangle$; then T' satisfies the conditions of Theorem 2. $\square$

5. Examples

1. Let A be a lattice of "atoms." Let $T(L) = \{1\} \amalg (A \times L)$. L_∞ is the lattice of stacks of A's. The image of $\{1\}$ is the empty stack.

2. Let A be a lattice of atoms. Let $T(L) = A \amalg (L \times L)$. L_∞ is the lattice of "first-rest" lists.

3. If we wish the null list to be distinguishable, then we may set $T(L) = \{1\} \amalg A \amalg (L \times L)$. The choice of T depends on the use to be made of the data type, the operations desired, and the type of partial information needed. Note that $\{1\} \amalg A \amalg (L \times L)$, $(\{1\} \amalg A) \amalg (L \times L)$, and $\{1\} \amalg (A \amalg (L \times L))$ are distinct, non-isomorphic lattices [1].

4. Let $\langle \Omega, r \rangle$ be a ranked set [2]. Let $T(L) = \amalg \{L^{r(s)} \mid s \in \Omega\}$. Then L_∞ is the lattice of ranked Ω-trees [11,12].

5. Let Hom: $\underline{CL}^{op} \times \underline{CL} \to \underline{CL}$ be the internal Hom-functor given by $\mathrm{Hom}(L,M) = [L \to M]$; if $f \in \underline{CL}(L,M)$ and $g \in \underline{CL}(N,P)$ then $\mathrm{Hom}(f,g) \in \underline{CL}([M \to N],[L \to P])$ is given by $\mathrm{Hom}(f,g)(h) = ghf$. Now let $T(L) = [L \to L]$; $T(\langle f,g \rangle) = \langle \mathrm{Hom}(g,f), \mathrm{Hom}(f,g) \rangle$. Then L_∞ should be a model for the lambda-calculus; in fact, $L_\infty = \{1\}$. To get Scott's model [10] one must set $L_0 = \{\bot, \top\}$ and use Theorem 2.

6. Let D be a lattice, let $T(L) = D \amalg [L \to L]$, $T(\langle f,g \rangle) = \langle 1_D \amalg \mathrm{Hom}(g , f) , 1_D \amalg \mathrm{Hom}(f,g) \rangle$. Then L_∞ is a model for a programming language based on the primitive data type D .

7. Hierarchical graphs (similar to [6]). Let G be a fixed set of unlabelled graphs. A hierarchical graph is to be a graph from G whose nodes are labelled with atoms A or other hierarchical graphs. For $g \varepsilon G$, let $|g|$ be the number of nodes in g . So a hierarchical graph is either an atom or a graph g with $|g|$ other hierarchical graphs as the node labels. So we have $T(L) = A \amalg \{L^{|g|} | g \varepsilon G\}$. This gives a representation of these objects as trees.

6. Conclusions

Scott's fixed-point construction is put in a categorical setting, following the approach of [9,10]. The main theorem is seen to be a generalization of the Tarski fixed-point theorem to the category <u>CLR</u> . The result is put in a form which is easy to apply to practical data structures.

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