A Decomposition Theorem for Convexity Spaces

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TECHNICAL REPORT NO. 273

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February 1989

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January 13, 1989

Abstract

We utilize the unifying framework of families of convexity spaces for the treatment of various notions of planar convexity and the associated convex hulls. Our major goal is to prove the refinement and decomposition theorems for families of convexity spaces. These general theorems are then applied to two examples: restricted-oriented convex sets and $NESW$-convex sets. The applications demonstrate the usefulness of these general theorems, since they give rise to simple algorithms for the computation of the associated convex hulls of polygons.

1 Introduction

Convexity spaces provide one algebraic abstraction of convexity in terms of the closure of convex sets under intersection. They have been called *convexity spaces* [11], *convexity structures* [20], *alignments* [8], or *algebraic closure systems* [2]. They capture the lattice-theoretic or algebraic properties of convex sets rather than their topological ones. We use them to provide a unifying framework for different notions of convexity that have a risen in recent computational studies of polygons and regions in the plane; that is, in

*This work was supported under a Natural Sciences and Engineering Research Council of Canada Grant No. A-5692 and under a grant from the Information Technology Research Centre.

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the relatively new area of computational geometry. Some examples of non-traditional convexity are: orthogonal convexity, see [13] for example; finitely-oriented convexity, see [15,25]; restricted-oriented convexity [17]; $NESW$-closure [12,21]; and link convexity, see [1,24], for example. However, if a unifying framework was all that convexity spaces provided, they would be interesting but have little import. Fortunately, they provide more, as we hope to demonstrate in this paper, the first of a series.

One similarity underlying many of the non-traditional notions of convexity is that we can compute the "convex" hull of a polygon, say, by decomposing the computation into the computation of "simpler" hulls and combining their results. For example, we can compute the orthogonal hull of a polygon by first computing its x-hull and y-hull and taking their union [18]. We could also compute the y-hull of the x-hull and be assured that we again obtain the correct result. A similar result obtains for $NESW$-closure [21]. In this paper, we consider the decomposition issue in the setting of families of convexity spaces over a common groundset. This study is practically motivated since the convex hull of an object has, typically, less complexity than the object itself and so it is much used in testing for intersections among objects [14,22]. The same reason suffices to explain the great popularity of the "bounding box" of an object in computer graphics and computer vision. Indeed, the convex hull was one of the first concepts studied in computational geometry [19] and so deserves especial attention.

In Section 2 we introduce convexity spaces and a useful technique for constructing them. In Section 3 we prove our main theorems, the refinement and decomposition theorems for families of convexity spaces over the same groundset. In the remaining three sections we consider three specific convexity spaces induced by restricted-oriented convex sets, $NESW$-convex sets, and $NEED$-convex sets. For restricted-oriented convex sets we prove a new result, the Orientation Decomposition Theorem, this generalizes the result of [18] for orthogonal convex sets; for $NESW$-convex sets we reprove, in a completely different way, an earlier result, the $NESW$ Decomposition Theorem, see [21]; and for $NEED$-convex sets we show that there is no decomposition theorem. This latter result demonstrates that convexity spaces over $\Re^2$ need not be decomposable, a conjecture that the previous two results had led us to make.

2 Convexity Spaces and Their Construction

Abstract convexity theory is concerned with collections of subsets of a set which obey two weak axioms. A convexity space, in the sense we use it here, is intended to be an abstraction of the more essential properties of convex sets in $\Re^n$.

Definition 2.1 *Given a set $\mathcal{S}$ and a family $\mathcal{C}$ of subsets of $\mathcal{S}$ the structure $(\mathcal{S},\mathcal{C})$ is said to be a* convexity space *if*

1. *$\emptyset, \mathcal{S} \in \mathcal{C}$; and*
2. *for all $C \subseteq \mathcal{C}$, $\bigcap C \in \mathcal{C}$, where $\bigcap C = \bigcap_{X \in C} X$.*

$\mathcal{S}$ is called the *groundset* of the convexity space and any element of the family $\mathcal{C}$ is said to be *$\mathcal{C}$-convex*. The interpretation of the family $\mathcal{C}$ is that it is the set of all convex sets over some space, where we have deferred the operational question of what we *mean* by convexity. The dominant characteristic of convex sets is taken to be closure under intersection.

Definition 2.2 *Given a convexity space $(\mathcal{S},\mathcal{C})$, we define the associated* hull operator *$\mathcal{C}$-hull as follows:*

$$\textit{For all } \mathbf{P} \subseteq \mathcal{S}, \ \mathcal{C}\text{-}hull(\mathbf{P}) = \bigcap\{\mathbf{Q} \mid \mathbf{P} \subseteq \mathbf{Q} \ \wedge \ \mathbf{Q} \in \mathcal{C}\}$$

We define the intersection to be the empty set, if there are no sets $\mathbf{Q}$ satisfying the conditions.

It is straightforward to show that $\mathcal{C}$-*hull*$(\mathbf{P})$ exists, is unique, and is the smallest $\mathcal{C}$-convex set which contains $\mathbf{P}$.

Two examples of convexity spaces that are worth noting are:

1. *The trivial convexity space* $(\mathcal{S}, \{\emptyset, \mathcal{S}\})$. In this convexity space there are only two convex sets—$\emptyset$ and $\mathcal{S}$—thus the hull of any non-empty subset of $\mathcal{S}$ is $\mathcal{S}$ itself.

2. *The complete convexity space* $(\mathcal{S}, \mathcal{P}(\mathcal{S}))$ where $\mathcal{P}(\mathcal{S})$ is the powerset of $\mathcal{S}$. In this convexity space every set is its own hull, or to put it another way, all subsets of $\mathcal{S}$ are convex since for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathbf{P} \in \mathcal{P}(\mathcal{S})$ and so
$$\mathcal{P}(\mathcal{S})\text{-}hull(\mathbf{P}) = \bigcap\{\mathbf{Q} \mid \mathbf{P} \subseteq \mathbf{Q} \ \wedge \ \mathbf{Q} \in \mathcal{P}(\mathcal{S})\} = \mathbf{P}.$$

The results stated in the following theorem are well-known in abstract convexity theory (see Kay and Womble [9]).

Theorem 2.1 *Given a convexity space $(\mathcal{S},\mathcal{C})$; then, for all $\mathbf{P}, \mathbf{Q} \subseteq \mathcal{S}$,*

1. *$\mathcal{C}$-hull$(\mathbf{P}) \in \mathcal{C}$;*
2. *$\mathbf{P} \subseteq \mathcal{C}$-hull$(\mathbf{P})$;*
3. *$\mathcal{C}$-hull$(\mathbf{P}) = \mathbf{P} \iff \mathbf{P} \in \mathcal{C}$;*
4. *$\mathbf{P} \subseteq \mathbf{Q} \Longrightarrow \mathcal{C}$-hull$(\mathbf{P}) \subseteq \mathcal{C}$-hull$(\mathbf{Q})$;*

5. $\mathcal{C}$-*hull*($\mathcal{C}$-*hull*($\mathbf{P}$)) = $\mathcal{C}$-*hull*($\mathbf{P}$).

In topology any operator which has properties (2), (4) and (5) is known as a *closure operator* [3]. Although we do not do it here, it is possible to show that a closure operator *induces* a convexity space. That is, given a set $\mathcal{S}$ and a closure operator $\mathcal{C}$, $\mathcal{S}$ together with the set of all $\mathcal{C}$-closed sets over $\mathcal{S}$ is a convexity space.

We need to establish that specific structures $(\Re^2, \mathcal{C})$, where $\mathcal{C}$ is a collection of subsets of $\Re^2$, do indeed form convexity spaces. Rather than providing many different but similar proofs, we prefer to provide a general result that is straightforward to particularize.

To this end, we begin with the following definition which is reminiscent of the definitions above for convexity spaces and hulls.

Definition 2.3 *Let $\mathcal{L}$ be a collection of subsets of a groundset $\mathcal{S}$ such that $\emptyset$ is not in $\mathcal{L}$—we call $(\mathcal{S}, \mathcal{L})$ a* line space. *Intuitively, we think of elements of $\mathcal{L}$ as line segments. For all $\mathbf{P} \subseteq \mathcal{S}$, let $\mathcal{L}(\mathbf{P}) = \bigcap\{\mathbf{L} \mid \mathbf{P} \subseteq \mathbf{L} \wedge \mathbf{L} \in \mathcal{L}\}$, and define* $\mathcal{L}$-hull($\mathbf{P}$) *by*

$$\mathcal{L}\text{-}hull(\mathbf{P}) = \begin{cases} \mathcal{L}(\mathbf{P}), & \text{if } \mathcal{L}(\mathbf{P}) \text{ is in } \mathcal{L} \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Intuitively, the $\mathcal{L}$-hull of a set $\mathbf{P}$ is the smallest line segment in $\mathcal{L}$ that contains $\mathbf{P}$. Note that, because $(\mathcal{S}, \mathcal{L})$ is not necessarily a convexity space, $\mathcal{L}(\mathbf{P})$ is not necessarily in $\mathcal{L}$.

Based on this notion, we now define convexity by way of line segments in a manner reminiscent of one definition of convex sets, namely, a set is convex if, for every two points in the set, the line segment joining them is also in the set.

Definition 2.4 *We say that $\mathbf{P} \subseteq \mathcal{S}$ is* $\mathcal{L}$-convex *if, for all $\mathbf{X} \subseteq \mathbf{P}$ such that $\mathcal{L}$-hull($\mathbf{X}$) is defined, $\mathcal{L}$-hull($\mathbf{X}$) $\subseteq \mathbf{P}$.*

We are now in a position to prove that line spaces induce convexity spaces.

Theorem 2.2 *Let $(\mathcal{S}, \mathcal{L})$ be a line space and $\mathcal{C}$ be the collection of all $\mathcal{L}$-convex sets together with $\emptyset$. Then, $(\mathcal{S}, \mathcal{C})$ is a convexity space.*

Proof: We have to prove that $(\mathcal{S}, \mathcal{C})$ satisfies the two axioms of a convexity space. Clearly, $\mathcal{C}$ contains $\emptyset$. That it contains $\mathcal{S}$ follows directly, because whenever $\mathcal{L}$-*hull*($\mathbf{X}$) is defined for any $\mathbf{X} \subseteq \mathcal{S}$, $\mathcal{L}$-*hull*($\mathbf{X}$) $\subseteq \mathcal{S}$ is immediate.

This leaves the second axiom. Let $\mathbf{P} = \bigcap \mathcal{C}' = \bigcap\{\mathbf{C} \mid \mathbf{C} \in \mathcal{C}'\}$, for some $\mathcal{C}' \subseteq \mathcal{C}$. If $\emptyset \in \mathcal{C}'$, then $\mathbf{P} = \emptyset \in \mathcal{C}$. Therefore, assume $\emptyset \notin \mathcal{C}'$. Now, $\mathbf{X} \subseteq \mathbf{P}$ if and only if $\mathbf{X} \subseteq \mathbf{C}$, for all $\mathbf{C} \in \mathcal{C}'$. Whenever $\mathcal{L}\text{-}hull(\mathbf{X})$ is defined, we have $\mathcal{L}\text{-}hull(\mathbf{X}) \subseteq \mathbf{C}$, for all $\mathbf{C} \in \mathcal{C}'$, since each $\mathbf{C}$ is $\mathcal{L}$-*convex*. In other words, $\mathcal{L}\text{-}hull(\mathbf{X}) \subseteq \mathbf{P}$, $\mathbf{P}$ is $\mathcal{L}$-*convex*, and $\mathbf{P} \in \mathcal{C}$ by definition. □

Based on this theorem we can now establish that some well known notions of convexity do indeed form convexity spaces.

1. $\mathcal{L}$ is the set of all line segments and singleton point sets in the plane. Clearly the $\mathcal{L}$-hull of a set $\mathbf{P}$ in the plane is in $\mathcal{L}$ if and only if $\mathbf{P}$ is a singleton set or it consists of collinear points. In other words, the induced convexity space consists of all planar convex sets.
2. $\mathcal{L}$ is the set of all horizontal and vertical line segments and all singleton point sets in the plane. This induces the convexity space consisting of all orthogonal convex sets in the plane; see [13], for example.

3 The Refinement and Decomposition Theorems

Our aim is to elucidate the geometry of families of convex sets each convex with respect to similar, but different notions. Each notion gives rise to a different collection of convex sets. Thus, it is natural for us to consider *families* of convexity spaces over a fixed groundset S. In the *Refinement Theorem* (Theorem 3.3) we show that the union of several hulls of a set, where each hull is formed in a distinct convexity space, is a subset of a *"composed"* hull formed from the separate hulls, and the composed hull is, in turn, a subset of the hull with respect to the intersection of the different convexity spaces. In the *Decomposition Theorem* (Theorem 3.4) we specialize the refinement theorem to convexity spaces which act as if they were independent of each other (so called *invariant* convexity spaces).

Definition 3.1 *Let $(S, \mathcal{C}_1)$ and $(S, \mathcal{C}_2)$ be two convexity spaces defined on the groundset S. Then, the space $(S, \mathcal{C}_2)$ is said to be a* refinement *of the space $(S, \mathcal{C}_1)$ if $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Alternatively, $(S, \mathcal{C}_1)$ is said to be* coarser *than $(S, \mathcal{C}_2)$. Intuitively, any $\mathcal{C}_1$-convex set is $\mathcal{C}_2$-convex.*

Notice that in the example convexity spaces defined above the complete convexity space is a refinement of the trivial convexity space. Indeed, the complete convexity space is a refinement of all convexity spaces over S and the trivial convexity space is coarser than any other convexity space over S.

Definition 3.2 *Let $(S, \mathcal{C}_1)$ and $(S, \mathcal{C}_2)$ be two convexity spaces defined on the groundset S. We use the notation $\mathcal{C}_1 \wedge \mathcal{C}_2$ to represent the subset of $\mathcal{C}_1$ produced by* composing *the two hull operators. That is,*

$$\mathcal{C}_1 \wedge \mathcal{C}_2 = \{\mathcal{C}_1\text{-}hull(\mathcal{C}_2\text{-}hull(\mathbf{P})) \mid \mathbf{P} \subseteq S\}$$

Observe that we may simplify this definition to

$$C_1 \wedge C_2 = \{C_1\text{-}hull(\mathbf{P}) \mid \mathbf{P} \in C_2\}$$

The following example shows that the composition of two convexity spaces is not necessarily a convexity space. Let $S = \{a, b, c\}$; $C_1 = \{\emptyset, S, \{a, c\}, \{b, c\}, \{c\}\}$; $C_2 = \{\emptyset, S, \{a\}, \{b\}\}$; then $C_1 \wedge C_2 = \{\emptyset, S, \{a, c\}, \{b, c\}\}$. But $C_1 \wedge C_2$ is not a convexity space since $\{a, c\} \cap \{b, c\} = \{c\} \notin C_1 \wedge C_2$.

Although the composition of two convexity spaces is not necessarily a convexity space we can extend the notion of a hull operator to such families of sets as follows:

> *Given a family of subsets C of S and a set $\mathbf{P} \subseteq S$, the C-hull of $\mathbf{P}$ is the intersection of all sets in C which contain $\mathbf{P}$.*

Note that if (S, C) is not a convexity space, the hull of a set may not be in C. Furthermore, the C-*hull* of $\mathbf{P}$ is undefined if $\mathbf{P}$ is not contained in some set in C.

Definition 3.3 *Let (S, C_1) and (S, C_2) be two convexity spaces defined on the groundset S. We use the notation $(C_1 \wedge C_2)$-hull to represent the* composition *of the two C-hull operators; that is,*

$$(C_1 \wedge C_2)\text{-}hull(\mathbf{P}) \equiv C_1\text{-}hull(C_2\text{-}hull(\mathbf{P}))$$

Note that, for all $\mathbf{P} \subseteq S$, $(C_1 \wedge C_2)$-hull$(\mathbf{P})$ is well-defined.

The following example shows that hull operators do not commute under composition. Let $S = \{a, b, c\}$; $C_1 = \{\emptyset, S, \{a\}\}$; $C_2 = \{\emptyset, S, \{a, b\}\}$. Then $(C_1 \wedge C_2)$-*hull*$(\{a\}) = S$, but $(C_2 \wedge C_1)$-*hull*$(\{a\}) = \{a, b\}$.

We now establish a weak form of commutativity of composed hull operators. We show that two hull operators commute if one of the convexity spaces is a subset of the other. In fact, in this case the composed hulls are both equal to the hull formed by the coarser of the two convexity spaces.

Theorem 3.1 *Let (S, C_1) and (S, C_2) be two convexity spaces defined on the groundset S; then*

$$\begin{aligned} C_1 \subseteq C_2 \quad &\Longleftrightarrow (1) \quad \textit{for all } \mathbf{P} \subseteq S, \quad C_2\text{-}hull(\mathbf{P}) \subseteq C_1\text{-}hull(\mathbf{P}) \\ &\Longleftrightarrow (2) \quad \textit{for all } \mathbf{P} \subseteq S, \quad (C_2 \wedge C_1)\text{-}hull(\mathbf{P}) = C_1\text{-}hull(\mathbf{P}) \\ &\Longleftrightarrow (3) \quad \textit{for all } \mathbf{P} \subseteq S, \quad (C_1 \wedge C_2)\text{-}hull(\mathbf{P}) = C_1\text{-}hull(\mathbf{P}) \end{aligned}$$

Proof:

1. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_1$-*hull*$(\mathbf{P}) \in \mathcal{C}_2$. Hence, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_2$-*hull*$(\mathbf{P}) \subseteq \mathcal{C}_1$-*hull*$(\mathbf{P})$.

   Conversely, suppose that, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_2$-*hull*$(\mathbf{P}) \subseteq \mathcal{C}_1$-*hull*$(\mathbf{P})$. If $\mathbf{P} \in \mathcal{C}_1$, then $\mathbf{P} \subseteq \mathcal{C}_2$-*hull*$(\mathbf{P}) \subseteq \mathcal{C}_1$-*hull*$(\mathbf{P}) = \mathbf{P}$. Therefore, $\mathbf{P} = \mathcal{C}_2$-*hull*$(\mathbf{P})$ and $\mathbf{P} \in \mathcal{C}_2$. Hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

2. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_1$-*hull*$(\mathbf{P}) \in \mathcal{C}_2$. Hence, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_2$-*hull*$(\mathcal{C}_1$-*hull*$(\mathbf{P})) = \mathcal{C}_1$-*hull*$(\mathbf{P})$.

   Conversely, suppose that, for all $\mathbf{P} \subseteq \mathcal{S}$, $(\mathcal{C}_2 \wedge \mathcal{C}_1)$-*hull*$(\mathbf{P}) = \mathcal{C}_1$-*hull*$(\mathbf{P})$. If $\mathbf{P} \in \mathcal{C}_1$, then $\mathcal{C}_1$-*hull*$(\mathbf{P}) = \mathbf{P}$, $\mathcal{C}_2$-*hull*$(\mathbf{P}) = \mathbf{P}$, and $\mathbf{P} \in \mathcal{C}_2$. Hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

3. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then from (1) we have that, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_2$-*hull*$(\mathbf{P}) \subseteq \mathcal{C}_1$-*hull*$(\mathbf{P})$. This implies that, for all $\mathbf{P} \subseteq \mathcal{S}$, $\mathcal{C}_1$-*hull*$(\mathcal{C}_2$-*hull*$(\mathbf{P})) \subseteq \mathcal{C}_1$-*hull*$(\mathcal{C}_1$-*hull*$(\mathbf{P})) = \mathcal{C}_1$-*hull*$(\mathbf{P})$. But, $\mathcal{C}_1$-*hull*$(\mathbf{P}) \subseteq \mathcal{C}_1$-*hull*$(\mathcal{C}_2$-*hull*$(\mathbf{P}))$. Therefore, $\mathcal{C}_1$-*hull*$(\mathcal{C}_2$-*hull*$(\mathbf{P})) = \mathcal{C}_1$-*hull*$(\mathbf{P})$.

   We prove the converse by proving its contrapositive. Suppose that, for all $\mathbf{P} \subseteq \mathcal{S}$, $(\mathcal{C}_1 \wedge \mathcal{C}_2)$-*hull*$(\mathbf{P}) = \mathcal{C}_1$-*hull*$(\mathbf{P})$. If $\mathbf{P} \notin \mathcal{C}_2$, then $\mathbf{P} \subset \mathcal{C}_2$-*hull*$(\mathbf{P})$. But, this implies that $\mathbf{P} \subset \mathcal{C}_1$-*hull*$(\mathcal{C}_2$-*hull*$(\mathbf{P})) = \mathcal{C}_1$-*hull*$(\mathbf{P})$. Therefore, $\mathbf{P} \notin \mathcal{C}_1$ and, hence, $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

□

Result (1) has been previously proved by Sierksma [20].

We now prove that convexity spaces over the same groundset are closed under intersection, but not under union. This is used in the proof of the refinement theorem below.

Lemma 3.2 *Let* $\{(\mathcal{S}, \mathcal{C}_i) \mid i \in I\}$ *be a non-empty family of convexity spaces defined on* $\mathcal{S}$*; then, the structure* $(\mathcal{S}, \bigcap_{i \in I} \mathcal{C}_i)$ *is a convexity space on* $\mathcal{S}$.

Proof: $\emptyset, \mathcal{S} \in \mathcal{C}_i$, for all $i \in I \Longrightarrow \emptyset, \mathcal{S} \in \bigcap_{i \in I} \mathcal{C}_i$.

$$\mathcal{C} \subseteq \bigcap_{i \in I} \mathcal{C}_i \Longrightarrow \mathcal{C} \subseteq \mathcal{C}_i, \text{ for all } i \in I \Longrightarrow \bigcap \mathcal{C} \in \mathcal{C}_i \text{ for all } i \in I \Longrightarrow \bigcap \mathcal{C} \in \bigcap_{i \in I} \mathcal{C}_i$$ □

The following example shows that a similar result does not hold for the union of convexity spaces. Let $\mathcal{S} = \{a, b, c\}$; $\mathcal{C}_1 = \{\emptyset, \mathcal{S}, \{a, b\}\}$; $\mathcal{C}_2 = \{\emptyset, \mathcal{S}, \{a, c\}\}$; and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 = \{\emptyset, \mathcal{S}, \{a, b\}, \{a, c\}\}$. Now, $(\mathcal{S}, \mathcal{C})$ is not a convexity space since $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{C}$.

Theorem 3.3 (The Refinement Theorem) *Given* $n \geq 1$ *convexity spaces* (S, C_i), $1 \leq i \leq n$; *then, for all* $\mathbf{P} \subseteq S$,

$$\bigcup_{i=1}^{n}(C_i\text{-}hull(\mathbf{P})) \subseteq (\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P}) \subseteq (\bigcap_{i=1}^{n} C_i)\text{-}hull(\mathbf{P})$$

Proof: The proof is by induction on n.
Basis: The theorem is vacuously true for $n = 1$. Consider the case $n = 2$. From Theorem 2.1(2) we have $\mathbf{P} \subseteq C_2\text{-}hull(\mathbf{P})$ and from Theorem 2.1(4) we have $C_1\text{-}hull(\mathbf{P}) \subseteq C_1\text{-}hull(C_2\text{-}hull(\mathbf{P}))$. Also, from Theorem 2.1(2) we have $C_2\text{-}hull(\mathbf{P}) \subseteq C_1\text{-}hull(C_2\text{-}hull(\mathbf{P}))$. Hence, $(C_1\text{-}hull(\mathbf{P}) \cup C_2\text{-}hull(\mathbf{P})) \subseteq C_1\text{-}hull(C_2\text{-}hull(\mathbf{P}))$.

Since $C_1 \cap C_2 \subseteq C_2$, Theorem 3.1(1) implies $C_2\text{-}hull(\mathbf{P}) \subseteq (C_1 \cap C_2)\text{-}hull(\mathbf{P})$. Hence, $C_1\text{-}hull(C_2\text{-}hull(\mathbf{P})) \subseteq C_1\text{-}hull((C_1 \cap C_2)\text{-}hull(\mathbf{P})) = (C_1 \cap C_2)\text{-}hull(\mathbf{P})$. Thus, $(C_1\text{-}hull(\mathbf{P}) \cup C_2\text{-}hull(\mathbf{P})) \subseteq C_1\text{-}hull(C_2\text{-}hull(\mathbf{P})) \subseteq (C_1 \cap C_2)\text{-}hull(\mathbf{P})$. Therefore, the theorem holds for $n = 2$.
Induction Hypothesis: Assume the theorem holds for all $k < n$, for some $n \geq 3$.
Induction Step: Given a composition of n hulls, consider the two innermost operators:

$$(\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P}) = C_1\text{-}hull(\ldots C_{n-1}\text{-}hull(C_n\text{-}hull(\mathbf{P}))\ldots)$$

From the theorem for $n = 2$ we know that

$$C_{n-1}\text{-}hull(C_n\text{-}hull(\mathbf{P})) \subseteq (C_{n-1} \cap C_n)\text{-}hull(\mathbf{P})$$

Since there are only $n - 2$ operators surrounding these inner operators we can apply the induction hypothesis to conclude that

$$\begin{aligned}(\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P}) &= (\bigwedge_{i=1}^{n-2} C_i)\text{-}hull(C_{n-1}\text{-}hull(C_n\text{-}hull(\mathbf{P}))) \\ &\subseteq (\bigcap_{i=1}^{n-2} C_i)\text{-}hull(C_{n-1}\text{-}hull(C_n\text{-}hull(\mathbf{P})))\end{aligned}$$

But both $(\bigcap_{i=1}^{n-2} C_i)\text{-}hull$ and $(C_{n-1} \cap C_n)\text{-}hull$ are hull operators (Lemma 3.2); therefore, from the basis we have

$$(\bigcap_{i=1}^{n-2} C_i)\text{-}hull(C_{n-1}\text{-}hull(C_n\text{-}hull(\mathbf{P}))) \subseteq (\bigcap_{i=1}^{n-2} C_i)\text{-}hull((C_{n-1} \cap C_n)\text{-}hull(\mathbf{P}))$$

$$\subseteq \quad ((\bigcap_{i=1}^{n-2} C_i) \cap (C_{n-1} \cap C_n))\text{-}hull(\mathbf{P})$$
$$= \quad (\bigcap_{i=1}^{n} C_i)\text{-}hull(\mathbf{P})$$

Thus,

$$(\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P}) \;\subseteq\; (\bigcap_{i=1}^{n} C_i)\text{-}hull(\mathbf{P})$$

and all that remains is to show that

$$\bigcup_{i=1}^{n}(C_i\text{-}hull(\mathbf{P})) \;\subseteq\; (\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P})$$

Consider the j^{th} hull operator in the composition:

$$(\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P}) = C_1\text{-}hull(\ldots(C_j\text{-}hull(\ldots(\mathbf{P})\ldots))\ldots)$$

and consider the set on which C_j-*hull* acts; that is, the set $(\ldots(\mathbf{P})\ldots)$ produced by the hull operators inside the inner pair of brackets. This set must contain $\mathbf{P}$ since each hull operator is expansive (Theorem 2.1(2)). Thus, from Theorem 2.1(4), we have, for each j, $1 \le j \le n$,

$$C_j\text{-}hull(\mathbf{P}) \;\subseteq\; \ldots(C_j\text{-}hull(\ldots(\mathbf{P})\ldots))\ldots$$

and, hence,

$$\bigcup_{i=1}^{n}(C_i\text{-}hull(\mathbf{P})) \;\subseteq\; (\bigwedge_{i=1}^{n} C_i)\text{-}hull(\mathbf{P})$$

□

Note that this result holds independently of the order in which the hulls are composed in the middle term. The following example shows that, in general, we cannot replace any of the containments by equality. Let $S = \{a, b, c, d\}$; $C_1 = \{\emptyset, S, \{a\}, \{a, b, c\}\}$; $C_2 = \{\emptyset, S, \{a, b\}\}$; and $C = C_1 \cap C_2 = \{\emptyset, S\}$. Then C_1-$hull(\{a\}) \cup C_2$-$hull(\{a\}) = \{a, b\}$; $(C_1 \wedge C_2)$-$hull(\{a\}) = \{a, b, c\}$; and C-$hull(\{a\}) = S$.

However, as a special case, if two hull operators are *invariant* in the sense defined below, then the second two terms are equal (that is, the composition hull equals the intersection hull).

Definition 3.4 *Given two convexity spaces* (S, C_1) *and* (S, C_2), C_2 *is said to be* C_1-invariant *if, for all* $\mathbf{P} \in C_2$, C_1-$hull(\mathbf{P}) \in C_2$.

The idea is that C_2 is C_1-invariant if the C_1-*hull* of any C_2-convex set does not destroy its C_2-convexity.

Theorem 3.4 (The Decomposition Theorem) *Let* (S, C_1) *and* (S, C_2) *be two convexity spaces defined on the groundset* S. *If* C_2 *is* C_1*-invariant, then for all* $\mathbf{P} \subseteq S$, $(C_1 \wedge C_2)$*-hull*$(\mathbf{P}) = (C_1 \cap C_2)$*-hull*$(\mathbf{P})$.

Proof: Let $\mathbf{P}$ be a subset of S. By definition, C_2-*hull*$(\mathbf{P}) \in C_2$. If C_2 is C_1-invariant, then C_1-*hull*$(C_2$-*hull*$(\mathbf{P})) \in C_2$. But C_1-*hull*$(C_2$-*hull*$(\mathbf{P})) \in C_1$ and, therefore, C_1-*hull*$(C_2$-*hull*$(\mathbf{P})) \in C_1 \cap C_2$. But $(C_1 \cap C_2)$-*hull*$(\mathbf{P})$ is a subset of all $(C_1 \cap C_2)$-convex sets which contain $\mathbf{P}$. Therefore, $(C_1 \cap C_2)$-*hull*$(\mathbf{P}) \subseteq (C_1 \wedge C_2)$-*hull*$(\mathbf{P})$. Now, from the refinement theorem, we know that, for all $\mathbf{P} \subseteq S$, $(C_1 \wedge C_2)$-*hull*$(\mathbf{P}) \subseteq (C_1 \cap C_2)$-*hull*$(\mathbf{P})$. Hence the result follows. □

The following example shows that the union of two hulls is not necessarily equal to their composition (and their intersection) even if the convexity spaces are invariant.

Let $S = \{a, b, c, d\}$; $C_1 = \{\emptyset, S, \{a, b\}\}$; $C_2 = \{\emptyset, S, \{a, c\}\}$; and $C = C_1 \cap C_2 = \{\emptyset, S\}$. Then C_2 is C_1-invariant but C_1-*hull*$(\{a\}) \cup C_2$-*hull*$(\{a\}) = \{a, b, c\}$; $(C_1 \wedge C_2)$-*hull*$(\{a\}) = S$; and C-*hull*$(\{a\}) = S$.

4 Example I: Restricted-Orientation Convexity

The *orientation of a directed line* is the counterclockwise angle made with the horizontal in a directed plane (in the goniometric sense). The *orientation of an undirected line* is the smaller of the two possible orientations. We only discuss undirected lines in this section. We use the symbol $\mathcal{O}$, with or without subscripts, to refer to *a set* (possibly empty) *of orientations*. We normally assume that the set $\mathcal{O}$ is symmetric about the horizontal, although our results hold even if this is not the case.

A collection of lines, segments and rays is said to be $\mathcal{O}$*-oriented* if the set of orientations of the elements of the collection is a subset of $\mathcal{O}$. Thus, we speak of $\mathcal{O}$*-lines*, $\mathcal{O}$*-segments*, and $\mathcal{O}$*-rays* to mean $\mathcal{O}$-oriented lines, segments and rays. By extension, we call a polygon an $\mathcal{O}$*-polygon* if its edges are $\mathcal{O}$-segments.

The notion of $\mathcal{O}$-orientation has been previously defined, but only for finite $\mathcal{O}$, in [6,16,26,25] and, in a slightly related form, in [4]. There is a vast literature concerning the special case of $\mathcal{O} = \{0°, 90°, 180°, 270°\}$. $\{0°, 90°, 180°, 270°\}$-objects are more usually called *orthogonal* (also; rectilinear, isothetic, iso-oriented, x-y or aligned) objects; see [14,27] for further references.

Definition 4.1 *We say that a set* $\mathbf{P} \subseteq \Re^2$ *is* $\mathcal{O}$-convex *if the intersection of* $\mathbf{P}$ *and any* $\mathcal{O}$*-line is either empty or connected.*

This is a natural generalization of orthogonal convexity and normal convexity. Figure 1 contains some example figures which are $\mathcal{O}$-convex for various $\mathcal{O}$. Figure 1 (a) is not $\mathcal{O}$-convex for any non-empty $\mathcal{O}$, but is $\mathcal{O}$-convex if $\mathcal{O} = \emptyset$, as are all the other figures. Figures (b) and (c) are convex with respect to any horizontal line, as are (d), (e) and (f), so they are all 0°-convex besides being $\emptyset$-convex. Note that (b) and (c) are not convex for any other orientation. Figures (d), (e) and (f) are convex with respect to any vertical line as well and so they are also $\{0°, 90°, 180°, 270°\}$-convex. Note that (d) is not convex for any other orientation. Figures (e) and (f) are convex with respect to any line with orientation in the ranges $[90°, 180°]$ and $[270°, 360°]$, and so they are also $\{\theta \mid 0° \leq \theta \leq 90° \text{or} 180° \leq \theta \leq 270°\}$-convex. Note that (e) is not convex for any other orientation. Figure (f) is $\mathcal{O}$-convex for any $\mathcal{O}$.

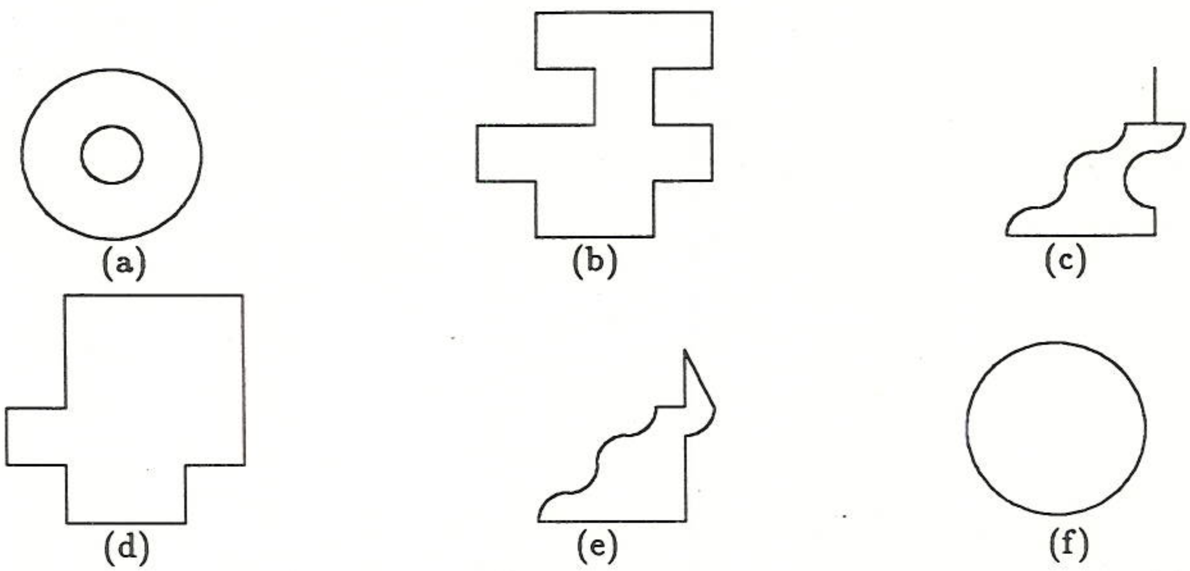

Figure 1: Some Examples of $\mathcal{O}$-convex Sets

We have the following immediate relation between convexity and $\mathcal{O}$-convexity.

Lemma 4.1 *For any set* $\mathcal{O}$ *of orientations, if* $\mathbf{P}$ *is convex then* $\mathbf{P}$ *is* $\mathcal{O}$*-convex.*

The following sets are convex, and hence $\mathcal{O}$-convex for any $\mathcal{O}$: the empty set, $\Re^2$, and, any point, line, line segment, ray or halfplane in $\Re^2$.

We first note that, for any set of orientations $\mathcal{O}$, the set of all $\mathcal{O}$-convex sets over the groundset $\Re^2$ forms a convexity space. This follows because we can equally well define $\mathcal{O}$-convex sets by:

> A set $\mathbf{P} \subseteq \Re^2$ is $\mathcal{O}$-convex if, for all points $p, q \in P$, if the line through p and q is an $\mathcal{O}$-line, then the line segment joining them is wholly in $\mathbf{P}$.

Immediately, letting $\mathcal{L}_{\mathcal{O}}$ be the set of all $\mathcal{O}$-line segments, $(\Re^2, \mathcal{L}_{\mathcal{O}})$ is a line space and, therefore, we obtain:

Theorem 4.2 *For any set $\mathcal{O}$ of orientations, let $\mathcal{C}_{\mathcal{O}}$ be the set of all $\mathcal{O}$-convex sets in $\Re^2$.*

Then $(\Re^2, \mathcal{C}_{\mathcal{O}})$ is a convexity space.

$\mathcal{O}$-convexity is a generalization of both normal convexity and orthogonal convexity. We show that the $\mathcal{O}$-convex hull obeys a strong decomposition theorem (Theorem 4.8) when restricted to connected sets.

Definition 4.2 *The intersection of all $\mathcal{O}$-convex sets containing* $\mathbf{P}$ *is the* $\mathcal{O}$-hull *of* $\mathbf{P}$ *and is denoted by $\mathcal{O}$-hull($\mathbf{P}$) (cf. the definition of the $\mathcal{C}$-hull in Section 3).*

Observe that if $\mathcal{O}$ or $\mathbf{P}$ is empty, then $\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P}$. When $\mathcal{O} = \theta$, for some orientation θ, and $\mathbf{P}$ is a polygon, the $\mathcal{O}$-*hull* of $\mathbf{P}$ has been called the θ-*visibility hull* of $\mathbf{P}$ in [18,23]. As examples of hulls observe that in Figure 1, (f) is the $\mathcal{O}$-*hull* of (a), for any non-empty $\mathcal{O}$, and (d) and (e) are the 90°-*hulls* of (b) and (c), respectively.

Theorem 4.2 establishes that every choice of $\mathcal{O}$ gives rise to a convexity space; thus, we may employ the results of Section 3. Immediately, from Theorem 2.1 we have:

Corollary 4.3 *For all* $\mathcal{O}, \mathbf{P}, \mathbf{Q}$,

$$\mathbf{P} \subseteq \mathcal{O}\text{-}hull(\mathbf{P})$$

$$\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P} \iff \mathbf{P} \text{ is } \mathcal{O}\text{-convex}$$

$$\mathbf{P} \subseteq \mathbf{Q} \Longrightarrow \mathcal{O}\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}\text{-}hull(\mathbf{Q})$$

Observe that if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then, for all $\mathbf{P}$, $\mathcal{O}_1\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_2\text{-}hull(\mathbf{P})$ as was proved in Theorem 3.1. The reader should note, however, that the inclusion is *reversed* since if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then the set of $\mathcal{O}_2$-convex sets is a refinement of the set of $\mathcal{O}_1$-convex sets. That is, if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every $\mathcal{O}_2$-convex set is $\mathcal{O}_1$-convex.

Note that, as a set $\mathcal{O}$ of orientations "grows" to include all possible orientations, $\mathcal{O}\text{-}hull(\mathbf{P})$ "grows" to be the convex hull of $\mathbf{P}$. Indeed we have the following theorem.

Theorem 4.4 *The set of all $\mathcal{O}$-convex sets over $\Re^2$, for all $\mathcal{O}$, form a lattice under refinement. The set of $[0°, 360°)$-convex sets is the supremum of this lattice and the set of $\emptyset$-convex sets is the infimum.*

Proof: Grätzer ([7] Exercise 9.(i), page 7) shows that the set of all subsets of a set form a lattice under inclusion. We need merely observe that refinement is equivalent to inclusion for $\mathcal{O}$-convex sets. □

We now state the *Separation Theorem* proved in [16], which we use to prove the *Orientation Decomposition Theorem.*

Theorem 4.5 (The Separation Theorem) *Let* $\mathbf{P}$ *be connected and* $p \notin \mathbf{P}$. *Then,* $p \in \mathcal{O}$*-hull*$(\mathbf{P})$ *if and only if there exists a* $\theta \in \mathcal{O}$ *such that the* θ*-line through* p *intersects* $\mathbf{P}$ *in, at least, two points on either side of* p.

This theorem is false if $\mathbf{P}$ is not connected; see [16].

As an interesting aside this theorem yields:

Corollary 4.6 *Let* $\mathbf{P}$ *be connected. If* $p \notin \mathcal{O}$*-hull*$(\mathbf{P})$, *then there exists an* $\mathcal{O}$*-stairline separating* p *and* $\mathbf{P}$.

When $\mathcal{O} = [0°, 180°)$, all $\mathcal{O}$-stairlines are lines and this corollary is the usual separability property.

Intuitively, we may think of forming the $\mathcal{O}$*-hull* of a set $\mathbf{P}$ by sweeping a line of each orientation in $\mathcal{O}$ across $\mathbf{P}$ and adding suitable line segments to the hull formed so far so that it is convex in each orientation in $\mathcal{O}$ (if $\mathcal{O}$ is empty, then we do not add anything to $\mathbf{P}$). Thinking of it this way, it appears reasonable that the hull we eventually produce is unchanged if we choose a different sweeping order. As we prove in Theorem 4.8 this is, in fact, the case but only for *connected* sets. In general, Lemma 4.7 is the strongest possible result.

As a by-product of the following theorems we establish the validity of two assumptions made in the literature for orthogonal hulls. Sack [18] showed, in the orthogonal case, that the horizontal hull of the vertical hull of an orthogonal polygon (or alternately the vertical hull of the horizontal hull) is equivalent to the union of both hulls. It was taken as self-evident that their union is the smallest horizontally and vertically convex polygon enclosing the orthogonal polygon. Theorem 4.8 validates this assumption. Toussaint and Sack [23] made the observation that the convex hull is the union of the "visibility hulls" over all directions of visibility. Theorem 4.8 supplies the proof for this observation.

The following lemma follows from the Refinement Theorem (Theorem 3.3). Observe that the hull with respect to the intersection of the family of convexity spaces has been replaced by the hull with respect to the union of the sets of orientations. This is because when we form the union of two sets of orientations the hull with respect to their union is convex with respect to *both* sets of orientations.

Lemma 4.7 *Given $n \geq 1$ sets $\mathcal{O}_i$ of orientations, $1 \leq i \leq n$; then, for all* $\mathbf{P}$,

$$\bigcup_{i=1}^{n}(\mathcal{O}_i\text{-}hull(\mathbf{P})) \subseteq (\bigwedge_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P}) \subseteq (\bigcup_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P})$$

Simple counterexamples show that this result is the best possible, in that there exist sets for which the respective converses are false. However, we can strengthen Lemma 4.7 considerably by restricting $\mathbf{P}$ to be connected. As we prove below, when $\mathbf{P}$ is connected the containment relations in Lemma 4.7 become equalities. Thus, in the language of Section 3, with respect to connected subsets of $\Re^2$, any two orientation convexity spaces are mutually invariant.

Theorem 4.8 (The Orientation Decomposition Theorem)
Given $n \geq 1$ sets $\mathcal{O}_i$ of orientations, $1 \leq i \leq n$; if $\mathbf{P}$ is connected, then

$$\bigcup_{i=1}^{n}(\mathcal{O}_i\text{-}hull(\mathbf{P})) = (\bigwedge_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P}) = (\bigcup_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P})$$

Proof: Because of Lemma 4.7 we need establish only that if $\mathbf{P}$ is connected, then $(\bigcup_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P}) \subseteq \bigcup_{i=1}^{n}(\mathcal{O}_i\text{-}hull(\mathbf{P}))$. If $\mathbf{P}$ or $(\bigcup_{i=1}^{n} \mathcal{O}_i)$ is empty, then this holds, so assume that both are non-empty.

Let $p \in (\bigcup_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P})$. If $p \in \mathbf{P}$, then $p \in \bigcup_{i=1}^{n}(\mathcal{O}_i\text{-}hull(\mathbf{P}))$. So suppose that $p \in (\bigcup_{i=1}^{n} \mathcal{O}_i)\text{-}hull(\mathbf{P}) \setminus \mathbf{P}$. From Theorem 4.5, we know that there must exist a $\theta \in (\bigcup_{i=1}^{n} \mathcal{O}_i)$ such that the θ-line through p cuts $\mathbf{P}$ on both sides of p. This implies that p must be in $\mathcal{O}_i\text{-}hull(\mathbf{P})$, for some i such that $\theta \in \mathcal{O}_i$. Hence, $p \in \bigcup_{i=1}^{n}(\mathcal{O}_i\text{-}hull(\mathbf{P}))$ and the result follows. □

This decomposition result immediately yields an algorithm to find the hull of any connected set given that we can find the hull in one orientation and that we can find the union of two or more hulls. As it turns out, however, connected $\mathcal{O}$-convex sets have considerably more structure than this which we can exploit to obtain optimal algorithms to find the hull of any connected set; see [15].

5 Example II: $NESW$-Convexity

As an illustration of the generality of our framework we discuss how they apply to a problem arising from locked transactions in databases.

In attempting to solve a database concurrency problem, Yannakakis *et al.* [28] found a correspondence between the safety of a locked transaction system and what was later called the $NESW$*-closure* of a collection of orthogonal rectangles but is, in reality, their $NESW$-hull. Lipski and

Papadimitriou [12] found an $O(n \lg n \lg \lg n)$ time and $O(n \lg n)$ space algorithm to find the $NESW$-hull of a set of orthogonal rectangles. Later, Soisalon-Soininen and Wood [21] proved that the $NESW$-hull can be decomposed into two simpler hulls. With this decomposition result in hand they were able to derive a simple—and optimal—$O(n \lg n)$ time and $O(n)$ space algorithm to find the $NESW$-hull. We re-prove their decomposition result for connected sets, using a completely different technique.

Definition 5.1 *A horizontal ray is an E-*ray *if it lies to the east of some vertical line. Similarly, we can define N-, S-, and W-*rays.

*The SW-*line *at a point p consists of a N-ray and a W-ray that have their endpoints at point p. The point p is said to be the* vertex *of the SW-line. The notions of NE-, SE-, and NW-*lines *are defined similarly.*

Given two points p and q in the plane, they determine a *unique NE-line* if either p is to the left of q and above q or vice versa. Therefore, it makes sense to say that p, q define a NE-line in these cases.

Definition 5.2 *We say that* $\mathbf{P}$ *is NE-*convex *if, for all $p, q \in \mathbf{P}$ that define a NE-line, the vertex of this line is in* $\mathbf{P}$. *We define SW-convex, SE-convex, and NW-convex similarly.*

We say that a set $\mathbf{P}$ *is $NESW$-*convex *if it is both NE- and SW-convex.*

Let $\mathcal{L}_{\mathcal{N}}$ be the collection of all sets $\{p, q, v\}$ such that p, q define a NE-line with vertex v. Then, $(\Re^2, \mathcal{L}_{\mathcal{N}})$ is a line space and each set $\mathbf{P} \subseteq \mathcal{S}$, $\mathbf{P} \neq \emptyset$, is $\mathcal{L}_{\mathcal{N}}$-convex if and only if it is NE-convex. In a similar manner we define $\mathcal{L}_{\mathcal{S}}$, the collection of all triples that define a SW-line. We define $\mathcal{L}_{\mathcal{NS}}$ to be the union of these two sets.

By Theorem 2.2, we have:

Theorem 5.1 *For $\mathcal{X}$ the collection of all NE-convex sets, SW-convex sets, and $NESW$-convex sets, $(\Re^2, \mathcal{X})$ is a convexity space.*

Soisalon-Soininen and Wood proved a decomposition theorem for $NESW$-convex sets by examining the boundaries of the NE- and SW-hulls. We re-prove their results, for the special case of connected sets, by proving a decomposition theorem that is similar in spirit to Theorem 4.8. We approach the proof along similar lines to the proof of the Orientation Decomposition Theorem in the previous section.

Lemma 5.2 *Let* $\mathbf{P}$ *be a connected set. If a point $p \in NESW$-hull$(\mathbf{P}) \setminus \mathbf{P}$, then either the NE-line or the SW-line at p intersect* $\mathbf{P}$ *on both sides of p.*

Proof: Assume that $p \in NESW$-*hull*$(\mathbf{P}) \setminus \mathbf{P}$ and neither the NE-line nor the SW-line at p cut $\mathbf{P}$ on each side of p. Then, we can remove all the points in either the NW-quadrant at p or the SE-quadrant at p. We argue as follows.

Assume the vertical line through p cuts $\mathbf{P}$ on each side of p. Because $\mathbf{P}$ is connected and $p \notin \mathbf{P}$, the E-ray at p or the W-ray at p must cut $\mathbf{P}$. But this implies that either the NE-line or the SW-line at p cuts $\mathbf{P}$ on each side of p — a contradiction. A similar argument holds for the horizontal line through p.

Now assume, without loss of generality, that the S-ray at p does not cut $\mathbf{P}$. Then, either the E-ray or the W-ray at p do. If the W-ray does, remove from $NESW\text{-}hull(\mathbf{P})$, all points in the closed quadrant containing p and formed by the E-ray and the S-ray at p. If the E-ray cuts p, then neither the W-ray nor the N-ray cut $\mathbf{P}$. In this case, remove all points in the closed quadrant containing p and formed by the N-ray and the W-ray at p.

In both cases, since no points in $\mathbf{P}$ were deleted this set still contains $\mathbf{P}$ and is $NESW$-convex. But $NESW\text{-}hull(\mathbf{P})$ is the smallest set which both contains $\mathbf{P}$ and is $NESW$-convex, a contradiction. Thus the lemma is proved. □

Theorem 5.3 *For all connected sets* $\mathbf{P}$,

$$\begin{aligned} NE\text{-}hull(\mathbf{P}) \cup SW\text{-}hull(\mathbf{P}) &= NE\text{-}hull(SW\text{-}hull(\mathbf{P})) \\ &= SW\text{-}hull(NE\text{-}hull(\mathbf{P})) \\ &= NESW\text{-}hull(\mathbf{P}) \end{aligned}$$

Proof: We only need prove that $NESW\text{-}hull(\mathbf{P}) \subseteq NE\text{-}hull(\mathbf{P}) \cup SW\text{-}hull(\mathbf{P})$, since the other containments follow from Theorem 3.3.

If $\mathbf{P} = \emptyset$, this holds vacuously, so assume that $\mathbf{P} \neq \emptyset$. Consider $p \in NESW\text{-}hull(\mathbf{P})$. If $p \in \mathbf{P}$, the $p \in NE\text{-}hull(\mathbf{P}) \cup SW\text{-}hull(\mathbf{P})$. Therefore, assume $p \in NESW\text{-}hull(\mathbf{P}) \setminus \mathbf{P}$. From Lemma 5.1 either the NE-line or the SW-line at p cuts $\mathbf{P}$ on each side of p. Hence, p is in either the $NE\text{-}hull(\mathbf{P})$ or the $SW\text{-}hull(\mathbf{P})$, respectively; that is, $p \in NE\text{-}hull(\mathbf{P}) \cup SW\text{-}hull(\mathbf{P})$. The result follows. □

6 Example III: $NEED$-Convexity

We now provide a final example to demonstrate that connectedness does not imply decomposability. Using the notions of the previous section, a NW-line rotated counterclockwise by 45° gives what we call *an ED-line* or an *eastern diamond line.* Clearly, if two points p and q do not both lie on the same ray of an ED-line, then they determine at most one ED-line. Hence, for ED-lines, we say that p, q *define a an ED-line*, if they determine an ED-line and this line is unique.

Definition 6.1 *A set* $\mathbf{P} \subseteq \Re^2$ *is* ED-convex *if for all points* $p, q \in \mathbf{P}$ *that define an ED-line,its vertex is in* $\mathbf{P}$.
A set $\mathbf{P} \subseteq \Re^2$ *is* $NEED$-convex *if it is both NE-convex and ED-convex.*

Let $\mathcal{L}_{\mathcal{E}}$ be the collection of all sets $\{p, q, r\}$ such that p, q define an ED-line with vertex v. Then, $(\Re^2, \mathcal{L}_{\mathcal{E}})$ is a line space and each set $\mathbf{P} \subseteq \Re^2$, $\mathbf{P} \neq \emptyset$, is $\mathcal{L}_{\mathcal{E}}$-convex if and only if it is ED-convex. Now, let $\mathcal{L}_{\mathcal{NE}}$ be the union of $\mathcal{L}_{\mathcal{N}}$ and $\mathcal{L}_{\mathcal{E}}$; again it induces a line space. Hence, we obtain:

Theorem 6.1 *For* $\mathcal{X}$ *the collection of all ED-convex sets and NEED-convex sets,* $(\Re^2, \mathcal{X})$ *is a convexity space.*

However, as we now show there is no decomposition theorem for $NEED$-convex sets. Consider an orthogonal unit square in the plane. Its NE-hull is itself; its ED-hull is, after a little thought, a pentagon with the same bottom, top, and left edges, but with two additional edges given by the ED-line defined by the topmost and bottommost righmost corner points. The NE-hull of the ED-hull is the pentagon with its top edge extended a half unit to the right and from this terminating point a line segment is dropped a half unit downwards. But, its $NEED$-hull is a quadrilateral with the same bottom and left edges, its top edge extended again by a half unit (so it is two units long), and the right edge joins the two rightmost corner points.

Therefore, there is no decomposition theorem, since we do not have invariance.

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