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Visibility, Skulls, and Kernels in Convexity Spaces

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Visibility, Skulls, and Kernels in Convexity Spaces*

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Abstract

In this paper we pursue the notion of a convexity space as a unifying framework for the treatment of various notions of convexity in the plane. In particular, we suggest how to capture the notion of visibility within the general framework of convexity spaces, and investigate the relationship between visibility, kernels and skulls. We prove the Kernel Theorem and the Cover Kernel Theorem, both of which relate kernels and skulls.

1 Introduction

Convexity spaces represent an abstraction of the structure of convex sets in Euclidian space. Leaving aside all topological concerns we require convex sets to be closed under intersection, thus capturing their lattice-theoretic and algebraic properties. It is not surprising that such a general concept has arisen in many different contexts and has led to a number of names such as *convexity spaces, convexity structures,* or *algebraic closure systems.*

The main aim of abstract convexity is to provide a unifying framework that goes beyond the concept of convexity based on line segments as in real vector spaces. In fact, there is an astonishing variety of "non-standard" notions of convexity in the plane that have been considered in computational geometry in the past few years: *restricted orientation convexity* [13], *NESW-convexity* [12, 20], *rectangular convexity* [10, 20], and *geodesic convexity* [6, 24], to name the most prominent ones.

There is a close relationship between questions concerning visibility and those concerning convexity. In most convexity spaces the definition of a

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convex set is based on the property that all points of a convex set can "see" each other. So only the seeing relationship between two points has to be specified in order to define the whole convexity space. We will pursue this special notion of convexity in Section 3.2.

The original motivation for an abstract treatment of convexity is to investigate the relationships between the Helly, Radon, and Carathéodory numbers in an axiomatic setting [11, 9, 16]. Additionally, the exchange number of a convexity space has been introduced and several generalizations of the Helly and Radon numbers have been considered [4, 18].

The observation that many convexity spaces can be decomposed into simpler components has led to a second line of investigation. The main goal of this approach is to deduce properties of the convexity space from those of its components. All the results obtained by this approach are concerned with the characteristics of the convex hull operator [3, 5, 14, 17]. Little attention has been paid to problems which involve visibility.

On the other hand, visibility is a well studied concept in the context of real vector spaces [1, 23, 25]. But, as the above mentioned examples illustrate, this is often too restrictive a setting. So, in order to define and investigate notions of convexity that are not based on the properties of straight lines, it is desirable to take the more general approach of convexity spaces and make use of general results obtained in this context. This leads to a number of questions which have not been considered or even cannot be formulated in the framework of real vector spaces.

In Section 2 we introduce the concept of visibility in the setting of the usual convexity in the plane. This is followed in Section 3, the main section of this paper, with the presentation of our results on kernels and skulls in convexity and aligned spaces. This culminates in the proofs of the Kernel Theorem and the Cover Kernel Theorem. Finally, in Section 4 we consider four examples of convexity spaces: real vector spaces; orthogonal convex sets; geodesic convex sets; and $NESW$-convex sets.

2 Visibility and Convexity

Given a simple polygon $\mathbf{P}$ in the plane, we say that two points p and q in $\mathbf{P}$ *see each other* if the line segment $[p, q]$ joining them is wholly in $\mathbf{P}$. The visibility relation is denoted by $sees_{\mathbf{P}}$ and we write $p\ sees_{\mathbf{P}}\ q$.

Clearly, $sees_{\mathbf{P}}$ is reflexive and symmetric, but not necessarily transitive. Indeed, it is transitive if and only if $\mathbf{P}$ is convex. Because $sees_{\mathbf{P}}$ is not transitive in general, it gives rise to weaker notions of convexity that we do not pursue here. Our goal is to consider the relationship between maximal convex subsets (see below) and kernels; we will further investigate when kernels are convex.

Recall that the *star of a polygon* $\mathbf{P}$ with respect to a point p in $\mathbf{P}$ is the set $\{q \mid p\ sees_{\mathbf{P}}\ q\}$ of points in $\mathbf{P}$ that are seen by p. This set is denoted by $star(p, \mathbf{P})$. We say that $\mathbf{P}$ is *star-shaped* if $star(p, \mathbf{P}) = \mathbf{P}$, for some p. The *kernel of* $\mathbf{P}$ is the set of all points that see $\mathbf{P}$, namely, $kernel(\mathbf{P}) = \{p \mid star(p, \mathbf{P}) = \mathbf{P}\}$. Clearly, $kernel(\mathbf{P}) \neq \emptyset$ if and only if $\mathbf{P}$ is star-shaped. It is well known that $kernel(\mathbf{P})$ is convex.

Let us now define the less familiar notion of a skull. Let $\mathbf{P}$ be a simple polygon and $\mathbf{Q}$ be a convex set such that $\mathbf{Q} \subseteq \mathbf{P}$; $\mathbf{Q}$ is an *inscribed convex set*. We say that $\mathbf{Q}$ is *maximal* if there is no inscribed convex set $\mathbf{R}$ with $\mathbf{Q} \subset \mathbf{R} \subseteq \mathbf{P}$. Such a maximal convex set is called a *skull of* $\mathbf{P}$. The family of skulls of a polygon $\mathbf{P}$ is denoted by $skulls(\mathbf{P})$ and is defined to be the set $\{\mathbf{S} \mid \mathbf{S} \text{ is a skull of } \mathbf{P}\}$.

We now state an elegant kernel theorem that relates skulls and kernels [21, 22].

Theorem 2.1 *For all polygons* $\mathbf{P}$ *in the plane,*

$$kernel(\mathbf{P}) = \bigcap_{\mathbf{S} \in skulls(\mathbf{P})} \mathbf{S}.$$

Since, by definition, each skull is convex, this result implies that $kernel(\mathbf{P})$ is convex. Furthermore, whenever $skulls(\mathbf{P})$ is finite, it gives rise to an algorithm, albeit an inefficient one, to compute $kernel(\mathbf{P})$. In this paper, we explore the relationship between skulls and kernels in the general setting of convexity spaces. We want to characterize those convexity spaces for which a Kernel Theorem holds; further it is our aim to establish the relationship between skulls and kernels in arbitrary convexity spaces, and we want to characterize when kernels are convex.

In the course of this investigation the proof of the above theorem will turn out to be a corollary of a much more general theorem which we state and prove in Section 3.4. However, the above theorem can, of course, be proved directly and the reader is invited to do so.

3 Kernels in Convexity Spaces

This section contains the main results of the paper. It is structured as follows: First, in Section 3.1 we define what we understand by a convexity space and establish some of the properties of the convex hull operator. In Section 3.2 we abstract a suitable notion of visibility in a convexity space and illustrate some of its consequences. Finally, Section 3.3 deals with skulls and aligned spaces. We are then in a position to state and prove the Kernel Theorem in Section 3.4 and explore some natural follow up questions.

3.1 Convexity Spaces

We base the following investigation on the concept of a convexity space whose formal definition was first introduced by Levi [11]. A convexity space is intended to abstract some of the essential properties of convex sets in n dimensional Euclidian space.

Definition 3.1 *Let $\mathcal{X}$ be a set and $\mathcal{C}$ be a collection of subsets of $\mathcal{X}$. Then, $(\mathcal{X}, \mathcal{C})$ is a* convexity space *if:*

1. *$\emptyset$ and $\mathcal{X}$ are in $\mathcal{C}$; and*
2. *for all $\mathcal{C}' \subseteq \mathcal{C}$, we have $\bigcap \mathcal{C}' \in \mathcal{C}$, where by $\bigcap \mathcal{C}'$ we mean $\bigcap_{\mathbf{C} \in \mathcal{C}} \mathbf{C}$.*[1]

$\mathcal{X}$ is called the *groundset* of the convexity space and $\mathcal{C}$ contains the "convex sets" of $\mathcal{X}$. Each set in $\mathcal{C}$ is called *$\mathcal{C}$-convex* (or convex for short if the convexity space is understood). So, the only characteristic required of convex sets is their closure under intersection. It is obvious that additional properties are needed to generalize the more intricate properties of $\mathbb{E}^n$ since a wide variety of structures satisfy the above definition. Simple examples are:

1. *The discrete convexity space* $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$, where $\mathcal{P}(\mathcal{X})$ is the powerset of $\mathcal{X}$. Obviously, any intersection of subsets of $\mathcal{X}$ is again a subset of $\mathcal{X}$.
2. *The topological convexity space.* Let $(\mathcal{X}, \mathcal{T})$ be a topological space; then $(\mathcal{X}, \mathcal{C})$ with $\mathcal{C} = \{\mathbf{C} \subseteq \mathcal{X} \mid \mathbf{C} \text{ is closed in } (\mathcal{X}, \mathcal{T})\}$ is a convexity space since the intersection of closed sets remains closed.

Immediately associated with a convexity space is the convex hull operator.

Definition 3.2 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space; then, for all $\mathbf{X} \subseteq \mathcal{X}$, the* $\mathcal{C}$-hull *of $\mathbf{X}$, which is denoted by $\mathcal{C}$-hull$(\mathbf{X})$, is defined by*

$$\mathcal{C}\text{-}hull(\mathbf{X}) = \bigcap \{\mathbf{C} \in \mathcal{C} \mid \mathbf{X} \subseteq \mathbf{C}\}.$$

It is easy to see that the *$\mathcal{C}$-hull* operator is well defined and unique; some of its more pleasing properties are summarized in the following theorem.

Theorem 3.1 *Given a convexity space $(\mathcal{X}, \mathcal{C})$; then, for all $\mathbf{X}, \mathbf{Y} \subseteq \mathcal{X}$:*

1. *$\mathcal{C}$-hull$(\mathbf{X}) \in \mathcal{C}$;*
2. *$\mathbf{X} \subseteq \mathcal{C}$-hull$(\mathbf{X})$;*

[1] In this paper we will use $\bigcap \mathcal{F}$ to denote the intersection of all sets in a family $\mathcal{F}$ and $\bigcup \mathcal{F}$ to denote the union of all sets in $\mathcal{F}$.

3. *$\mathcal{C}$-hull*$(\mathbf{X}) = \mathbf{X}$ *if and only if* $\mathbf{X} \in \mathcal{C}$;

4. $\mathbf{X} \subseteq \mathbf{Y}$ *implies* $\mathcal{C}$*-hull*$(\mathbf{X}) \subseteq \mathcal{C}$*-hull*$(\mathbf{Y})$; *and*

5. $\mathcal{C}$*-hull*$(\mathcal{C}$*-hull*$(\mathbf{X})) = \mathcal{C}$*-hull*$(\mathbf{X})$.

It can be shown that any operator φ that satisfies properties (2), (4) and (5) induces a convexity space in the following way: For a given set $\mathcal{X}$ let $\mathcal{C} = \{\mathbf{C} \subseteq \mathcal{X} \mid \varphi(\mathbf{C}) = \mathbf{C})\}$, then $(\mathcal{X}, \mathcal{C})$ is a convexity space.

In most notions of convexity singleton sets are convex. This doesn't hold in convexity spaces, in general. For this reason, convexity spaces with this property are singled out.

Definition 3.3 *A convexity space* $(\mathcal{X}, \mathcal{C})$ *is called* simple *if for all* $x \in \mathcal{X}$ *we have* $\{x\} \in \mathcal{C}$.

3.2 Visibility in Convexity Spaces

Given two distinct points p and q in the plane, their convex hull is the line segment joining them. Since this is the basis of visibility in polygons, our abstract definition of visibility is analogous.

Definition 3.4 *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space and* $\mathbf{X} \subseteq \mathcal{X}$. *We say that two points* x *and* y *in* $\mathbf{X}$ see each other *if* $\mathcal{C}$*-hull*$(\{x, y\}) \subseteq \mathbf{X}$. *We write* x $sees_{\mathbf{X}}$ y *in this case.*

Observe that $sees_{\mathbf{X}}$ is symmetric, but not necessarily reflexive or transitive. We have reflexivity, for all $\mathbf{X} \subseteq \mathcal{X}$, if and only if $(\mathcal{X}, \mathcal{C})$ is simple.

Once having established a consistent definition of visibility it is easy to generalize the notions of starshaped sets and kernels for convexity spaces which we need in the remainder of this paper.

Definition 3.5 *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space and* $\mathbf{X} \subseteq \mathcal{X}$.

1. *For* $x \in \mathbf{X}$, *we define* $\mathcal{C}$*-star*$(x, \mathbf{X}) = \{y \in \mathbf{X} \mid x \ sees_{\mathbf{X}} \ y\}$.

2. $\mathbf{X}$ *is* star-shaped *if* $\mathbf{X} = \mathcal{C}$*-star*$(x, \mathbf{X})$ *for some* $x \in \mathbf{X}$.

3. $\mathcal{C}$*-kernel*$(\mathbf{X}) = \{x \in \mathbf{X} \mid \mathcal{C}$*-star*$(x, \mathbf{X}) = \mathbf{X}\}$.

Note that, in general, $\mathcal{C}$*-kernel*$(\mathbf{X})$ is not convex. As an example we introduce the following convexity space.

Example Let $\mathcal{X} = I\!N$ and $\mathcal{C} = \{\mathbf{C} \subseteq I\!N \mid \mathbf{C} \text{ is finite}\}$. Now consider $\mathbf{X}$ = set of odd numbers and let $x, y \in \mathbf{X}$. Since $|\{x, y\}| \leq 2 < \infty$, we get $\mathcal{C}$-*hull*$(\{x, y\}) = \{x, y\}$ and this is surely contained in $\mathbf{X}$. Hence, all $x, y \in \mathbf{X}$

can see each other by our above definition of visibility and $\mathcal{C}$-*kernel*$(\mathbf{X}) = \mathbf{X}$. But $\mathbf{X}$ is not finite and, therefore, $\mathbf{X} \notin \mathcal{C}$.

Because $\mathcal{C}$-*hull* is an expansive operator (Statement 2 of Theorem 3.1), we deduce immediately that, for a convex set $\mathbf{C}$ and any two points x and y in $\mathbf{C}$, $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathcal{C}$-*hull*$(\mathbf{C}) = \mathbf{C}$. Thus, for all points x and y in $\mathbf{C}$, we have x *sees*$_{\mathcal{C}}$ y. The converse does not hold in general as the above example illustrates. This kind of unexpected behaviour of convexity spaces leads us to the next definition.

Definition 3.6 *A convexity space* $(\mathcal{X}, \mathcal{C})$ *is said to be* complete *if, for all* $\mathbf{X} \subseteq \mathcal{X}$ *such that, for all* $x, y \in \mathbf{X}$, $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathbf{X}$, *then we have* $\mathbf{X} \in \mathcal{C}$.

Hence, in a complete convexity space, all-seeingness and convexity are equivalent notions. This property makes complete convexity spaces a natural setting for questions concerning visibility. Moreover, most of the convexity spaces that arise in practice are based on the the definition of the convex hull of two points and extend this definition to a complete convexity space. As an example consider the definition of convexity in the plane: A set $\mathbf{X}$ is convex if it contains the line segment between any two points x and y in $\mathbf{X}$; that is, if all x and y in $\mathbf{X}$ can see each other.

Although not all convexity spaces are complete, we can always embed an incomplete convexity space in a complete one, in a natural way, as we now demonstrate.

Definition 3.7 *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space. The* completion *of* $(\mathcal{X}, \mathcal{C})$ *is denoted by* $(\mathcal{X}, \tilde{\mathcal{C}})$ *and is defined by* $\tilde{\mathcal{C}} = \mathcal{C} \cup \{\mathbf{C} \subseteq \mathbf{X} \mid$ *for all* $x, y \in \mathbf{C}$, $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathbf{C}\}$.

Theorem 3.2 *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space. Then, the completion* $(\mathcal{X}, \tilde{\mathcal{C}})$ *of* $(\mathcal{X}, \mathcal{C})$ *is a convexity space and it is complete.*

Proof: 1. We begin by proving the intersection property for $(\mathcal{X}, \tilde{\mathcal{C}})$, since this is sufficient to guarantee that it is a convexity space. Take any $\mathcal{C}' \subseteq \tilde{\mathcal{C}}$ and any two points $x, y \in \bigcap \mathcal{C}'$. If x and y can see each other via $\mathcal{C}$-*hull*$(\{x, y\})$ in $\bigcap \mathcal{C}'$, then we know that $\bigcap \mathcal{C}'$ is $\tilde{\mathcal{C}}$-convex, since x and y are arbitrary. But, for all $\mathbf{C} \in \mathcal{C}'$, we have $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathbf{C}$, so $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \bigcap \mathcal{C}'$. Hence, $\bigcap \mathcal{C}' \in \tilde{\mathcal{C}}$.

2. We next show that $(\mathcal{X}, \tilde{\mathcal{C}})$ is complete, that is, if $\tilde{\mathcal{C}}$-*hull*$(\{x, y\}) \subseteq \mathbf{C}$ for all $x, y \in \mathbf{C}$, then $\mathbf{C} \in \tilde{\mathcal{C}}$. To this end we prove that, for all $x, y \in \mathbf{X}$, $\tilde{\mathcal{C}}$-*hull*$(\{x, y\}) = \mathcal{C}$-*hull*$(\{x, y\})$ and the claim follows by the definition of $\tilde{\mathcal{C}}$.

From the definition of $\tilde{\mathcal{C}}$ it follows immediately that $\tilde{\mathcal{C}}$-*hull*$(\{x, y\}) \subseteq \mathcal{C}$-*hull*$(\{x, y\})$. To see the reverse inclusion observe that a set is $\tilde{\mathcal{C}}$-convex only if it contains $\mathcal{C}$-*hull*$(\{u,v\})$, for all $u,v \in \mathbf{X}$. But, this implies that $\bar{\mathcal{C}}$-*hull*$(\{x,y\})$ contains $\mathcal{C}$-*hull*$(\{x,y\})$. □

As a last visibility-related concept we need the notion of a $\mathcal{C}$-*join*.

Definition 3.8 *Let* $(\mathcal{X},\mathcal{C})$ *be a convexity space,* $\mathbf{C} \subseteq \mathcal{X}$*, and* $x \in \mathcal{X}$*; we define the* $\mathcal{C}$-*join*() *of* x *and* $\mathbf{C}$ *by*

$$\mathcal{C}\text{-}join(x,\mathbf{C}) = \bigcup_{c\in\mathbf{C}} \mathcal{C}\text{-}hull(\{x,c\}).$$

The $\mathcal{C}$-*join* of a convex set $\mathbf{C}$ and a point x consists, intuitively speaking, of all the line segments between x and points c in $\mathbf{C}$. It is easy to show that the $\mathcal{C}$-*join* in the plane is *always* convex if we consider normal convexity. This, however, is not true for arbitrary convexity spaces. (We will look at some examples in the next section.) This leads to:

Definition 3.9 *Let* $(\mathcal{X},\mathcal{C})$ *be a convexity space.* $(\mathcal{X},\mathcal{C})$ *is said to satisfy the* $\mathcal{C}$-join condition *if, for all* $x \in \mathcal{X}$ *and* $\mathbf{C} \in \mathcal{C} \setminus \{\emptyset\}$*, we have*

$$\mathcal{C}\text{-}join(x,\mathbf{C}) \textit{ is convex.}$$

Note that the inclusion $\mathcal{C}$-*join*$(x,\mathbf{C}) \subseteq \mathcal{C}$-*hull*$(\{x\} \cup \mathbf{C})$ holds for any convexity space while the reverse inclusion only holds for convexity spaces that satisfy the $\mathcal{C}$-*join* condition. Hence, another way to state the $\mathcal{C}$-*join* condition is to require that $\mathcal{C}$-*join*$(x,\mathbf{C}) = \mathcal{C}$-*hull*$(\{x\} \cup \mathbf{C})$, for all $\mathbf{C} \in \mathcal{C}\setminus\{\emptyset\}$. Now the reason to exclude the empty set from the definition becomes apparent, $\mathcal{C}$-*join*$(x,\emptyset) = \bigcup_{c\in\emptyset} \mathcal{C}$-*hull*$(x,c) = \emptyset \neq \mathcal{C}$-*hull*$(\{x\} \cup \emptyset)$.

3.3 Skulls and Aligned Spaces

The definition of skulls in a convexity space is exactly analogous to the definition in the plane.

Definition 3.10 *Let* $(\mathcal{X},\mathcal{C})$ *be a convexity space and* $\mathbf{X} \subseteq \mathcal{X}$*; then*

1. $\mathbf{S} \subseteq \mathbf{X}$ *is a* $\mathcal{C}$-skull *of* $\mathbf{X}$ *if* $\mathbf{S} \in \mathcal{C}$ *and there is no* $\mathbf{S}' \in \mathcal{C}$ *such that* $\mathbf{S} \subset \mathbf{S}' \subseteq \mathbf{X}$.

2. $\mathcal{C}$-*skulls*$(\mathbf{X}) = \{\mathbf{S} \mid \mathbf{S}$ *is a* $\mathcal{C}$-*skull of* $\mathbf{X}\}$.

As we have seen before, convexity spaces can behave in a rather unexpected manner. This also applies to the definition of skulls. Given a set $\mathbf{X} \subseteq \mathcal{X}$ in a convexity space $(\mathcal{X},\mathcal{C})$ and a convex subset $\mathbf{C}$ of $\mathbf{X}$, we cannot always assume that there is a $\mathcal{C}$-*skull* $\mathbf{S}$ of $\mathbf{X}$ that contains $\mathbf{C}$ as the next example illustrates.

Example Let $\mathcal{X}$ be the closed set of reals $[0,1]$ and let $\mathcal{C}$ be all closed intervals in $\mathcal{X}$. Let $\mathbf{X}$ be the open interval $(0,1)$. As it can be seen easily, $\mathbf{X}$ contains convex sets, $[1/4, 3/4]$ for instance, but there is no maximal convex set in $\mathbf{X}$. For, consider any convex set $[\alpha, \beta] \subseteq \mathbf{X}$; it is contained in $\mathbf{X}$ and, therefore, $\alpha > 0$. Now $[\alpha, \beta] \subset [\alpha/2, \beta] \subset \mathbf{X}$ and $[\alpha, \beta]$ cannot be maximal.

The above example suggests that we require the union of nested chains of convex sets to be convex sets once more.

Definition 3.11 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space. We call $(\mathcal{X}, \mathcal{C})$ an* aligned space *if, for every nested chain $\mathcal{N} \subseteq \mathcal{C}$, the union of $\mathcal{N}$ is also convex; that is, $\bigcup \mathcal{N} \in \mathcal{C}$.*

Aligned spaces are well studied objects in the literature [7, 8, 19] but usually for quite different reasons than those stated here. The following lemma shows that the existence.of skulls is, indeed, guaranteed by forcing nested unions to be convex. For this reason, whenever we require that such skulls exist, we restrict ourselves to aligned spaces.

Lemma 3.3 *Let $(\mathcal{X}, \mathcal{C})$ be an aligned space. Then, for all $\mathbf{X} \subseteq \mathcal{X}$ and for all $\mathbf{C} \in \mathcal{C}$ with $\mathbf{C} \subseteq \mathbf{X}$, there is a $\mathcal{C}$-skull $\mathbf{S}$ of $\mathbf{X}$ that contains $\mathbf{C}$.*

Proof: To see this, take any $\mathbf{X} \subseteq \mathcal{X}$ and $\mathbf{C} \in \mathcal{C} \setminus \{\emptyset\}$ with $\mathbf{C} \subseteq \mathbf{X}$.

Let $\mathcal{S}' = \{\mathbf{C}' \in \mathcal{C} \mid \mathbf{C} \subseteq \mathbf{C}' \subseteq \mathbf{X}\}$. Now $\mathcal{S}'$ is a partially-ordered set and any chain $\mathcal{C}' \subseteq \mathcal{S}'$ has an upper bound $\bigcup \mathcal{C}' \in \mathcal{C}$ with $\mathbf{C} \subseteq \bigcup \mathcal{C}' \subseteq \mathbf{X}$, since $(\mathcal{X}, \mathcal{C})$ is an aligned space. By Zorn's lemma, there is a maximal element $\mathbf{S}$ in $\mathcal{S}'$, so $\mathbf{S}$ is a $\mathcal{C}$-*skull* of $\mathbf{X}$ containing $\mathbf{C}$. □

Another counter-intuitive observation is that $\mathcal{C}$-*skulls*$(\mathbf{X})$ need not necessarily cover $\mathbf{X}$, in the sense that $\bigcup \mathcal{C}$-*skulls*$(\mathbf{X})$ is not necessarily $\mathbf{X}$ itself, but only a subset. Whenever an aligned space is simple, however, we obtain a cover since every point is convex subset and, therefore, there exists a maximal convex subset containing it.

We have introduced the notions of complete and aligned spaces above. Fortunately, there is a straightforward relationship between them.

Lemma 3.4 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space. If $(\mathcal{X}, \mathcal{C})$ is complete, then $(\mathcal{X}, \mathcal{C})$ is an aligned space.*

Proof: Let $\mathcal{N}$ be a nested chain of convex sets and let $\mathbf{C} = \bigcup \mathcal{N}$. Further, let x and y be arbitrary points in $\mathbf{C}$. It suffices to show that x and y see each other in $\mathbf{C}$ to prove that $\mathbf{C}$ is convex because $(\mathcal{X}, \mathcal{C})$ is complete. For x and y there are $\mathbf{C}_x$ and $\mathbf{C}_y \in \mathcal{N}$ such that $x \in \mathbf{C}_x$ and $y \in \mathbf{C}_y$. Since $\mathcal{N}$ is a nested chain we can assume that $\mathbf{C}_x \subseteq \mathbf{C}_y$. So $\mathbf{C}_x \cup \mathbf{C}_y = \mathbf{C}_y$ is convex, and $\mathcal{C}\text{-}hull(\{x,y\}) \subseteq \mathbf{C}_y \subseteq \mathbf{C}$. □

Lemma 3.4 ensures that whenever we are dealing with complete convexity spaces skulls automatically exist. There is also a (weak) converse that requires the $\mathcal{C}$-*join* condition (see Theorem 2 in Kay and Womble [9]).

Lemma 3.5 *Let $(\mathcal{X},\mathcal{C})$ be an aligned space. If $(\mathcal{X},\mathcal{C})$ satisfies the $\mathcal{C}$-join condition, then $(\mathcal{X},\mathcal{C})$ is complete.*

Proof: Consider a set $\mathbf{X} \subseteq \mathcal{X}$ such that $\mathbf{X} \neq \emptyset$ and $\mathcal{C}\text{-}hull(\{x,y\}) \subseteq \mathbf{X}$, for all $x, y \in \mathbf{X}$. We prove that $\mathbf{X}$ is convex.

Because $\mathbf{X} \neq \emptyset$, $\mathbf{X}$ contains at least one point x, say, and because $\mathcal{C}\text{-}hull(\{x,x\}) \subseteq \mathbf{X}$, by assumption, there is a $\mathcal{C}$-*skull* $\mathbf{S}_x$ with $x \in \mathbf{S}_x$ and $\mathbf{S}_x \subseteq \mathbf{X}$. Now, take $y \in \mathbf{X}$; then, we have $\mathcal{C}\text{-}hull(\{s,y\}) \subseteq \mathbf{X}$, for all $s \in \mathbf{S}_x$, since y can see all the points in $\mathbf{X}$. In other words, $\mathcal{C}\text{-}join(y, \mathbf{S}_x) \subseteq \mathbf{X}$.

But, $\mathcal{C}\text{-}join(y, \mathbf{S}_x)$ is convex and $\mathcal{C}\text{-}join(y, \mathbf{S}_x) \subseteq \mathbf{S}_x$, by the definition of a $\mathcal{C}$-*skull*. Thus, $y \in \mathbf{S}_x$ and $\mathbf{S}_x = \mathbf{X}$; hence $\mathbf{X}$ is convex. □

3.4 The Kernel Theorem

After this preparation we can now turn to stating and proving the Kernel Theorem. It gives a complete characterization of those convexity spaces for which the kernel of a set $\mathbf{X}$ is given by the intersection of all skulls in $\mathbf{X}$; we make crucial use of the concepts introduced so far.

Theorem 3.6 (The Kernel Theorem) *Let $(\mathcal{X},\mathcal{C})$ be a convexity space. Then, we have, for all $\mathbf{X} \subseteq \mathcal{X}$,*

$$\mathcal{C}\text{-}kernel(\mathbf{X}) = \bigcap \mathcal{C}\text{-}skulls(\mathbf{X})$$

if and only if the following three conditions hold:

i. $(\mathcal{X},\mathcal{C})$ is an aligned space.

ii. For all $x \in \mathcal{X}$, for all $\mathbf{C} \in \mathcal{C}$, $\mathcal{C}\text{-}join(x, \mathbf{C})$ is convex.

iii. For all $x,\ y \in \mathcal{X}$, $\mathcal{C}\text{-}hull(\{y\}) \subseteq \mathcal{C}\text{-}hull(\{x\}) \cup \{y\}$.

Proof: Let $K = \mathcal{C}\text{-}kernel(\mathbf{X})$ and $I = \bigcap \mathcal{C}\text{-}skulls(\mathbf{X})$.

if. We split the proof into two parts.

$K \subseteq I$. If $K = \emptyset$, this holds vacuously, so assume that $K \neq \emptyset$. Consider $p \in K$; we prove that $p \in I$. Let $\mathbf{S}$ be a skull in $\mathcal{C}\text{-}skulls(\mathbf{X})$ and s a point in $\mathbf{S}$; since $p \in \mathcal{C}\text{-}kernel(\mathbf{X})$, we have p $sees_{\mathbf{X}}$ s. Thus, $\mathcal{C}\text{-}hull(\{p,s\}) \subseteq \mathbf{X}$ and so $\mathcal{C}\text{-}join(p, \mathbf{S}) = \bigcup_{s \in \mathbf{S}} \mathcal{C}\text{-}hull(\{p,s\}) \subseteq \mathbf{X}$;

furthermore, $\mathcal{C}$-*join*$(p, \mathbf{S})$ is convex, by assumption. But $\mathbf{S}$ is a maximal inscribed convex set of $\mathbf{X}$; therefore, $\mathcal{C}$-*join*$(p, \mathbf{S}) = \mathbf{S}$, $p \in \mathbf{S}$, and hence $p \in I$.

$I \subseteq K$. Again assume that $I \neq \emptyset$ and consider $p \in I$ and an arbitrary point $x \in \mathbf{X}$. We have to show that p *sees*$_{\mathbf{X}}$ x. Since $\mathcal{C}$-*hull*$(\{p\}) \subseteq I \subseteq \mathbf{X}$, we have $\mathcal{C}$-*hull*$(\{x\}) \subseteq \{x\} \cup \mathcal{C}$-*hull*$(\{p\}) \subseteq \mathbf{X}$ by Condition (iii). Also, since $\mathcal{C}$-*hull*$(\{x\}) \subseteq \mathbf{X}$, we know that there is an $\mathbf{S}_x \in \mathcal{C}$-*skulls*$(\mathbf{X})$ with $\mathcal{C}$-*hull*$(\{x\}) \subseteq \mathbf{S}_x$. Now $p \in I \subseteq \mathbf{S}_x$ and, thus, $\mathcal{C}$-*hull*$(\{p, x\}) \subseteq \mathbf{S}_x \subseteq \mathbf{X}$. Therefore, p *sees*$_{\mathbf{X}}$ x and $p \in K$.

only if. We prove the necessity of the three conditions in the order in which they appear in the theorem.

Condition i: We have to show that $(\mathcal{X}, \mathcal{C})$ is an aligned space. Let $\mathcal{N} \subseteq \mathcal{C}$ be a nested chain of convex sets and $\mathbf{Q}$ denote $\bigcup \mathcal{N}$. Consider $x, y \in \mathbf{Q}$. As in the proof of Lemma 3.4, we obtain $\mathbf{C} \in \mathcal{N}$ with $x, y \in \mathbf{C}$; thus $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathbf{C} \subseteq \mathbf{Q}$. Hence $\mathcal{C}$-*kernel*$(\mathbf{Q}) = \mathbf{Q}$ and since $\mathcal{C}$-*kernel*$(\mathbf{Q})$ is convex (it is the intersection of convex sets), we have $\mathbf{Q} \in \mathcal{C}$.

Condition ii: We now prove that the $\mathcal{C}$-*join* is convex. Consider an arbitrary point $x \in \mathbf{X}$ and an arbitrary $\mathbf{C} \in \mathcal{C} \setminus \emptyset$; we show that $\mathbf{X}$ denoting $\mathcal{C}$-*join*$(x, \mathbf{C})$ is convex. For all $y \in \mathbf{X}$, we have $y \in \mathcal{C}$-*hull*$(\{x, c\})$, for some $c \in \mathbf{C}$. But, this implies that $\mathcal{C}$-*hull*$(\{x, y\}) \subseteq \mathcal{C}$-*hull*$(\{x, c\}) \subseteq \mathbf{X}$ and x *sees*$_{\mathbf{X}}$ y. In other words, $x \in \mathcal{C}$-*kernel*$(\mathbf{X})$ and, hence, $x \in \bigcap \mathcal{C}$-*skulls*$(\mathbf{X})$. Therefore, all skulls of $\mathbf{X}$ contain x. Because $(\mathcal{X}, \mathcal{C})$ is an aligned space, there is a skull $\mathbf{S}$ of $\mathbf{X}$ that contains $\mathbf{C}$. $\mathbf{S}$ also contains x, so $\mathbf{X} = \mathcal{C}$-*join*$(x, \mathbf{C}) \subseteq \mathcal{C}$-*hull*$(\{x\} \cup \mathbf{C}) \subseteq \mathbf{S} \subseteq \mathbf{X}$ and $\mathbf{X}$ is convex.

Condition iii: Consider two arbitrary points x and y in $\mathbf{X}$ and let $\mathbf{X} = \mathcal{C}$-*hull*$(\{x\}) \cup \{y\}$. We prove that $\mathcal{C}$-*hull*$(\{y\}) \subseteq \mathbf{X}$ by contradiction.

Assume that $\mathcal{C}$-*hull*$(\{y\}) \not\subseteq \mathbf{X}$. Then $\mathcal{C}$-*hull*$(\{y, z\}) \not\subseteq \mathbf{X}$, for all $z \in \mathbf{X}$, and, hence, y is not visible from any point in $\mathbf{X}$ (including itself). Thus, $\mathcal{C}$-*kernel*$(\mathbf{X}) = \emptyset$. But, by similar reasoning, for every $\mathbf{S} \in \mathcal{C}$-*skulls*$(\mathbf{X})$, we have $y \notin \mathbf{S}$. This implies that $\mathbf{S} \subseteq \mathcal{C}$-*hull*$(\{x\})$ and $\mathcal{C}$-*hull*$(\{x\})$ is the only skull of $\mathbf{X}$. Thus, $\bigcap \mathcal{C}$-*skulls*$(\mathbf{X}) = \mathcal{C}$-*hull*$(\{x\}) \neq \emptyset$ and we have a contradiction; so $\mathcal{C}$-*hull*$(\{y\}) \subseteq \mathbf{X}$.

□

As an immediate consequence we get the following corollary.

Corollary 3.7 *Let $(\mathcal{X}, \mathcal{C})$ be a convexity space that satisfies the conditions of the Kernel Theorem; then, for all $\mathbf{X} \subseteq \mathcal{X}$, $\mathcal{C}$-kernel$(\mathbf{X})$ is convex.*

To give a more elaborate example we will illustrate the application of the Kernel Theorem in the case of *partition convexity*.

Example Let $\mathcal{X}$ be some nonempty set and $\Pi = \{\mathbf{X}_i \mid i \in I\}$ be a partition of $\mathbf{X}$ ($\mathbf{X} = \bigcup \Pi$ and $\mathbf{X}_i \cap \mathbf{X}_j \neq \emptyset$ implies $i = j$). Then, $(\mathcal{X}, \mathcal{C})$ is a convexity space, where $\mathcal{C} = \{\emptyset, \mathbf{X}\} \cup \Pi$, because

$$\bigcap \mathcal{C}' = \begin{cases} \emptyset & \text{if } \emptyset \in \mathcal{C}' \text{ or } \mathbf{X}_i \text{ and } \mathbf{X}_j \in \mathcal{C}',\ i \neq j. \\ \mathbf{X}_i & \text{if } \mathcal{C}' \text{ is either } \{\mathbf{X}_i\} \text{ or } \{\mathbf{X}_i, \mathcal{X}\}. \\ \mathcal{X} & \text{if } \mathcal{C}' = \{\mathcal{X}\}. \end{cases}$$

Consider $\mathbf{X}$ such that $\emptyset \subset \mathbf{X} \subset \mathbf{X}_i$, for some $\mathbf{X}_i \in \Pi$; then there are points x and y in $\mathbf{X}$ that do not see each other, since $\mathcal{C}\text{-}hull(\{x, y\}) = \mathbf{X}_i$. Conversely, all pairs of points in $\emptyset$, $\mathcal{X}$, and $\mathbf{X} \in \Pi$ see each other, so $(\mathcal{X}, \mathcal{C})$ is complete and, hence, it is a complete aligned space.

If there is an $\mathbf{X}_i$ that consists of at least two points, say x and y, then consider $\mathbf{Q} = \mathbf{X}_j \cup \{x\}$, for $\mathbf{X}_j \neq \mathbf{X}_i$. Immediately, $\mathcal{C}\text{-}kernel(\mathbf{Q}) = \emptyset$ and $\mathcal{C}\text{-}skulls(\mathbf{Q}) = \{\mathbf{X}_j\}$; the Kernel Theorem does not hold.

Otherwise, each $\mathbf{X}_i$ is a singleton set and $(\mathcal{X}, \mathcal{C})$ is simple. Consider three points p, q, and $r \in \mathcal{X}$, If $p = q = r$, then $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q, r\})) = \{p\}$, otherwise $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q, r\})) = \mathcal{X}$. In both cases, the $\mathcal{C}\text{-}join$ is convex; thus, the Kernel Theorem holds in $(\mathcal{X}, \mathcal{C})$.

The Kernel Theorem gives rise to two natural follow up questions.

1. Can we guarantee the convexity of kernels under weaker conditions than those necessary for the Kernel Theorem?
2. Is there a relationship between kernels and skulls in general convexity spaces?

Let us consider the convexity of kernels first. Algorithms that compute kernels of polygons often make crucial use of their convexity. Therefore, it is desirable to have a characterization of those convexity spaces for which this is true. Unfortunately, there is a large gap between the sufficient and necessary conditions for the convexity of kernels as stated in the following two theorems.

Theorem 3.8 *Let $(\mathcal{X}, \mathcal{C})$ be a complete convexity space. If $(\mathcal{X}, \mathcal{C})$ satisfies the $\mathcal{C}$-join condition, then $\mathcal{C}\text{-}kernel(\mathbf{X})$ is convex, for all $\mathbf{X} \subseteq \mathcal{X}$.*

Proof: Without loss of generality we can assume that $\mathcal{C}\text{-}kernel(\mathbf{X}) \neq \emptyset$, so let $x, y \in \mathcal{C}\text{-}kernel(\mathbf{X})$. Since x and y can see each other, we have $\mathcal{C}\text{-}hull(\{x, y\}) \subseteq \mathbf{X}$. Take any $z \in \mathbf{X}$. Then, $\mathcal{C}\text{-}join(x, \mathcal{C}\text{-}hull(\{y, z\})) \subseteq \mathbf{X}$, since $\mathcal{C}\text{-}hull(\{y, z\}) \subseteq \mathbf{X}$ and x can see all points in $\mathcal{C}\text{-}hull(\{y, z\})$. Since,

$\mathcal{C}$-*join*$(x, \mathcal{C}$-*hull*$(\{y,z\})) = \mathcal{C}$-*hull*$(\{x,y,z\}) \supseteq \mathcal{C}$-*join*$(z, \mathcal{C}$-*hull*$(\{x,y\}))$, we know that, for any $w \in \mathcal{C}$-*hull*$(\{x,y\})$, we have $\mathcal{C}$-*hull*$(\{w,z\}) \subseteq \mathbf{X}$. But this implies that $w \in \mathcal{C}$-*kernel*$(\mathbf{X})$, since z was arbitrary. □

This immediately yields an analogous result for aligned spaces in view of Lemma 3.5.

Corollary 3.9 *Let* $(\mathcal{X}, \mathcal{C})$ *be an aligned space that satisfies the* $\mathcal{C}$-*join condition; then,* $\mathcal{C}$-*kernel*$(\mathbf{X})$ *is convex, for all* $\mathbf{X} \subseteq \mathcal{X}$.

The assumption of a complete space cannot be relaxed as the next theorem shows.

Theorem 3.10 *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space such that* $\mathcal{C}$-*kernel*$(\mathbf{X})$ *is convex, for all* $\mathbf{X} \subseteq \mathcal{X}$*; then,* $(\mathcal{X}, \mathcal{C})$ *is complete.*

Proof: Let $\mathbf{C} \subseteq \mathcal{X}$ and, for all $p, q \in \mathbf{C}$, let $\mathcal{C}$-*hull*$(\{p,q\}) \subseteq \mathbf{C}$. We have to show that $\mathbf{C}$ is convex; that is, $\mathbf{C} \in \mathcal{C}$. Since every point $p \in \mathbf{C}$ can see all the other points q in $\mathbf{C}$, we obtain $\mathcal{C}$-*kernel*$(\mathbf{C}) = \mathbf{C}$. But all the kernels in $(\mathcal{X}, \mathcal{C})$ are convex and, hence, $\mathbf{C} \in \mathcal{C}$. □

We turn now to the second question: What is the relationship between $\mathcal{C}$-*skulls*$(\mathbf{X})$ and $\mathcal{C}$-*kernel*$(\mathbf{X})$, for $\mathbf{X} \subseteq \mathcal{X}$, in an arbitrary convexity space? It is obvious that we have to require the existence of skulls in order to be able to make a statement that relates skulls and kernels. Taking this condition into account the next theorem gives the most general connection between skulls and kernels possible.

But before we can state and prove a generalized version of the Kernel Theorem, we need the notion of a skull cover.

Definition 3.12 *Given a convexity space* $(\mathcal{X}, \mathcal{C})$ *and a set* $\mathbf{X} \subseteq \mathcal{X}$*, we say that a subset* $\mathcal{S}$ *of* $\mathcal{C}$-*skulls*$(\mathbf{X})$ *is a* skull cover *of* $\mathbf{X}$ *if* $\bigcup_{\mathbf{S} \in \mathcal{S}} \mathbf{S} = \mathbf{X}$. *We define the collection of skull covers of* $\mathbf{X}$ *as the set* $\{\mathcal{S} \mid \mathcal{S} \subseteq \mathcal{C}$-*skulls*$(\mathbf{X})$ *and* $\mathcal{S}$ *is a skull cover of* $\mathbf{X}\}$ *and we denote it by* $\Sigma(\mathbf{X})$.

The theorem can now be stated as follows.

Theorem 3.11 (The Cover Kernel Theorem) *Let* $(\mathcal{X}, \mathcal{C})$ *be a convexity space. Then, we have, for all* $\mathbf{X} \subseteq \mathcal{X}$,

$$\mathcal{C}\text{-}kernel(\mathbf{X}) = \bigcup_{\mathcal{S} \in \Sigma(\mathbf{X})} \left(\bigcap \mathcal{S}\right)$$

if and only if, for all convex subsets $\mathbf{C}$ *of* $\mathbf{X}$*, there is an* $\mathbf{S} \in \mathcal{C}$-*skulls*$(\mathbf{X})$ *with* $\mathbf{C} \subseteq \mathbf{S}$.

Proof: **if.** Let $K = \mathcal{C}$-*kernel*$(\mathbf{X})$ and $I = \bigcup_{\mathcal{S} \in \Sigma(\mathbf{X})} (\bigcap \mathcal{S})$. We first prove that $K \subseteq I$ and then prove that $I \subset K$.

$K \subseteq I$. Let $x \in K$ and define $\mathcal{S}_x = \{\mathbf{S} \in \mathcal{C}\text{-}skulls(\mathbf{X}) \mid x \in \mathbf{S}\}$. Since x can see all points in $\mathbf{X}$, we have that, for all $y \in \mathbf{X}$, $\mathcal{C}\text{-}hull(\{x,y\}) \subseteq \mathbf{X}$; that is, there exists an $\mathbf{S} \in \mathcal{S}_x$ such that $\{x,y\} \subseteq \mathcal{C}\text{-}hull(\{x,y\}) \subseteq \mathbf{S} \in \mathcal{S}_x$ since the existence of skulls for any convex subset $\mathbf{C}$ of $\mathbf{X}$ is guaranteed. Hence, $\mathcal{S}_x$ is a cover of $\mathbf{X}$ with $x \in \bigcap \mathcal{S}_x \subseteq I$.

$I \subseteq K$. Let $x \in I$; then, there is an $\mathcal{S} \in \Sigma(\mathbf{X})$ such that $x \in \bigcap \mathcal{S}$. Let y be an arbitrary point in $\mathbf{X}$; then, since $\mathcal{S}$ is a cover of $\mathbf{X}$, there is a skull $\mathbf{S} \in \mathcal{S}$ such that $y \in \mathbf{S}$. Because $x \in \bigcap \mathcal{S}$, we know that $x \in \mathbf{S}$, so $\{x,y\} \subseteq \mathbf{S}$ and, by expansiveness of the $\mathcal{C}\text{-}hull$ operator, $\mathcal{C}\text{-}hull(\{x,y\}) \subseteq \mathbf{S} \subseteq \mathbf{X}$; that is, x $sees_{\mathbf{X}}$ y and $x \in \mathcal{C}\text{-}kernel(\mathbf{X})$.

only if. Let $\mathbf{X} \subseteq \mathcal{X}$ and $\mathbf{C}$ a convex subset of $\mathbf{X}$. We have to show that there is a skull of $\mathbf{X}$ that contains $\mathbf{C}$. Consider the family $\mathcal{S}_{\mathbf{C}} = \{\mathbf{C}' \in \mathcal{C} \mid \mathbf{C} \subseteq \mathbf{C}' \subseteq \mathbf{X}\}$ and let $\mathbf{Q}$ be the set $\bigcup \mathcal{S}_{\mathbf{C}}$. Clearly, $\mathbf{C} \subseteq \mathcal{C}\text{-}kernel(\mathbf{Q})$ and since $\mathcal{C}\text{-}kernel(\mathbf{Q}) = \bigcup_{\mathcal{S} \in \Sigma(\mathbf{Q})}(\bigcap \mathcal{S})$, there has to be at least one skull $\mathbf{S}$ contained in $\mathbf{Q}$ with $\mathbf{C} \subseteq \mathbf{S}$. Since all convex sets that contain $\mathbf{C}$ of $\mathbf{X}$ are also contained in $\mathbf{Q}$ we have $\mathbf{S} \in \mathcal{C}\text{-}skulls(\mathbf{X})$. □

Clearly, we can restrict ourselves to minimal skull covers in the above theorem. This gives us a slightly stronger version of the first Kernel Theorem:

$$\begin{aligned}
\mathcal{C}\text{-}kernel(\mathbf{X}) &= \bigcap \mathcal{C}\text{-}skulls(\mathbf{X}) && \text{by the Kernel Theorem} \\
&\subseteq \bigcap \mathcal{S} && \text{with } \mathcal{S} \in \Sigma(\mathbf{X}),\ \mathcal{S} \text{ a minimal cover;} \\
&\subseteq \bigcup_{\mathcal{S} \in \Sigma(\mathbf{X})}(\bigcap \mathcal{S}) && \\
&= \mathcal{C}\text{-}kernel(\mathbf{X}) &&
\end{aligned}$$

and, hence, $\mathcal{C}\text{-}kernel(\mathbf{X}) = \bigcap \mathcal{S}$, for *any* minimal cover $\mathcal{S}$ of $\mathbf{X}$.

4 Examples

In this section we provide four examples in order to illustrate some of the properties of convexity spaces that we have considered; three of these are taken from the literature of computational geometry. Since we want to establish that the examples are not only aligned spaces, but also that the $\mathcal{C}\text{-}join$ condition holds, we begin by proving a general result for complete convexity spaces.

Theorem 4.1 *Let $(\mathcal{X}, \mathcal{C})$ be a complete convexity space. Then, the following two conditions are equivalent.*

1. *For all points p, q, and r in $\mathcal{X}$, $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q,r\}))$ is convex.*
2. *For all points p in $\mathcal{X}$ and for all $\mathbf{C}$ in $\mathcal{C}$, $\mathcal{C}\text{-}join(p, \mathbf{C})$ is convex.*

Proof: $1 \Rightarrow 2$. Consider some point p in $\mathcal{X}$ and some $\mathbf{C}$ in $\mathcal{C}$, and let $\mathbf{Q} = \mathcal{C}\text{-}join(p, \mathbf{C})$. For every pair x and y of points in $\mathbf{Q}$, we must show that $\mathcal{C}\text{-}hull(\{x, y\}) \subseteq \mathbf{Q}$. Because $(\mathcal{X}, \mathcal{C})$ is complete, this implies that $\mathbf{Q}$ is convex.

Now there are c_1 and c_2 in $\mathbf{C}$ with $q_i \in \mathcal{C}\text{-}hull(\{p, c_i\})$, $i = 1, 2$, by definition. Because c_1 and c_2 are in $\mathbf{C}$, $\mathcal{C}\text{-}hull(\{c_1, c_2\}) \subseteq \mathbf{C}$; therefore, $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{c_1, c_2\})) \supseteq \mathcal{C}\text{-}hull(\{p, c_1\}) \cup \mathcal{C}\text{-}hull(\{p, c_2\}) \supseteq \{q_1, q_2\}$. But, $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{c_1, c_2\})) \in \mathcal{C}$, by assumption; hence, $\mathcal{C}\text{-}hull(\{q_1, q_2\}) \subseteq \mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{c_1, c_2\})) \subseteq \mathcal{C}\text{-}join(p, \mathbf{C}) = \mathbf{Q}$.

$2 \Rightarrow 1$. Trivial. □

4.1 Real Vector Spaces

As we mentioned earlier, visibility has been studied mostly in the framework of real vector spaces. In the following we will show that the structure of the family $\mathcal{C}$ of convex sets in a real vector space $\mathcal{V}$ is very well behaved. In fact, $(\mathcal{V}, \mathcal{C})$ satisfies all the properties that we have considered in the previous section. Hence, for instance, the Kernel Theorem holds.

Of course, this property could have been proven directly, but the question of characterizing all convexity structures that satisfy the Kernel Theorem would have been well beyond the scope of the theory of real vector spaces. Thus real vector spaces turn out to be a special case among simple and complete convexity spaces that satisfy the $\mathcal{C}\text{-}join$ condition.

Theorem 4.2 *Let $\mathcal{V}$ be a real vector space with the normal definition of convexity, that is, $\mathbf{C} \subseteq \mathcal{V}$ is convex if $\alpha p + (1 - \alpha) q \in \mathbf{C}$, for all $p, q \in \mathbf{C}$ and $\alpha \in [0, 1]$. Let $\mathcal{C} = \{\mathbf{C} \subseteq \mathcal{V} \mid \mathbf{C}$ is convex $\}$.*

Then, $(\mathcal{V}, \mathcal{C})$ is a complete, simple convexity space that satisfies the $\mathcal{C}\text{-}join$ condition.

Proof: The completeness of $(\mathcal{V}, \mathcal{C})$ is an immediate consequence of the definition of a convex set since $\mathcal{C}\text{-}hull(\{p, q\})$ obviously equals $\{x \in \mathcal{V} \mid$ there is an $\alpha \in [0, 1]$ with $x = \alpha p + (1 - \alpha) q\}$ and $\mathbf{C} \subseteq \mathcal{V}$ is convex if and only if $\mathcal{C}\text{-}hull(\{p, q\}) \subseteq \mathbf{C}$, for all $p, q \in \mathbf{C}$.

The simplicity of $(\mathcal{V}, \mathcal{C})$ is equally easy to see. Note that simplicity implies Condition (iii) in the Kernel Theorem.

We will use Theorem 4.1 to show that $(\mathcal{V}, \mathcal{C})$ satisfies the $\mathcal{C}\text{-}join$ condition.

Let p, q, and $r \in \mathcal{V}$ and consider $\mathbf{Q} = \mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q, r\}))$. Since $\mathcal{V}$ is a vector space we can assume without loss of generality that $p = \mathbf{0}$, where $\mathbf{0}$ is the origin of $\mathcal{V}$. Take two arbitrary points $s, t \in \mathbf{Q}$. If we can show that s and t can see each other in $\mathbf{Q}$ we can conclude by the completeness of $(\mathcal{V}, \mathcal{C})$ that $\mathbf{Q}$ is convex. The definition of $\mathbf{Q}$ ensures that there are points $q_s, q_t \in \mathcal{C}\text{-}hull(\{q, r\})$ such that $s \in \mathcal{C}\text{-}hull(\{p, q_s\})$ and $t \in \mathcal{C}\text{-}hull(\{p, q_t\})$.

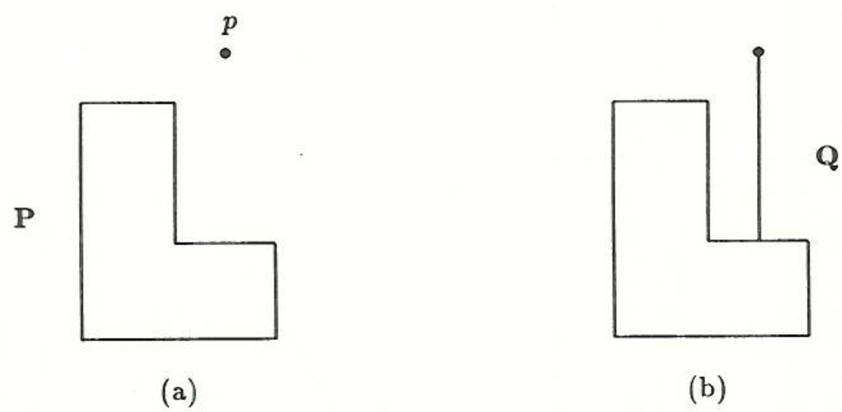

Figure 1: The *join* of an ortho-convex set and a point.

In particular, let $\alpha_s \in [0,1]$ with $s = \alpha_s q_s + (1-\alpha_s)p = \alpha_s q_s$ and let α_t be defined in the analogous way. Since $\mathbf{P} = \mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q_s, q_t\})) \subseteq \mathbf{Q}$ it suffices to show that s sees t in $\mathbf{P}$. Now choose any $u = \beta s + (1-\beta)t \in \mathcal{C}\text{-}hull(\{s,t\})$, $\beta \in [0,1]$. If we define λ to be $\beta\alpha_s/(\beta\alpha_s + (1-\beta)\alpha_t) \in [0,1]$ and $\mu = 1/(\beta\alpha_s + (1-\beta)\alpha_t) \in [0,1]$, it is easy to see that $u = \mu(\lambda q_s + (1-\lambda)q_t) + (1-\mu)p \in \mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q_s, q_t\}))$ and hence, $\mathcal{C}\text{-}hull(\{s,t\}) \subseteq \mathbf{P} \subseteq \mathbf{Q}$. □

From our previous results we now conclude that the Kernel Theorem holds in all real vector spaces.

4.2 Orthogonal Convexity

As an example of a convexity space that fails to satisfy the Kernel Theorem we study the notion of *ortho-convexity*.

We say that a subset $\mathbf{X}$ of $\mathbb{E}^2$ is *ortho-convex* if its intersection with a horizontal or vertical line is empty, a single point or a line segment. Letting $\mathcal{C}$ be the set of all ortho-convex sets, $(\mathbb{E}^2, \mathcal{C})$ is a convexity space; see [15]; The convexity space $(\mathbb{E}^2, \mathcal{C})$ is simple and complete. However, it does not satisfy the $\mathcal{C}$-*join* condition, so there is no Kernel Theorem. We provide a counterexample. Let $\mathbf{P}$ be the polygon in Figure 1(a) and p be the point shown outside $\mathbf{P}$. Then $\mathbf{Q} = \mathcal{C}\text{-}join(p, \mathbf{P})$ is given in Figure 1(b), and this is, clearly, not convex. The Cover Kernel Theorem holds, however, as is seen by examining some of the skulls of $\mathbf{Q}$, $\mathbf{P}$ is itself a skull of $\mathbf{Q}$ as are the subsets of $\mathbf{Q}$ shown in Figure 2. Observe that $\mathbf{P}$ and $\mathbf{S}_2$ as well as $\mathbf{S}_1$ and $\mathbf{S}_2$ cover $\mathbf{Q}$. The union of their respective intersections: $(\mathbf{P} \cap \mathbf{S}_2) \cup (\mathbf{S}_1 \cap \mathbf{S}_2)$

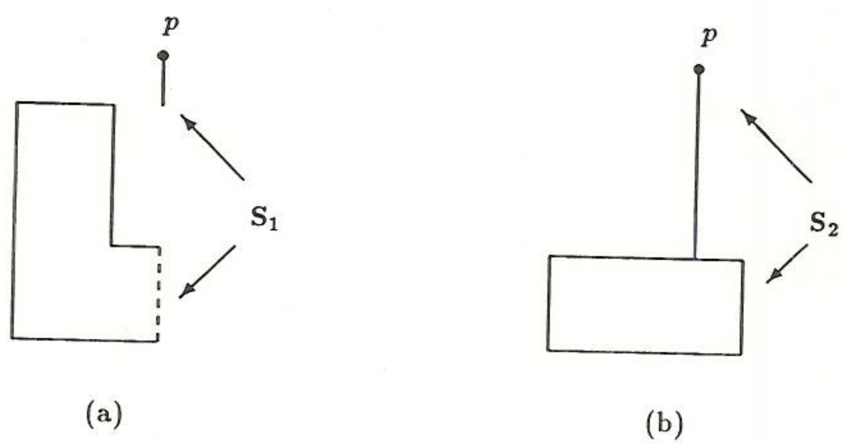

Figure 2: The ortho-skulls of **Q**.

is shown in Figure 3 — it is already the kernel of **Q**.

4.3 Geodesic Convexity

Geodesic convexity provides an example of a convexity space that has highly non-linear characteristics but still satisfies all the properties we considered in Section 3.

Let $\mathcal{X}$ be a polygon in the plane and for any two points x and y in $\mathcal{X}$ let $<x,y>_{\mathcal{X}}$ denote the shortest path lying wholly in $\mathcal{X}$ that connects x and y. (We use the standard L_2 metric to define distance.) A set $\mathbf{C} \subset \mathcal{X}$ is *g-convex* if, for every two points $x, y \in \mathbf{C}$, we have $<x,y>_{\mathcal{X}}$ belongs to $\mathbf{C}$. Figure 4 gives an example. Letting $\mathcal{C}$ be the set of all g-convex sets in $\mathcal{X}$, then $(\mathcal{X}, \mathcal{C})$ is a convexity space.

Clearly, $(\mathcal{X}, \mathcal{C})$ is simple and it is also complete; thus, $(\mathcal{X}, \mathcal{C})$ is a complete convexity space. Furthermore, it satisfies the $\mathcal{C}$*-join* condition. To see this consider any three points p, q and r in $\mathcal{X}$ and let $\mathbf{Q}$ be $\mathcal{C}\text{-}join(p, \mathcal{C}\text{-}hull(\{q,r\}))$. We must prove that $\mathbf{Q}$ is convex; that is, for all x, y in $\mathbf{Q}$, $<x,y>_{\mathcal{X}}$ is in $\mathbf{Q}$.

In the proof of the $\mathcal{C}$*-join* property we make use of basic observations about shortest paths in simple polygons.

i. There is a *unique* shortest path between any two points $p, q \in \mathcal{X}$.

ii. The shortest path between two points is a polygonal chain.

iii. If one endpoint of a shortest path is moved continuously, then the path alters its shape in a continuous way.

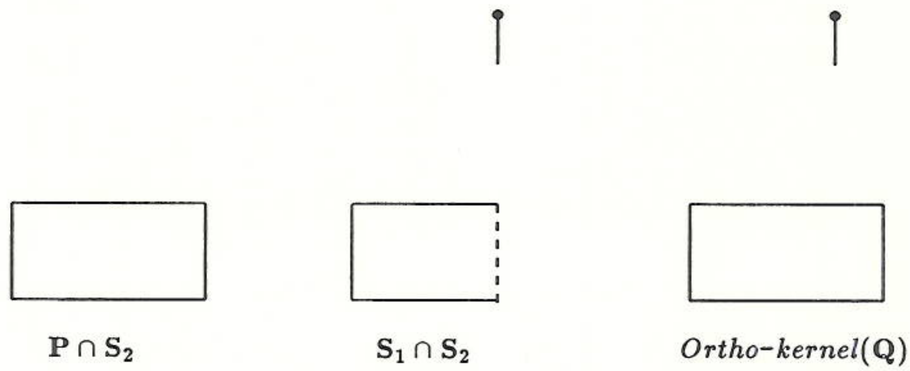

Figure 3: The ortho-kernel of **Q**.

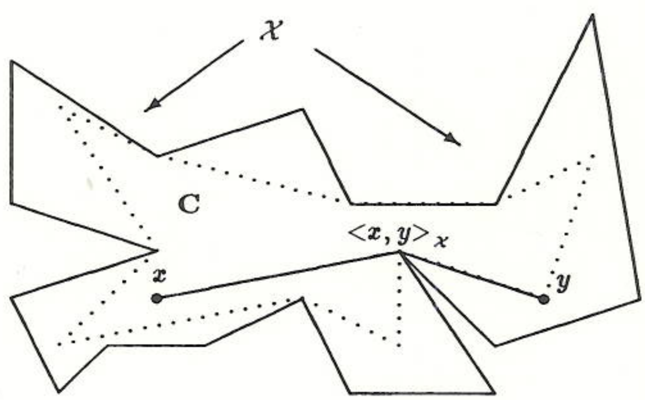

Figure 4: A g-convex set.

iv. If two shortest paths end in the same point, then they do not cross.

v. Subpaths of shortest paths are again shortest paths between their endpoints.

Hence, the boundary of *C-join*$(p, C\text{-}hull(\{q, r\}))$ is a simple polygon whose edges may overlap but do not cross. It consists of $<p,q>_{\mathcal{X}}$, $<q,r>_{\mathcal{X}}$ and $<r,p>_{\mathcal{X}}$. Because of the third observation, it is clear that $\mathbf{Q}$ is exactly the region delimited by the three shortest paths $<p,q>_{\mathcal{X}}$, $<q,r>_{\mathcal{X}}$ and $<r,p>_{\mathcal{X}}$; that is, $\mathbf{Q}$ contains no holes.

Now take the two points $x, y \in \mathbf{Q}$. If $<x,y>_{\mathcal{X}} \not\subseteq \mathbf{Q}$, then $<x,y>_{\mathcal{X}}$ must cross one of the three paths $<p,q>_{\mathcal{X}}, <q,r>_{\mathcal{X}}$, or $<r,p>_{\mathcal{X}}$ which is impossible according to Observations (i) and (v) since we would have two distinct shortest paths between the cross points.

From this we can conclude that the Kernel Theorem holds in any geodesic convexity space.

4.4 *NESW*-Convexity

NESW-convexity is a well studied convexity space that has been of especial interest since it can be shown to satisfy the decomposition theorem of [14]. But, unfortunately, it fails to satisfy the *C-join* condition.

Definition 4.1 *A horizontal ray is an* E-ray *if it lies to the east of some vertical line. Similarly, we can define N-, S-, and W-rays.*

The SW-line at a point p *consists of a N-ray and a E-ray that have their endpoints at point* p. *The point* p *is said to be the* vertex *of the SW-line. The notions of NE-, SE-, and NW-lines are defined similarly.*

Given two points p and q in the plane, they determine a *unique NE-line* if either p is to the left of and not below q or vice versa; p and q define a *NE*-line in these cases.

Definition 4.2 *Let* $\mathbf{P} \subseteq \mathbb{E}^2$ *and* $p, q \in \mathbf{P}$.

i. We say p *and* q NE-see *each other in* $\mathbf{P}$ *if they do not define a NE-line or, if they do, the vertex of the NE-line is in* $\mathbf{P}$.

ii. $\mathbf{P}$ *is* NE-convex *if all* $p, q \in \mathbf{P}$ *NE-see each other.*

We say that a set $\mathbf{P}$ *is* NESW-convex *if it is both NE- and SW-convex.*

Again *NESW*-convex sets form a simple, complete convexity space, but the *C-join* condition does not hold. Consider two points p and q such that p is to the left and below q. Therefore $\{p, q\}$ is *NESW*-convex and *NESW-hull*$(\{p, q\}) = \{p, q\}$. Let r be to the left of p and above q; then, *NESW-join*$(r, NESW\text{-}hull(\{p, q\}))$ consists of three line segments, as shown in Figure 5(a), and this is not *NESW*-convex.

$NESW\text{-}join(r, NESW\text{-}hull(\{p, q\}))$ $NESW\text{-}hull(\{p, q, r\})$

(a) (b)

Figure 5: The *NESW-join* of three points.

5 Conclusions

We have introduced a definition of visibility in abstract convexity spaces and explored some of its consequences. Our main concern has been to characterize the convexity spaces that satisfy the Kernel Theorem. While we have been able to prove a very general relationship between skulls and kernels for arbitrary convexity spaces, the problem of finding necessary and sufficient conditions for convexity spaces, in which all kernels are convex, remains elusive. We feel that the conditions considered in this paper cannot capture the convexity of kernels and new concepts will have to be introduced to answer this question.

A second concern is the computational implications of the discovered results. We showed that if the Kernel Theorem holds, it suffices to consider any minimal cover of skulls to find the kernel. For polygons in the plane this means that we have to consider at most n skulls if n is the number of edges of the polygon; this holds although there may be an infinite number of skulls altogether. There are, of course, much more efficient algorithms to compute the kernel of polygon in the plane. But it seems doubtful if the framework of convexity spaces allows us to develop more efficient algorithms without making crucial use of the topological properities of the Euclidian plane or the n-dimensional Euclidian space.

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