

Computer Science Department
Indiana University
Lindley Hall 101
Bloomington, Indiana 47401

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REALIZING DATA STRUCTURES AS LATTICES

MITCHELL WAND

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Abstract

We prove that if T is any endofunctor on the category of complete lattices which satisfies a weak continuity condition, then there is a canonical solution to the isomorphism $L \cong T(L)$.

The idea of defining data types recursively dates back at least to [4]^(*), e.g., "A list is either an atom or a pair of lists." In general, we want to find an object X such that $X = T(X)$, in this case, $X = A \cup X \times X$. Scott pointed out [8,9 & 10] that certain transformations T , such as $T(X) = X^X$, had no solutions in the category of sets, but there was a solution to the (weakened) equation $X \cong T(X)$ in the category of complete lattices. Scott also provided solutions for a number of interesting T 's. Reynolds [7] pointed out that Scott's constructions could be unified. In this note, we prove that the unified construction is essentially categorical in nature: if T is an endofunctor on the category of complete lattices, and T satisfies a certain continuity condition, then the equation $X \cong T(X)$ has a solution L_∞ which is canonical, i.e., if $M \cong T(M)$, then there exists a canonical morphism $L_\infty \rightarrow M$. This continues the program suggested by Scott [10].

*Of course, defining sets by induction is a much older idea, in general. Here we mean the application of this idea to computer programming.

1. Definitions

We presume familiarity with the following standard concepts: category, morphism, functor, and colimit. There are a number of elementary expositions of this material, notably [5] and [3, appendix].

On the other hand, there are several distinct notions of the category of complete lattices. A complete lattice is a partially ordered set (L, \leq) with the property that every subset S of L has a least upper bound, denoted $\sqcup S$. A subset D of L is called directed iff every finite subset of D has an upper bound in D . (Thus every directed set is nonempty). If (L, \leq) and (M, \leq') are complete lattices, with least upper bounds denoted \sqcup and \sqcup' , we say a function $f: L \rightarrow M$ is continuous iff for any directed $D \subseteq L$,

$$f(\sqcup D) = \sqcup' \{f(x) \mid x \in D\}$$

The set of all continuous maps from L to M forms a complete lattice under the ordering

$$f \leq g \text{ iff } (\forall x \in L)[f(x) \leq' g(x)]$$

This lattice is denoted $[L \rightarrow M]$. We will use the terms "lattice" and "complete lattice" interchangeably.

Let CL denote the category of complete lattices, with morphisms the continuous maps. Let CLP be the category whose objects are those of CL and with morphism sets given by

$$\underline{CLP}(L, M) = \underline{CL}(L, M) \times \underline{CL}(M, L) . \text{ If } \phi = \langle f, g \rangle \in \underline{CLP}(L, M) , \text{ let}$$

$$\phi^+ = \langle g, f \rangle \in \underline{CLP}(M, L) \text{ with } \langle f, g \rangle \langle f', g' \rangle = \langle ff', g'g \rangle . \text{ Let } \underline{CLR}$$

(the category of retractions) be the subcategory of CLP of

morphisms $\phi \in \underline{\text{CLP}}(L, M)$ such that $\phi\phi^+ \leq 1_M$ and $\phi^+\phi = 1_L$, where the ordering is the natural one on $\underline{\text{CLP}}(L, M) = [L \rightarrow M] \times [M \rightarrow L]$.

We say a functor $T: \underline{\text{CLP}} \rightarrow \underline{\text{CLP}}$ is continuous on morphism sets iff $\bigsqcup T(\phi_i) = T(\bigsqcup \phi_i)$ for any directed set of morphisms. Let

$\underline{\omega}$ be the free category generated by the graph

$\{\omega, \{(n, n+1) | n \in \omega\}\}$. Note that any functor $F: \underline{\omega} \rightarrow \underline{C}$ is uniquely specified by its values on the objects and its values on the (unique) arrows $n \rightarrow n+1$.

Proposition 1 (i) $(\phi \cdot \psi)^+ = \psi^+ \cdot \phi^+$

(ii) $\langle f, g \rangle \in \underline{\text{CLP}}(L, M)$ is a retraction iff $fg \leq 1_M$ and $gf = 1_L$

(iii) $\phi \in \underline{\text{CLR}}(L, M)$ is an isomorphism iff $\phi\phi^+ = 1_M$

(iv) If $T: \underline{\text{CLP}} \rightarrow \underline{\text{CLP}}$ is an endofunctor continuous on the morphism sets, then T preserves retractions.

(v) If $\langle f, g \rangle$ and $\langle f', g \rangle$ are retractions, then $f = f'$

(vi) $\{1\}$ is an initial object of $\underline{\text{CLR}}$

Proof. All trivial except (v): If $\langle f, g \rangle \in \underline{\text{CLR}}(L, M)$, then $f(x) = \bigsqcap \{y \in M | g(y) = x\}$. (vi) follows from (v) and the fact that $\{1\}$ is a final object in $\underline{\text{CL}}$.

If there is a retraction $\langle f, g \rangle \in \underline{\text{CLR}}(L, M)$, then L is embeddable in M via f , and every $x \in M$ has a unique best approximation in the image of L , given by $fg(x)$. So if $\underline{\text{CLR}}(L, M)$ is nonempty, we may think of L as a sub-datatype of M .

2. Lattice-Theoretic Lemmata

Throughout the following let F be a functor $\omega \rightarrow \underline{CLR}$ given by $F(n) = L_n$ and $F(n \rightarrow n+1) = \theta_n = \langle f_n, g_n \rangle$. Let L_∞ denote the complete lattice whose underlying set is given by

$$\{(x_0, x_1, \dots) \mid x_i \in L_i \text{ \& } x_i = g_i(x_{i+1})\}$$

under the ordering $x \leq y$ iff $(\forall i)[x_i \leq y_i]$. If $n \leq m$, we write $\langle f_{nm}, g_{mn} \rangle$ for the image under F of the unique morphism in $\omega(n, m)$.

Lemma 1. L_∞ is a complete lattice.

Proof. Let $S \subseteq L_\infty$. Let $S_k = \{x_k \mid x \in S\} \subseteq L_k$.

Then for each k , S_k has a least upper bound $\bigsqcup S_k \in L_k$.

Let $y_k = \bigsqcup_{n \geq k} g_{nk}(\bigsqcup S_n)$. Setting $n = k$, we see $y_k \geq \bigsqcup S_k$,

and y is a member of L_∞ by the construction of the g_{nk} .

If $x \in S$, then for each k , $x_k \leq \bigsqcup S_k \leq y_k$, so $x \leq y$.

Hence y is an upper bound for S in L_∞ . Let z be another upper bound for S in L_∞ . Then for every n , $\bigsqcup S_n \leq z_n$.

Now $z \in L_\infty$, so for every $n \geq k$, $z_k = g_{nk}(z_n)$. So $z_k =$

$$\bigsqcup_{n \geq k} g_{nk}(z_n) \geq \bigsqcup_{n \geq k} g_{nk}(\bigsqcup S_n) = y_k. \text{ So } y \leq z, \text{ and } y = \bigsqcup S. \quad \square$$

Define a morphism $\phi_n = \langle f_{n\infty}, g_{\infty n} \rangle \in \underline{CLP}(L_n, L_\infty)$ by

$$g_{\infty n}((x_0, x_1, \dots)) = x_n$$

$$(f_{n\infty}(y))_k = g_{nk}(y) \text{ if } k \leq n$$

$$f_{nk}(y) \text{ if } k > n$$

Thus

$$g_{\infty k} f_{n\infty} = g_{nk} \text{ if } k \leq n \text{ and } f_{nk} \text{ if } k > n, \text{ and } g_{\infty n} f_{n\infty} = 1_{L_n}.$$

Lemma 2. $\phi_n \in \underline{CLR}(L_n, L_\infty)$.

Proof. We must first show that the range of $f_{n\infty}$ is in fact L_∞ and not just ΠL_n , that is, that $(f_{n\infty}(y))_k = g_k(f_{n\infty}(y))_{k+1}$. If $k < n$, then $g_k(f_{n\infty}(y))_{k+1} = g_k(g_{n, k+1}(y)) = g_{nk}(y) = (f_{n\infty}(y))_k$. If $k \geq n$, then $g_k(f_{n\infty}(y))_{k+1} = g_k f_{n, k+1}(y) = g_{k+1, k} f_{n, k+1}(y) = g_{k+1, k} f_{k, k+1} f_{n, k}(y) = f_{nk}(y) = (f_{n\infty}(y))_k$. We may now show that ϕ_n is a retraction:

$$\begin{aligned} \phi_n \phi_n^+ &= \langle f_{n\infty}, g_{\infty n} \rangle \cdot \langle g_{\infty n}, f_{n\infty} \rangle = \langle f_{n\infty} g_{\infty n}, f_{n\infty} g_{\infty n} \rangle. f_{n\infty} g_{\infty n}(x_0, x_1, \dots) = \\ f_{n\infty}(x_n) &= \langle g_{n0}(x_n), g_{n1}(x_n), \dots, x_n, f_{n, n+1}(x_n), \dots, f_{nk}(x_n), \dots \rangle \end{aligned}$$

For $k \leq n$, $g_{nk}(x_n) = x_k$; for $k > n$, $f_{nk}(x_n) = f_{nk} g_{kn}(x_k) \leq x_k$, since $\langle f_{nk}, g_{kn} \rangle \in \underline{CLR}(L_n, L_k)$. So $\phi_n \phi_n^+ \leq 1_{L_\infty}$.

$\phi_n^+ \phi_n = \langle g_{\infty n} f_{n\infty}, g_{\infty n} f_{n\infty} \rangle$. So $g_{\infty n} f_{n\infty}(y) = (f_{n\infty}(y))_n = g_{nn}(y) = y$. Hence $\phi_n^+ \phi_n = 1_{L_n}$. So ϕ_n is a retraction. \square

Lemma 3. $\bigsqcup_n f_{n\infty} g_{\infty n} = 1_{L_\infty}$.

Proof. For each n , $f_{n\infty} g_{\infty n} \leq 1_{L_\infty}$, so $\bigsqcup_n f_{n\infty} g_{\infty n} \leq 1_{L_\infty}$.

Furthermore, if $x = (x_0, x_1, \dots) \in L_\infty$, then $x_n = g_{\infty n}(x) = g_{\infty n} f_{n\infty} g_{\infty n}(x) = (f_{n\infty} g_{\infty n}(x))_n \leq (\bigsqcup_n f_{n\infty} g_{\infty n}(x))_n$. So

$x \leq \bigsqcup_n f_{n\infty} g_{\infty n}(x)$, and thus $1_{L_\infty} \leq \bigsqcup_n f_{n\infty} g_{\infty n}(x)$. \square

Corollary. $\bigsqcup_n \phi_n \phi_n^+ = 1_{L_\infty}$

Lemma 4. The morphisms $\phi_n \phi_n^+ \in \underline{\text{CLR}}(L_\infty, L_\infty)$ form a directed set.

Proof. It will suffice to show that for each n and k , $g_{\infty k} f_{n\infty} g_n \leq g_{\infty k} f_{n+1, \infty}$, for then we would have $f_{n\infty} g_n \leq f_{n+1, \infty}$, and consequently $f_{n\infty} g_{\infty n} = f_{n\infty} g_n g_{\infty, n+1} \leq f_{n+1, \infty} g_{\infty, n+1}$. This, in turn, guarantees that $\phi_n \phi_n^+ \leq \phi_{n+1} \phi_{n+1}^+$, from which the lemma follows.

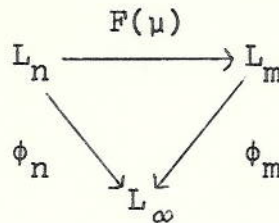
If $k \leq n$, then $g_{\infty k} f_{n\infty} g_n = g_{nk} g_n = g_{n+1, k} = g_{\infty k} f_{n+1, \infty}$.

If $k \geq n+1$, then $g_{\infty k} f_{n\infty} g_n = f_{nk} g_n = f_{n+1, k} f_n g_n \leq f_{n+1, k} = g_{\infty k} f_{n+1, \infty}$ \square .

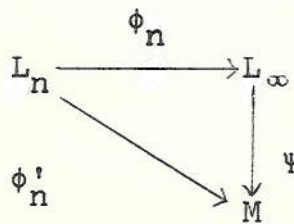
3. Existence of limits

Theorem 1. Let $F: \underline{\omega} \rightarrow \underline{CLR}$, L_ω , and ϕ_n be as before. Then L_ω is the colimit of F , and the ϕ_n form the limiting cone.

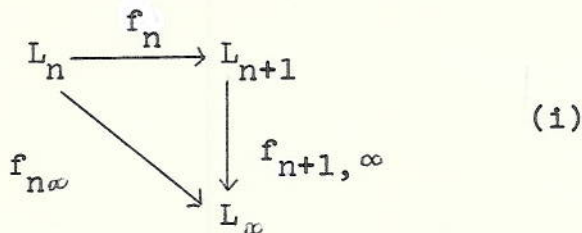
Proof. We must prove (A) that the ϕ_n form a cone, i.e. for all $n, m \in \underline{\omega}$ and $\mu \in \underline{\omega}(n, m)$



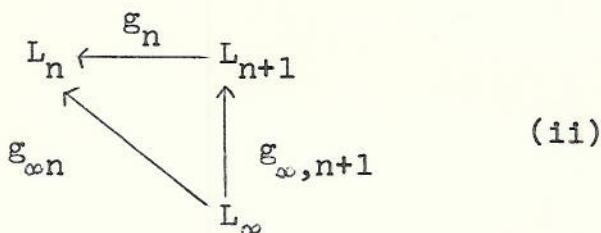
and (B) if ϕ' is a cone from F to some M , then there is a unique $\Psi \in \underline{CLR}(L_\omega, M)$ such that for all $n \in \underline{\omega}$



To show (A), we need only show that for each n



and



To show (i): Let $y \in L_n$. If $k \leq n$, $(f_{n+1,\infty}(f_n(y)))_k = g_{n+1,k}(f_n(y)) = g_{nk} \cdot g_n \cdot f_n(y) = g_{nk}(y) = (f_{n\infty}(y))_k$. If $k > n$, $(f_{n+1,\infty}(f_n(y)))_k = f_{n+1,k} \cdot f_n(y) = f_{nk}(y) = (f_{n\infty}(y))_k$. So for all k , $(f_{n+1,\infty}(f_n(y)))_k = (f_{n\infty}(y))_k$, and $f_{n+1,\infty} \cdot f_n = f_{n\infty}$.

To show (ii), if $x \in L_\infty$, $g_n(g_{\infty,n+1}(x)) = g_n(x_{n+1}) = x_n = g_{\infty n}(x)$.

To show (B), let ϕ' be a cone from F to M , with ϕ'_n given by $L_n \xrightarrow[f_{nM}]{f_{nM}}$ M . Let $\Psi = \bigsqcup \phi'_n \phi_n^+$, that is $\Psi = \langle h, j \rangle$, where

$$h = \bigsqcup f_{nM} g_{\infty n}$$

$$j = \bigsqcup f_{n\infty} g_{Mn}$$

We must first show that Ψ is a retraction:

$$\begin{aligned} j \cdot h &= \left(\bigsqcup_n f_{n\infty} g_{Mn} \right) \cdot \left(\bigsqcup_n f_{nM} g_{\infty n} \right) \\ &= \bigsqcup_n f_{n\infty} g_{Mn} f_{nM} g_{\infty n} && \text{(by continuity of composition)} \\ &= \bigsqcup_n f_{n\infty} g_{\infty n} && \text{(since } \langle f_{nM}, g_{Mn} \rangle \text{ is a retraction)} \\ &= 1_{L_\infty} && \text{(by Lemma 3)} \\ h \cdot j &= \left(\bigsqcup_n f_{nM} g_{\infty n} \right) \cdot \left(\bigsqcup_n f_{n\infty} g_{Mn} \right) \\ &= \bigsqcup_n f_{nM} g_{\infty n} f_{n\infty} g_{Mn} \\ &\leq \bigsqcup_n f_{nM} g_{Mn} \\ &\leq 1_{L_n} \end{aligned}$$

To show that Ψ is the required mediating arrow, we must show for each n

$$\begin{array}{ccc}
 L_n & \xrightarrow{f_{n\infty}} & L_\infty \\
 & \searrow f_{nM} & \downarrow h \\
 & & M
 \end{array} \quad (iii)$$

$$\begin{array}{ccc}
 L_n & \xleftarrow{g_{\infty n}} & L_\infty \\
 & \swarrow g_{Mn} & \uparrow j \\
 & & M
 \end{array} \quad (iv)$$

To show (iii): If $y \in L_n$,

$$\begin{aligned}
 h(f_{n\infty}(y)) &= \bigsqcup_p f_{pM} g_{\infty p} f_{n\infty}(y) \\
 &= \bigsqcup (\{f_{pM} f_{np}(y) | p \geq n\} \cup \{f_{pM} g_{np}(y) | p < n\}) \\
 &= \bigsqcup (\{f_{nM}(y) | p > n\} \cup \{f_{pM} g_{np}(y) | p < n\})
 \end{aligned}$$

The last equality comes from the fact that the f_{nM} form a cone. So (iii) will be established if only we can show that for $p < n$, $f_{pM} g_{np} < f_{nM}$. But $f_{pM} g_{np} = f_{nM} f_{pn} g_{np} \leq f_{nM}$. Here the equality comes from the cone property and the inequality from the definition of a retraction. This completes the proof of (iii).

To show (iv):

$$\begin{aligned}
 g_{\infty n} \cdot j(m) &= g_{\infty n} (\bigsqcup_p f_{p\infty} g_{Mp}(m)) \\
 &= \bigsqcup_p g_{\infty n} f_{p\infty} g_{Mp}(m) \\
 &= \bigsqcup_p (f_p \cdot g_{Mp}(m))_n \\
 &= \bigsqcup (\{g_{pn} g_{Mp}(m) | p < n\} \cup \{f_{pn} g_{Mp}(m) | p > n\}) \\
 &= \bigsqcup (\{g_{Mn}(m)\} \cup \{f_{pn} g_{Mp}(m) | p > n\})
 \end{aligned}$$

Again, we need only verify that for $p > n$, $f_{pn} g_{Mp} \leq g_{Mn}$.

If $p \geq n$, since the g 's form a cone, we have $f_{pn} g_{np} = f_{pn} g_{np} g_{Mn} \leq g_{Mn}$. This completes (iv).

Last, we must verify that Ψ is unique. It will suffice to show j is unique. So let $\Psi' = \langle h', j' \rangle$ be another mediating

arrow. Then $g_{\omega n} j' = g_{Mn}$. So $j' = (\bigsqcup_{n \in \omega} g_{\omega n}) j' = \bigsqcup_{n \in \omega} g_{\omega n} j' = \bigsqcup_{n \in \omega} g_{Mn} = j$. \square

Theorem 2. Let $T: \underline{CLP} \rightarrow \underline{CLP}$ be a functor continuous on the morphism sets with $T(\phi^+) = (T(\phi))^+$. Let L_0 be a complete lattice, and $\theta_0 \in \underline{CLR}(L_0, T(L_0))$. Define a functor

$F: \omega \rightarrow \underline{CLR}$ as follows:

$$F(0) = L_0$$

$$F(k+1) = T(F(k)) \quad k \geq 0$$

$$F(0 \rightarrow 1) = \theta_0$$

$$F(k+1 \rightarrow k+2) = T(F(k \rightarrow k+1)) \quad k \geq 0$$

Let L_n denote $F(n)$, $\theta_n = \langle f_n, g_n \rangle$ denote $F(n \rightarrow n+1)$, and let L_∞ be as before. Then $L_\infty \cong T(L_\infty)$.

Proof. By Proposition 1, the range of F does lie in \underline{CLR} . Let $\phi_n \in \underline{CLR}(L_n, L_\infty)$ be the components of the limiting cone. Define retractions $\phi'_n: L_n \rightarrow T(L_\infty)$ by

$$\phi'_n = \begin{cases} T(\phi_{n-1}) & \text{if } n > 0 \\ T(\phi_0) \cdot \theta_0 & \text{if } n = 0 \end{cases}$$

We claim that the ϕ'_n form a cone from F to $T(L_\infty)$, that is for each n :

$$\begin{array}{ccc} & \theta_n & \\ & \longrightarrow & \\ L_n & & L_{n+1} \\ & \searrow \phi'_n & \swarrow \phi'_{n+1} \\ & T(L_\infty) & \end{array}$$

For $n = 0$, $\phi'_0 = T(\phi_0) \cdot \theta_0 = \phi'_1 \cdot \theta_0$.

For $n > 0$, we have $\phi'_n = T(\phi_{n-1}) = T(\phi_n \cdot \theta_{n-1}) = T(\phi_n) \cdot T(\theta_{n-1}) = \phi'_{n+1} \cdot \theta_n$. So, by Theorem 1, we have a retraction $\psi: L_\infty \rightarrow T(L_\infty)$.

By Proposition 1, we need only show that $\psi\psi^+ = 1_{T(L_\infty)}$.

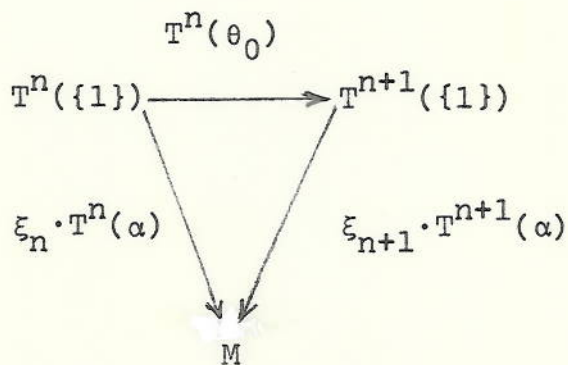
$$\begin{aligned}
 \psi\psi^+ &= (\bigcup \phi'_n \phi_n^+) (\bigcup \phi'_n \phi_n^+)^+ \\
 &= (\bigcup \phi'_n \phi_n^+) \cdot (\bigcup \phi_n^{++} \cdot (\phi'_n)^+)^+ \\
 &= (\bigcup \phi'_n \phi_n^+) \cdot (\bigcup \phi_n (\phi'_n)^+)^+ \\
 &= \bigcup \phi'_n \phi_n^+ \phi_n (\phi'_n)^+ \\
 &= \bigcup \phi'_n (\phi'_n)^+ \\
 &= \bigcup T(\phi_{n-1}) (T(\phi_{n-1}))^+ \\
 &= \bigcup T(\phi_{n-1}) T(\phi_{n-1}^+) \\
 &= \bigcup T(\phi_{n-1} \phi_{n-1}^+) \\
 &= T(\bigcup \phi_{n-1} \phi_{n-1}^+) \\
 &= T(1_{L_\infty}) \\
 &= 1_{T(L_\infty)} \quad \square
 \end{aligned}$$

4. Uniqueness Results

Theorem 3 With the hypothesis of the previous theorem, let $L_0 = \{1\}$ and let θ_0 be the unique retraction $\{1\} \rightarrow T(\{1\})$. Let M be any "partial solution" to $\chi \cong T(\chi)$, that is, there is a $\xi \in \underline{\text{CLR}}(T(M), M)$.

Then there is a retraction $\psi \in \underline{\text{CLR}}(L_\infty, M)$. Furthermore, there is a unique ψ with the following property: let α be the unique member of $\underline{\text{CLR}}(\{1\}, M)$. Define $\xi_k \in \underline{\text{CLR}}(T^k(M), M)$ by $\xi_0 = 1_M$; $\xi_{k+1} = \xi \cdot T(\xi_k)$. Then for each n , $\psi \cdot \phi_n = \xi_n \cdot T^n(\alpha)$.

Proof. All this follows merely from the definition of colimit and the fact that the $\xi_n \cdot T^n(\alpha)$ form a cone from F to M . To verify the cone property, we need only show that for each $n \geq 0$



i.e., $\xi_n \cdot T^n(\alpha) = \xi_{n+1} \cdot T^{n+1}(\alpha) \cdot T^n(\theta_0)$.

For $n = 0$, we have $\xi_0 \cdot T^{(0)}(\alpha) = \alpha = \xi_1 \cdot T(\alpha) \cdot \theta_0$ by uniqueness of α . Assume the required identity holds for n . Then

$$\begin{aligned}
 \xi_{n+2} \cdot T^{n+2}(\alpha) \cdot T^{n+1}(\theta_0) &= \xi \cdot T(\xi_{n+1}) \cdot T^{n+2}(\alpha) \cdot T^{n+1}(\theta_0) \\
 &= \xi \cdot T(\xi_{n+1} \cdot T^{n+1}(\alpha) \cdot T^n(\theta_0)) \\
 &= \xi \cdot T(\xi_n \cdot T^n(\alpha)) \quad (\text{by IH}) \\
 &= \xi \cdot T(\xi_n) \cdot T^{n+1}(\alpha) \\
 &= \xi_{n+1} \cdot T^{n+1}(\alpha) \quad \square
 \end{aligned}$$

Theorem 4. (Main Result). Let $T: \underline{CL} \rightarrow \underline{CL}$ be any endofunctor continuous on the morphism sets. Then there exists a solution L_∞ to the equation $\chi \cong T(\chi)$ which is canonical in the sense that if $M \cong T(M)$, then there is a retraction $\psi \in \underline{CLR}(L_\infty, M)$.

Proof. T extends to $T' : \underline{CLP} \rightarrow \underline{CLP}$ via $T'(\langle f, g \rangle) = \langle Tf, Tg \rangle$; then T' satisfies the conditions of Theorem 2. \square

5. Examples

1. Let A be a lattice of "atoms." Let $T(L) = \{1\} \amalg (A \times L)$. L_∞ is the lattice of stacks of A 's. The image of $\{1\}$ is the empty stack.

2. Let A be a lattice of atoms. Let $T(L) = \text{All}(L \times L)$. L_∞ is the lattice of "first-rest" lists.

3. If we wish the null list to be distinguishable, then we may set $T(L) = \{1\} \amalg \text{All}(L \times L)$. The choice of T depends on the use to be made of the data type, the operations desired, and the type of partial information needed. Note that $\{1\} \amalg \text{All}(L \times L)$, $(\{1\} \amalg A) \amalg (L \times L)$, and $\{1\} \amalg (\text{All}(L \times L))$ are distinct, non-isomorphic lattices [1].

4. Let $\langle \Omega, r \rangle$ be a ranked set [2]. Let $T(L) = \amalg \{L^{r(s)} \mid s \in \Omega\}$. Then L_∞ is the lattice of ranked Ω -trees [11,12].

5. Let $\text{Hom}: \underline{CL}^{\text{op}} \times \underline{CL} \rightarrow \underline{CL}$ be the internal Hom-functor given by $\text{Hom}(L, M) = [L \rightarrow M]$; if $f \in \underline{CL}(L, M)$ and $g \in \underline{CL}(N, P)$ then $\text{Hom}(f, g) \in \underline{CL}([M \rightarrow N], [L \rightarrow P])$ is given by $\text{Hom}(f, g)(h) = ghf$. Now let $T(L) = [L \rightarrow L]$; $T(\langle f, g \rangle) = \langle \text{Hom}(g, f), \text{Hom}(f, g) \rangle$. Then L_∞ should be a model for the lambda-calculus; in fact, $L_\infty = \{1\}$. To get Scott's model [10] one must set $L_0 = \{1, \tau\}$ and use Theorem 2.

6. Let D be a lattice, let $T(L) = D \parallel [L \rightarrow L]$, $T(\langle f, g \rangle) = \langle 1_D \parallel \text{Hom}(g, f), 1_D \parallel \text{Hom}(f, g) \rangle$. Then L_∞ is a model for a programming language based on the primitive data type D .

7. Hierarchical graphs (similar to [6]). Let G be a fixed set of unlabelled graphs. A hierarchical graph is to be a graph from G whose nodes are labelled with atoms A or other hierarchical graphs. For $g \in G$, let $|g|$ be the number of nodes in g . So a hierarchical graph is either an atom or a graph g with $|g|$ other hierarchical graphs as the node labels. So we have $T(L) = A \parallel \{L^{|g|} \mid g \in G\}$. This gives a representation of these objects as trees.

6. Conclusions

Scott's fixed-point construction is put in a categorical setting, following the approach of [9,10]. The main theorem is seen to be a generalization of the Tarski fixed-point theorem to the category CLR. The result is put in a form which is easy to apply to practical data structures.

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