LOGICS WHICH ARE CHARACTERIZED BY SUBRESIDUATED LATTICES

George Epstein
Computer Science Department
Indiana University
Bloomington, Indiana 47401

Alfred Horn
Department of Mathematics
University of California
Los Angeles, California 90024

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GEORGE EPSTEIN
ALFRED HORN
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by George Epstein in Bloomington, Indiana

and Alfred Horn in Los Angeles, California (U.S.A.)

A subresiduated lattice (abbreviated s.r. lattice) is a pair (A,Q), where A is a bounded distributive lattice (with largest element 1 and smallest element 0), and Q is a sublattice of A containing $0,1$ such that for each $x,y \in A$ there is an element $z \in Q$ with the property that for all $q \in Q$, $x \wedge q \leq y$ if and only if $q \leq z$. This z is denoted by $x \overset{Q}{\rightarrow} y$, or simply $x \rightarrow y$. We shall sometimes refer to the s.r. lattice by A. When $Q = A$, A is usually called a Heyting algebra (or pseudo-Boolean algebra). The set of complemented elements of A is called the center of A, and is denoted by $B(A)$. If $x \in B(A)$, its complement is denoted by $-x$.

Suppose we have a propositional calculus with symbols $\&$, $\vee$, $\supset$ and $\sim$ for conjunction, disjunction, implication and negation, and possibly with a symbol $\Box$ for affirmation or necessitation. If we assign to the propositional variables values in a s.r. lattice, then we obtain a valuation $v(\alpha)$ for each formula α by the rules $v(\alpha \;\&\; \beta) = v(\alpha) \wedge v(\beta)$, $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$, $v(\alpha \supset \beta) = v(\alpha) \rightarrow v(\beta)$, $v(\sim\alpha) = v(\alpha) \rightarrow 0$ and $v(\Box\alpha) = 1 \rightarrow v(\alpha)$. α is said to be valid in A if $v(\alpha) = 1$ for every assignment in A. A logic is said to be characterized by a class K of s.r. lattices if it consists of those formulas which are valid in every member of K.

This framework provides a unified method of classifying several known calculi and leads naturally to new calculi which are of some algebraic and philosophic interest. For example, it is well known that the intuitionist propositional calculus is characterized by the class of all Heyting algebras.

In the Lewis systems S4 and S5 of modal logic, there are two kinds of implication: classical or material implication, denoted by $\alpha \supset \beta$, and strict implication, denoted by $\Box(\alpha \supset \beta)$. There are also classical negation, denoted by $\sim\alpha$, and strict negation, denoted by $\Box \sim \alpha$. It will be convenient for us to change the notation as follows: we shall use $\alpha \supset \beta$ and $\sim \alpha$ for strict implication and negation. The notation for classical negation will be $-\alpha$ and classical implication, previously denoted by $\alpha \supset \beta$ will be denoted by $-\alpha \vee \beta$. With this notation, the set of all theorems of S4 (or S5) which involve the strict connectives $\supset$ and $\sim$ together with & and $\vee$ is called the Lewy calculus for S4 (or S5) by Hacking [4]. We shall use R4 and R5 to denote these Lewy calculi. Hacking gave a set of axioms for R4 and R5 using Gentzen methods. We shall see that R4 is characterized by the class of all s.r. lattices, and R5 is characterized by the class of all s.r. lattices (A,Q) such that Q is a Boolean subalgebra of the center of A. The Lewis S4 and S5 systems are characterized by the class of all s.r. lattices (A,Q) such that A is a Boolean algebra (and for the case of S5, Q is a Boolean subalgebra of A) provided we add the rule $v(-\alpha) = -v(\alpha)$. We shall give an algebraic derivation of Hacking's axioms for R4 and R5.

The calculus obtained by adding to S4 the axiom schema $(\Box\alpha \supset \Box\beta) \vee (\Box\beta \supset \Box\alpha)$ is called S4.3 by Dummett and Lemmon [2]. (Recall we are using $\supset$ for strict implication.) Accordingly we denote by R4.3 the logic obtained by adding this axiom to R4.

We shall see that R4.3 is characterized by the class of all s.r. lattices such that Q is a chain.

The logic characterized by the class of all s.r. lattices (A,Q) such that Q is the center of A is new. We shall give an axiomatization of this calculus.

Of special interest is the logic PC characterized by the class of all s.r. lattices (A,Q) such that A is a P-algebra [3], (or alternately, where A is a chain), and Q is the center of A. In such a lattice, both $x \overset{A}{\to} y$ and $x \overset{Q}{\to} y$ exist and they are connected by the relations $x \overset{A}{\to} y = y \vee (x \overset{Q}{\to} y)$, $x \overset{Q}{\to} y = \Box(x \overset{A}{\to} y)$. This leads to two types of implication in PC : $\alpha \supset \beta$ corresponding to $x \overset{Q}{\to} y$ and $\alpha \supset' \beta$ corresponding to $x \overset{A}{\to} y$. Either of these connectives could be used as primitive in an axiomatization of PC. In terms of $\supset'$ as primitive, we shall see that PC is a kind of strong modal extension of Dummett's LC[1].

1. Subresiduated lattices

If x is a member of a s.r. lattice A, we set $!x = 1 \to x$ and $\neg x = x \to 0$. We list some statements which hold identically in every s.r. lattice.

(1) $x \leq y$ if and only if $x \to y = 1$

(2) $x \leq y$ implies $x \to z \geq y \to z$, $z \to x \leq z \to y$, $!x \leq !y$ and $\neg x \geq \neg y$

(3) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(4) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$

(5) $(x \to y) \wedge (y \to z) \leq x \to z$

(6) $\neg 0 = !1 = 1$, $!0 = \neg 1 = 0$

(7) $!x \leq x$

(8) $!(x \wedge y) = !x \wedge !y$

(9) $\neg(x \vee y) = \neg x \wedge \neg y$

(10) $!!x = !x \leq \neg\neg !x \leq \neg\neg x$

(11) $!\neg x = \neg x \leq \neg\neg\neg x \leq \neg !x$

(12) $\neg\neg\neg\neg x = \neg\neg x$

(13) $\neg\neg\neg !x = \neg !x$

(14) $\neg x \to x = \neg\neg x$

(15) $x \to \neg x = \neg x$

The identities (10)-(13) follow from the fact that Q is a Heyting algebra. They show that there are at most seven elements which can be formed by starting with x and applying the operations $\neg$ and $!$. These are $x, \neg x, \neg\neg x, \neg\neg\neg x, !x, \neg !x$, and $\neg\neg !x$. It is easy to construct examples to show that these seven elements can be all different.

In a s.r. lattice, the operation $\to$ determines the set Q, since $Q = \{x : x = !x\}$. Therefore a s.r. lattice may be regarded as an algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$. We use the terms s.r. sublattice, s.r. homomorphism and s.r. congruence relation from this point of view. For example, a s.r. sublattice of A is a sublattice of A containing 0 and 1 which is closed under $\to$. The following theorem shows that the class of s.r. lattices is equational.

THEOREM 1. An algebra $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a s.r. lattice if and only if $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the following hold identically:

(a) $(x \wedge y) \to y = 1$

(b) $x \to y \leq z \to (x \to y)$

(c) $x \wedge (x \to y) \leq y$

(d) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$

<u>Proof</u>. The necessity of (a)-(d) is clear. Suppose these identities hold in A, and let $Q = \{x : x = 1 \to x\}$. By (c) we have

(e) $$1 \to y \leq y,$$

and by (d),

(f) $$x \leq y \text{ implies } z \to x \leq z \to y.$$

Suppose $q,r \in Q$. They by (f) and (e),

$$q = 1 \to q \leq 1 \to (q \vee r) \leq q \vee r,$$

and similarly $r \leq 1 \to (q \vee r)$. Hence $q \vee r = 1 \to (q \vee r)$, so $q \vee r \in Q$. By (d), we also have $q \wedge r \in Q$. Also $1 \in Q$ and $0 \in Q$ by (a) and (e). For any $x,y \in A$, we have $x \to y \in Q$ by (b) and (e). It remains to show: if $q \in Q$ and $x \wedge q \leq y$ then $q \leq x \to y$. Indeed

$$\begin{aligned} q = 1 \to q &\leq x \to (1 \to q) \quad \text{by (b)} \\ &\leq x \to q \\ &\leq x \to (x \wedge q) \quad \text{by (d), since } x \to x = 1, \\ &\leq x \to y \quad \text{by (f).} \end{aligned}$$

THEOREM 2. Let (A,Q) be a s.r. lattice. There is an order isomorphism between the set of all s.r. congruence relations θ in A and the set of all filters F in Q. Under this correspondence

(i) $$F = \{x \in Q : (x,1) \in \theta\}, \text{ and}$$

(ii) $$\theta = \{(x,y) : (x \to y) \wedge (y \to x) \in F\}$$

<u>Proof</u>. Given θ, define F by (i). Then F is a filter in Q and $(x,y) \in \theta$ implies $x \to y \in F$ and $y \to x \in F$. If

$(x \to y) \wedge (y \to x) \equiv 1 \pmod{\theta}$, then $x \wedge y \equiv x \wedge (x \to y) \wedge y$ $= x \wedge (x \to y) \equiv x$, and similarly $x \wedge y \equiv y$. Hence (ii) holds. If we start with a filter F in Q and define θ by (ii), then θ is a congruence relation since (5) holds and

$$x \to y \leq (y \to z) \to (x \to z)$$
$$x \to y \leq (z \to x) \to (z \to y)$$
$$x \to y \leq (x \wedge z) \to (y \wedge z)$$
$$x \to y \leq (x \vee z) \to (y \vee z)$$

Also (i) holds because $(1 \to q) \wedge (q \to 1) = q$ for all $q \in Q$.

COROLLARY 3. A s.r. lattice (A,Q) is subdirectly irreducible if and only if $\{x \in Q : x < 1\}$ has a largest element.

Proof. By Theorem 2, (A,Q) is subdirectly irreducible if and only if Q has a smallest filter properly containing $\{1\}$. This is easily seen to be equivalent to the given condition.

2. The R4 calculus and R4.3.

We first determine axioms for the logic characterized by the class of all s.r. lattices. It is easy to see that the following axiom schemas are valid in every s.r. lattice (α, β and γ refer to arbitrary formulas):

(A1) $\alpha \supset \alpha$

(A2) $(\alpha \supset \beta) \supset (\gamma \supset (\alpha \supset \beta))$

(A3) $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

(A4) $(\alpha \mathbin{\&} \beta) \supset \alpha$

(A5) $(\alpha \,\&\, \beta) \supset \beta$

(A6) $(\alpha \supset \beta) \supset ((\alpha \supset \gamma) \supset (\alpha \supset (\beta \,\&\, \gamma)))$

(A7) $\alpha \supset (\alpha \vee \beta)$

(A8) $\beta \supset (\alpha \vee \beta)$

(A9) $(\alpha \supset \gamma) \supset ((\beta \supset \gamma) \supset ((\alpha \vee \beta) \supset \gamma))$

(A10) $(\alpha \,\&\, (\beta \vee \gamma)) \supset ((\alpha \,\&\, \beta) \vee (\alpha \,\&\, \gamma))$

(A11) $\sim \alpha \supset (\alpha \supset \beta)$

(A12) $(\alpha \supset \sim \alpha) \supset \sim \alpha$

(A13) $\Box \alpha \supset ((\alpha \supset \alpha) \supset \alpha)$

(A14) $((\alpha \supset \alpha) \supset \alpha) \supset \Box \alpha$

THEOREM 4. The logic characterized by the class of all s.r. lattices is axiomatized by (A1)-(A14), using modus ponens as the only rule of inference.

Proof. We use the standard device of forming the Lindenbaum algebra of equivalence classes of formulas. Using (A2) and (A3) it is not hard to prove

(16) $(\alpha \supset \beta) \supset ((\gamma \supset \alpha) \supset (\gamma \supset \beta))$,

and

(17) $(\alpha \supset \beta) \supset ((\beta \supset \gamma) \supset (\alpha \supset \gamma)$

It is also easy to prove a deduction theorem: If $\alpha_1, \ldots, \alpha_n \vdash \beta$ then $\alpha_1, \ldots, \alpha_{n-1} \vdash \alpha_n \supset \beta$, provided $\alpha_1, \ldots, \alpha_{n-1}$ are each hypothetical, that is, of the form $\gamma \supset \delta$. Next one can show

(18) $(\alpha \supset \beta) \supset (\sim \beta \supset \sim \alpha)$

using $\sim\beta \supset (\beta \supset \sim\alpha)$, (A12), and (17).

Now define α eq β if $\vdash \alpha \supset \beta$ and $\vdash \beta \supset \alpha$. It is easily seen that eq is an equivalence relation. Let $|\alpha|$ be the equivalence class containing α. We can define operations on the set E of all equivalence classes unambiguously by

$$|\alpha| \vee |\beta| = |\alpha \vee \beta|$$
$$|\alpha| \wedge |\beta| = |\alpha \,\&\, \beta|$$
$$|\alpha| \rightarrow |\beta| = |\alpha \supset \beta|$$
$$\neg |\alpha| = |\sim\alpha|$$
$$!\,|\alpha| = |\Box\,\alpha|$$

Although the schema $\alpha \supset (\beta \supset \alpha)$ is not provable, we can prove by induction on the length of the proof of α that: if $\vdash \alpha$ then $\vdash \beta \supset \alpha$. This holds when α is an axiom since every axiom is a hypothetical formula. Therefore the set of all provable formulas forms an equivalence class, which we denote by 1.

If we define a partial ordering so that $|\alpha| \leq |\beta|$ when $\vdash \alpha \supset \beta$, then it is easily seen that E is a distributive lattice whose largest member is 1, and in which $\vee$ and $\wedge$ are join and meet respectively. By (A11), $\neg 1 \leq 1 \rightarrow x \leq x$ for all $x \in E$. Hence $\neg 1$ is the smallest member of E, and we denote it by 0. Next we can prove E is a s.r. lattice by verifying (a)-(d) of Theorem 1. For $x \in E$, we have $\neg x \leq x \rightarrow 0$ by (A11), and $x \rightarrow 0 \leq x \rightarrow \neg x \leq \neg x$ by (A12), so that $\neg x = x \rightarrow 0$. Finally $!x = 1 \rightarrow x$ by (A13) and (A14).

If α is any formula, and we assign the value $|p|$ to each propositional variable p, then $v(\alpha) = |\alpha|$. Therefore if α is valid in every s.r. lattice, we must have $|\alpha| = 1$ and so α is provable. This concludes the proof of Theorem 4.

We could drop the affirmation operation $\Box$ from our primitive symbols and omit (A13) and (A14). $\Box\alpha$ could then be defined as a shorthand for $(\alpha \supset \alpha) \supset \alpha$. A more symmetric system would be obtained by dropping the operations $\sim$ and $\Box$ and adding logical constants T and F for truth and falsity. Axioms (A11)-(A14) would then be replaced by the axioms $\alpha \supset T$ and $F \supset \alpha$; and $\Box\alpha$, $\sim\alpha$ could be defined by $T \supset \alpha$ and $\alpha \supset F$ respectively.

DEFINITION 5. An interior algebra is a Boolean algebra with a unary operation I satisfying

$$Ix \leq x$$

$$IIx = Ix$$

$$I(x \wedge y) = Ix \wedge Iy$$

$$I1 = 1.$$

THEOREM 6. The R^4 calculus is characterized by the class of all s.r. lattices (A,Q) such that A is a Boolean algrbra.

Proof. It is well known that the Lewis $S4$ calculus is characterized by the class of all interior algebras if we use the rules $v(\Box\alpha) = Iv(\alpha)$, $v(-\alpha) = -\alpha$. Thus for formulas of R^4, we have $v(\alpha \supset \beta) = Iv(\sim\alpha \vee \beta)$ and $v(\sim\alpha) = I(-v(\alpha))$. If Q denotes the set $\{x : x = Ix\}$ of open elements of an interior algebra A, then (A,Q) is an s.r. lattice and $x \rightarrow y = I(-x \vee y)$,

$\neg x = I(-x)$ and $!x = Ix$. Conversely if (A,Q) is a s.r. lattice such that A is a Boolean algebra, then A will be an interior algebra if we define $Ix = !x$.

THEOREM 7. If (A,Q) is any s.r. lattice, there exists a Boolean algebra B such that (A,Q) is a s.r. sublattice of (B,Q).

Proof. We may regard A is a lattice of subsets of a set X which contains X and the empty set $\emptyset$. Let B be the Boolean algebra of subsets of X which is generated by A. Then (B,Q), will be a s.r. lattice if for each $x \in B$, there is a largest element $!x$ of Q which is $\leq x$. Now every element x of B has the form $\bigwedge_{i=1}^{n}(-x_i \vee y_i)$, where $x_i, y_i \in A$. If $q \in Q$, then $q \leq x$ if and only if $q \leq -x_i \vee y_i$ for all i, hence if and only if $qx_i \leq y_i$, so if and only if $q \leq x_i \to y_i$ in A. Therefore $!x$ exists and is equal to $\bigwedge_{i=1}^{n}(x_i \to y_i)$. Finally if $x, y \in A$, then $x \to y$ in (B,Q) is $!(-x \vee y)$, which is equal to $x \to y$ in (A,Q).

THEOREM 8. The R^4 calculus is characterized by the set of all s.r. lattices, and is axiomatized by (A1)-(A14).

Proof. By Theorem 7, if $v(\alpha) < 1$ for some assignment in some r.s. lattice (A,Q), then $v(\alpha) < 1$ for the same assignment in (B,Q). Therefore the theorem follows from Theorems 6 and 4.

The fact that (A1)-(A12) is an axiomatization of R^4 without $\Box$ was proved by Hacking [4] using Gentzen methods. Hacking also showed that the set of theorems of R^4 which have no connective other than $\supset$ can be axiomatized by (A1)-(A3). This can also be proved algebraically, but we will refrain from giving this proof.

The logic S4.3 (see [2]) is obtained by adding to S4 the axiom schema $(\Box\alpha \supset \Box\beta) \vee (\Box\beta \supset \Box\alpha)$. (Recall that we are using $\supset$ for strict implication). Thus S4.3 is characterized by the class of all interior algebras such that

$$(19) \qquad (x \to y) \vee (y \to x) = 1$$

for any open elements x,y. Now an interior algebra is subdirectly irreducible if and only if there is a largest open element < 1. This can be proved as in Theorem 2 and Corollary 3, using the fact that $x \to y \leq (-y) \to (-x)$. But in this case (19) holds only when $(x \to y) = 1$ or $(y \to x) = 1$. Therefore the set of all open elements is a chain. Thus S4.3 is characterized by the class of all interior algebras such that the set of open elements is a chain. If we let R4.3 be the calculus obtained by adding the axiom $(\Box\alpha \supset \Box\beta) \vee (\Box\beta \supset \Box\alpha)$ then R4.3 is characterized by the class of all s.r. lattices (A,Q) such that Q is a chain.

Recall that $B(A)$ stands for the center of A. By Theorem 7 the logic characterized by the class of all s.r. lattices (A,Q) such that $Q \subseteq B(A)$ is the same as that characterized by the class of all s.r. lattices. The same is true for the logic characterized by the class of all s.r. lattices (A,Q) such that $Q \supseteq B(A)$. This is a consequence of the following theorem.

THEOREM 9. The logic characterized by the class of all s.r. lattices (A,Q) with center $\{0,1\}$ is the same as that characterized by the class of all s.r. lattices.

Proof. Let (A,Q) be any s.r. lattice. Let $L = A \cup \{z\}$ and $R = Q \cup \{z\}$, where z is a new element such that $x < z < 1$ for all $x \in A$, $x \neq 1$. Then (L,R) is a s.r. lattice in which for $x,y \in A$, $x \overset{R}{\to} y = x \overset{Q}{\to} y$, $x \wedge y$ in L is equal to $x \wedge y$ in A, $z \overset{R}{\to} x = 1 \overset{Q}{\to} x$, and $x \vee y$ in L is equal to $x \vee y$ in A except when $x < 1$, $y < 1$ and $x \vee y = 1$ in A.

If α is any formula and we assign values in A to the variables, then $v_L(\alpha) = v_A(\alpha)$ if $v_A(\alpha) < 1$ and $v_L(\alpha) \geq z$ if $v_A(\alpha) = 1$. This can be proved by an easy induction on the length of α. Hence if α is not valid in (A,Q) it is not valid in (L,R). This proves the theorem since the center of L is $\{0,1\}$.

We have seen that R4 is characterized by the class of all s.r. lattices (A,Q) such that

(i) A and Q are arbitrary, or

(ii) $Q \subseteq B(A)$, or

(iii) $Q \supseteq B(A)$, or

(iv) $B(A) = \{0,1\}$, or

(v) $B(A) = A$

In the next sections we shall determine the logics characterized by the class of all s.r. lattices (A,Q) such that Q is a Boolean subalgebra of $B(A)$, or such that $Q = B(A)$.

3. The R5 Calculus and R5 algebras

LEMMA 10. If (A,Q) is a s.r. lattice then the following are equivalent:

(i) $(q \to x) \to q \leq q$ for all $q \in Q$, $x \in A$

(ii) $\neg\neg q \leq q$ for all $q \in Q$

(iii) $q \vee \neg q = 1$ for all $q \in Q$

(iv) $!x \vee \neg x \vee \neg(!x \vee \neg x) = 1$ for all $x \in A$

(v) Q is a Boolean subalgebra of $B(A)$

Proof. By (2) and (14), $(q \to x) \to q \leq (q \to 0) \to q = \neg\neg q$. Therefore (i) is equivalent to (ii). If (ii), then $\neg\neg(q \vee \neg q) \leq q \vee \neg q$. But $\neg\neg(q \vee \neg q) = \neg(\neg q \wedge \neg\neg q) = \neg 0 = 1$ by (9) and (6). Therefore (iii) holds. Clearly (iii) implies (iv) because $!x \vee \neg x \in Q$. If (iv), then for any $q \in Q$ we have $1 = q \vee \neg q \vee \neg(q \vee \neg q) = q \vee \neg q \vee (\neg q \wedge \neg\neg q) = q \vee \neg q$, which proves (iii). Clearly (iii) implies (v) since (iii) implies $\neg q$ is a complement of q. Conversely if (v), then $-q = \neg q$ and (ii) and (iii) follow.

DEFINITION 11. An R5 lattice is a s.r. lattice satisfying the conditions of Lemma 10. An S5 algebra is an interior algebra such that the complement of an open element is open.

THEOREM 12. The logic characterized by the class of all R5 lattices is axiomatized by (A1)-(A14) together with any one of the following

(A15) $((\alpha \supset \beta) \supset \gamma) \supset (\alpha \supset \beta)) \supset (\alpha \supset \beta)$

(A15)' $(\sim\sim\Box\alpha) \supset \Box\alpha$

(A15)" $\Box\alpha \vee \sim\Box\alpha$

(A15)"' $\Box\alpha \vee \sim\alpha \vee \sim(\Box\alpha \vee \sim\alpha)$

Proof. We apply Lemma 10 to the Lindenbaum algebra E in the proof of Theorem 4.

THEOREM 13. The R5 calculus is characterized by the class of all R5 lattices, and is therefore axiomatized as in Theorem 12.

Proof. The Lewis S5 calculus is characterized by the class of all S5 algebras. Hence as in Theorem 6, the R5 calculus is characterized by the class of all R5 lattices (A,Q) such that A is a Boolean algebra and Q is a Boolean subalgebra of A. The result now follows by applying Theorem 7.

In a version of R5 in which $\Box$ is not a primitive operation, we may replace $\Box\alpha$ by $(\alpha \supset \beta)$ in (A15)' or (A15)". The advantage of (A15) is that all theorems of R5 whose only connective is $\supset$ are consequences of (A1), (A2), (A3), and (A15), see [4].

DEFINITION 14. If L is a bounded distributive lattice, let L* be the R5 lattice (L,{0,1}). A s.r. lattice of this form is called a special R5 lattice.

Obviously a special R5 lattice is a member of the class of all s.r. lattices (A,Q) such that Q is a chain. We have previously noted that this class characterizes the R4.3 calculus.

LEMMA 15. If (A,Q) is an R5 lattice, $x,y \in A$ and $q,r \in Q$, then

(20) $$\neg q = -q$$

(21) $$\neg x \vee \neg\neg x = 1$$

(22) $$x \leq \neg\neg x$$

(23) $$\neg\neg\neg x = \neg x$$

(24) $$(x \wedge q) \to (y \vee r) = \neg q \vee r \vee (x \to y)$$

(25) $$q \wedge x = 0 \text{ implies } q \wedge \neg\neg x = 0$$

(26) $$\neg\neg(q \wedge x) = q \wedge \neg\neg x$$

Proof. (20) and (21) follow from Lemma 10 (iii). (22) is proved by taking the meet of x with both sides of (21). To prove (24) observe that if $b \in Q$, then $b \wedge x \wedge q \leq y \vee r$ if and only if $b \wedge x \wedge q \wedge \neg r \leq y$, hence if and only if $b \wedge q \wedge \neg r \leq x \to y$, or $b \leq \neg q \vee r \vee (x \to y)$. For (25), $q \wedge x = 0$ implies $q \leq \neg x$, hence by (2), $\neg\neg x \leq \neg q$ and so $q \wedge \neg\neg x = 0$. Finally, $\neg\neg(q \wedge x) \leq \neg\neg q \wedge \neg\neg x = q \wedge \neg\neg x$. Also $q \wedge x \wedge \neg(q \wedge x) = 0$, hence by (25), $q \wedge \neg(q \wedge x) \wedge \neg\neg x = 0$. Therefore $q \wedge \neg\neg x \leq \neg\neg(q \wedge x)$, and this proves (26).

LEMMA 16. An R5 lattice is subdirectly irreducible if and only if it is a special R5 lattice.

Proof. By Corollary 3, (A,Q) is subdirectly irreducible if and only $\{x \in Q : x < 1\}$ has a largest element. Since Q is a Boolean algebra, this holds if and only if $Q = \{0,1\}$.

COROLLARY 17. An R5 lattice is a subdirect product of special R5 lattices. Hence the R5 calculus is characterized by the class of all special R5 lattices.

Proof. This follows from Lemma 16 by Birkhoff's Theorem.

By Theorem 1 and Lemma 10, the class of all R5 lattices is an equational class of algebras $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$. We now determine the free algebras of this class with finitely many free generators.

THEOREM 18. Let (A,Q) be the free R5 lattice with n free generators, and let L be the free bounded distributive lattice with n free generators. Then (A,Q) is isomorphic with

$$\prod_{\theta \in C(L)} (L/\theta)^*$$

where $C(L)$ is the set of all lattice congruence relations on L.

Proof. Note that the quotients L/θ include copies (with repetitions) of all possible bounded distributive lattices with at most n generat ors.

Let $v_1,\ldots,v_n$ be free generators of (A,Q) and let L be the sublattice of A generated by $v_1,\ldots,v_n,0,1$. L is free because if D is any bounded distributive lattice and $a_1,\ldots,a_n \in D$, there exists a s.r. homomorphism $f = A \rightarrow D^*$ such that $f(v_i) = a_i$, and $f : A \rightarrow D$ is a lattice homomorphism.

Let $S_1 = \emptyset, S_2,\ldots,S_N$ be all the subsets of $\{v_1,\ldots,v_n\}$, and let $p_j = \bigwedge(S_j)$, $s_j = \bigvee(S_j)$, and $w_{jk} = p_j \rightarrow s_k$. We divide the rest of the proof into parts.

(i) Q is the Boolean algebra generated by $\{w_{jk} : 1 \leq j,k \leq N\}$ and A is the lattice generated by $L \cup Q$.

Let Q_1 be the Boolean subalgebra of Q generated by the

w_{jk}, and let A_1 be the lattice generated by $L \cup Q_1$. If $u,v \in A_1$ then u has the form $\bigvee_j (p_j \wedge b_j)$, where $b_j \in Q_1$, and v has the form $\bigwedge_j (s_j \vee c_j)$, where $c_j \in Q_1$. By (3), (4) and (24), $u \to v = \bigwedge_{j,k}(-b_j \vee c_k \vee w_{jk}) \in Q_1 \subseteq A_1$. Thus A_1 is a s.r. sublattice of A containing $\{v_1,\ldots,v_n\}$. Hence $A_1 = A$ and therefore $Q_1 = Q$ because each element of Q is of the form $u \to v$. This proves (i).

If K is any subset of $\{1,2,\ldots,N\}^2$, let.

$$t_K = \bigwedge_{(i,j)\in K} w_{ij} \wedge \bigwedge_{(i,j)\notin K} - w_{ij}$$

(ii) For any $K \subseteq \{1,\ldots,n\}^2$, the following are equivalent

(a) t_K is an atom of Q

(b) $$\bigwedge_{(i,j)\in K} w_{ij} \not\leq \bigvee_{(i,j)\notin K} w_{ij}$$

(c) There exists a bounded distributive lattice D and a s.r. homomorphism $f = A \to D^*$ such that $f(w_{ij}) = 1$ for $(i,j) \in K$ and $f(w_{ij}) = 0$ for $(i,j) \notin K$.

(d) There exists $\theta \in C(L)$ such that $p_i/\theta \leq s_j/\theta$ if and only if $(i,j) \in K$.

To prove (ii), first note that the atoms of Q are those t_K which are not 0. Therefore (a) is equivalent to (b). Obviously (c) implies (b) and (b) implies (c) by Corollary 17. Suppose (d). Let $f_\theta : A \to (L/\theta)^*$ be the s.r. homomorphism such that $f_\theta(v_i) = v_i/\theta$ for all i. Then clearly

$$f_\theta(w_{ij}) = v_i/\theta \to w_j/\theta = 1 \text{ for } (i,j) \in K, \text{ and}$$

(27)

$$f_\theta(w_{ij}) = 0 \text{ for } (i,j) \notin K.$$

This proves (c). Conversely if (c) holds then $\theta = \{(x,y) \in L^2 : f(x) = f(y)\}$ satisfies (d). This completes the proof of (ii).

If $\theta \in C(L)$, let $a(\theta)$ be the atom t_K, where K is defined by (d). If $a(\theta_1) = a(\theta_2)$, then for all $x,y \in L$, we have $x/\theta_1 \le y/\theta_1$ if and only if $x/\theta_2 \le y/\theta_2$ since x is a join of p_i and y is a meet of s_j. Hence $\theta_1 = \theta_2$. So the mapping $\theta \to a(\theta)$ is a bijection of $C(L)$ onto the set of all atoms of Q.

Let $f_\theta = A \to (L/\theta)^*$ be the homomorphism defined in the proof of (ii). By (27), we have

$$(28) \quad f_\theta(a(\theta)) = 1, \text{ and } f_{\theta_1}(a(\theta_2)) = 0 \text{ if } \theta_1 \neq \theta_2$$

(iii) If $x \in L$, then $f_\theta(x) = 1$ if and only if $x \ge a(\theta)$.

The sufficiency is obvious by (28). Suppose $f_\theta(x) = 1$. Since x is a meet of s_j, we may assume $x = s_j$. Then $1 = f_\theta(x) = s_j/\theta \ge p_1/\theta$. Hence $(1,j) \in K$ and so $a(\theta) \le w_{1j} = 1 \to x \le x$. This proves (iii).

Now there exists a s.r. homomorphism

$$f = A \to \prod_{\theta \in C(L)} (L/\theta)^*$$

such that the θ component of $f(x)$ is $f_\theta(x)$. To show f is injective, suppose $f(x) = 1$. By (i), x has the form

$$x = \bigvee_{\theta \in C(L)} x(\theta) \wedge a(\theta)$$

where $x(\theta) \in L$. By (28), for each θ, $1 = f_\theta(x) = f_\theta(x(\theta))$. Therefore by (iii), $x(\theta) \geq a(\theta)$, and so $x = \bigvee_\theta a(\theta) = 1$.

Finally to show f is surjective, let $x(\theta)/\theta$ be any element of A/θ, and let $x = \bigvee_\theta x(\theta) \wedge a(\theta)$. Then $f_\theta(x) = f_\theta(x(\theta))$ $= x(\theta)/\theta$ for all θ. This completes the proof.

If A is a Boolean algebra, then A^* may be identified with the S5 algebra such that $Ix = 0$ for all $x < 1$. The following can be proved in a manner similar to but simpler than the proof of Theorem 18.

THEOREM 19. If B is the free Boolean algebra with n free generators, and $\mathfrak{F}(B)$ is the set of all filters of B, then the free S5 algebra with n free generators is

$$\prod_{F \in \mathfrak{F}(B)} (B/F)^*$$

4. B algebras and PC

In this section we consider s.r. lattices (A,Q) such that Q is the center $B(A)$ of A. Such algebras were called B algebras in [3].

THEOREM 20. If (A,Q) is a s.r. lattice with center B then $Q = B$ if and only if

(29) $$!x \vee \neg x \vee \neg !(x \vee y) \vee \neg\neg(x \wedge y) = 1$$

for all $x, y \in A$.

<u>Proof</u>. If (29) holds and q is any element of Q, then setting $x = q$, $y = 0$, we get $q \vee \neg q = 1$. Therefore Q is a subalgebra of B and A is an R5 lattice. Now suppose $x \in B$. If we set $y = -x$ in (29), we find $\neg x \vee !x = 1$. This implies $x \wedge !x = x$, so $x \leq !x$. Thus $x = !x \in Q$ and so $B \subseteq Q$, hence $B = Q$.

Conversely suppose $Q = B$. Then by (20), for all $x, y \in A$ we have

$$(x \wedge \neg(x \wedge y) \wedge !(x \vee y)) \vee \neg !(x \vee y) \vee (x \wedge \neg\neg(x \wedge y)) \vee y$$

$$= (x \wedge \neg(x \wedge y)) \vee \neg !(x \vee y) \vee (x \wedge \neg\neg(x \wedge y)) \vee y$$

$$= \neg !(x \vee y) \vee x \vee y \geq \neg !(x \vee y) \vee !(x \vee y) = 1.$$

Also

$$(x \wedge \neg(x \wedge y) \wedge !(x \vee y)) \wedge (\neg !(x \vee y) \vee (x \wedge \neg\neg(x \wedge y)) \vee y) = 0$$

Therefore $x \wedge \neg(x \wedge y) \wedge !(x \vee y) \in B$, and so is in Q. Hence by (20) and (26),

$$x \wedge \neg(x \wedge y) \wedge !(x \vee y) = \neg\neg(x \wedge \neg(x \wedge y) \wedge !(x \vee y))$$

$$= \neg\neg x \wedge \neg(x \wedge y) \wedge !(x \vee y).$$

This implies $\neg\neg x \wedge \neg(x \wedge y) \wedge !(x \vee y) \leq x$, and therefore $\neg\neg x \wedge \neg(x \wedge y) \wedge !(x \vee y) \leq !x$. Thus

$$1 = !x \vee \neg(\neg\neg x \wedge \neg(x \wedge y) \wedge !(x \vee y))$$

$$= !x \vee \neg x \vee \neg\neg(x \wedge y) \vee \neg !(x \vee y).$$

The following is an immediate consequence of Theorem 20.

COROLLARY 21. The logic characterized by the class of all B algebras is axiomatized by (A1)-(A14) together with

(A16) $\qquad \Box\alpha \vee \sim\alpha \vee \sim\Box(\alpha \vee \beta) \vee \sim\sim(\alpha \;\&\; \beta).$

The class of B algebras in which

(30) $\qquad (x \to y) \vee (y \to x) = 1$

holds identically is of particular interest. Such algebras are called P algebras and their properties are discussed in [3]. There it is proved that any R5 lattice in which (30) holds is a B algebra [3, Corollary 3.3]. In any s.r. lattice (30) holds identically if and only if

(31) $\qquad x \to (y \vee x) \leq (x \to y) \vee (x \to z)$

is satisfied identically. Indeed given (30), we have by (3), (1) and (5),

$$x \to (y \vee z) = ((x \to (y \vee z)) \wedge (z \to y)) \vee ((x \to (y \vee z)) \wedge (y \to z))$$

$$= ((x \to (y \vee z)) \wedge ((y \vee z) \to y)) \vee ((x \to (y \vee z)) \wedge ((y \vee z) \to z))$$

$$\leq (x \to y) \vee (x \to z)$$

Conversely given (31), then

$$1 = (x \vee y) \to (x \vee y) \leq ((x \vee y) \to x) \vee ((x \vee y) \to y)) = (y \to x) \vee (x \to y)$$

It is also the case that (30) holds if and only if

$$(32) \qquad (x \to y) \to z \leq ((y \to x) \to z) \to z$$

holds identically. For setting $z = 0$ in (32) gives $-(x \to y) \leq --(y \to x) = (y \to x)$, so that (30) follows. Conversely given (30), then by (3),

$$z \geq 1 \to z = ((x \to y) \vee (y \to x)) \to z = ((x \to y) \to z) \wedge ((y \to x) \to z)$$

which implies (32).

Thus a s.r. lattice satisfying any of the conditions of Lemma 10 together with (30), (31) or (32) is a P algebra.

The logic characterized by the class of all P-algebras will be called PC. The subdirectly irreducible P-algebras are chains [3, Theorem 4.4], that is, are s.r. lattices (A,Q) such that A is a chain and $Q = \{0,1\}$. Therefore PC is also characterized by the class of all linearly ordered B algebras. A set of axioms for PC is (A1)-(A15) and either

(A17) $((\alpha \supset \beta) \supset \gamma) \supset (((\beta \supset \alpha) \supset \gamma) \supset \gamma)$

or

(A17)' $(\alpha \supset (\beta \vee \gamma)) \supset ((\alpha \supset \beta) \vee (\alpha \supset \gamma)$

In [3, Theorem 3.4] it is shown that if A is a P-algebra then A is a Heyting algebra — that is, $x \overset{A}{\to} y$ exists for all x,y. It is also shown that

(33) $x \overset{A}{\to} y = y \vee (x \to y),\ x \to y = !(x \overset{A}{\to} y).$

By (30) and (33), we have

(34) $(x \overset{A}{\to} y) \vee (y \overset{A}{\to} x) = 1$

(35) $x \overset{A}{\to} 0 = x \to 0$

From this we see that in PC, we may introduce another implication connective $\supset'$ by letting $\alpha \supset' \beta$ be an abbreviation for $\beta \vee (\alpha \supset \beta)$. By (35) it is not necessary to introduce an additional negation connective. By (34), PC has as a theorem $(\alpha \supset' \beta) \vee (\beta \supset' \alpha)$. It is well known that the intuitionist propositional calculus is characterized by the class of all Heyting algebras. Therefore the fragment of PC consisting of all formulas involving $\vee$, $\wedge$, $\sim$ and $\supset'$ (but not $\supset$) contains Dummett's LC[1]. Thus PC is a kind of modal extension of LC. We shall now derive a set of axioms for PC which reflects this point of view.

From [3, theorem 3.4 (vi)] it is an easy consequence that a Heyting algebra A satisfying (34) identically is a P-algebra if and only if there exists a unary operation ! on A satisfying

$$!x \leq x$$

$$\neg\neg !x \leq !x$$

$$!1 = 1$$

$$!(x \vee y) = !x \vee !y$$

If we choose as our primitive connectives $\vee$, &, $\sim$, $\supset'$ and $\Box$, then we can obtain a set of axioms for PC as follows

(a 0) Axioms for the intuitionist propositional calculus (using $\supset'$ for implication)

(a 1) $(\alpha \supset' \beta) \vee (\beta \supset' \alpha)$

(a 2) $\Box\alpha \supset' \alpha$

(a 3) $\sim\sim\Box\alpha \supset' \Box\alpha$

(a 4) $\Box(\alpha \vee \beta) \supset' (\Box\alpha \vee \Box\beta)$

(a 5) $\Box(\alpha \supset' \beta) \supset' (\Box\alpha \supset' \Box\beta)$

The rules of inference are

(R1) If $\vdash \alpha$ and $\vdash \alpha \supset' \beta$ then $\vdash \beta$

(R2) If $\vdash \alpha$ then $\vdash \Box\alpha$.

Rule (R2) can be avoided if we replace each axiom α by the axiom $\Box\alpha$, and add the axioms $\Box(\Box\alpha \supset \Box\Box\alpha)$ and $\Box\alpha \supset \alpha$.

The proof of completeness of these axioms is similar to that of Theorem 4. The purpose of (α 5) is to ensure that α eq β implies $\Box\alpha$ eq $\Box\beta$.

As a final remark we shall characterize the class of s.r. algebras (A,Q) which are Heyting algebras satisfying (33). Since $x \to y = !(y \vee (x \to y))$ in every s.r. lattice, we need only consider the first part of (33).

THEOREM 22. If (A,Q) is a s.r. lattice, then A is a Heyting algebra such that $x \overset{A}{\to} y = y \vee (x \to y)$ for all $x,y \in A$ if and only if

$$y \leq x \vee (x \to y) \tag{36}$$

for all $x,y \in A$.

<u>Proof</u>. Suppose (36) holds. Let z be any element of A such that $z \wedge x \leq y$. Since $z \leq x \vee (x \to z)$, we have $z \leq (z \wedge x) \vee (x \to z) = (z \wedge x) \vee (x \to (z \wedge x)) \leq y \vee (x \to y)$ by (4) and (2). Also $x \wedge (y \vee (x \to y)) \leq y$. Therefore $x \overset{A}{\to} y = y \vee (x \to y)$.

Conversely if (33) holds, than for all $x,y \in A$, $y \leq x \overset{A}{\to} y = x \overset{A}{\to} (x \wedge y) = (x \wedge y) \vee (x \to (x \wedge y)) \leq x \vee (x \to y)$.

COROLLARY. If (A,Q) is a s.r. lattice such that $(x \to y) \vee (y \to x) = 1$ for all x,y, then A is a Heyting algebra satisfying (33)

<u>Proof</u>. We have $y = (y \wedge (x \rightarrow y)) \vee (y \wedge (y \rightarrow x)) \leq (x \rightarrow y) \vee x$, which proves (36).

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