Decisive Venn Diagrams

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The use of Venn diagrams as a pictorial aid in representing Boolean functions is well known. For n variables, 2^n regions are pictured, each of which can assume one of two possible values. Such representations aid in the simplification or identification of Boolean functions, as do Karnaugh maps and other kinds of pictorial aids which are in current use.

Attempts to modify these 2-valued Venn diagrams are generally based upon approaches which either (1) change the number of values which can be assumed in each region, or (2) change the number of regions themselves. Decisive Venn diagrams fall into this latter class, where the number of regions can be changed but only 2 values can be assumed in each region. These diagrams are referred to as decisive, after a similar usage by N. Rescher for truth tables[p. 61,14].

Consider first the usual Venn diagram for the single variable p, as shown in Figure 1. Here $\Box p$ pictures the interior of the disk and $\sim p$ pictures the exterior of the disk. The unary operator $\Box$ is an interior operator which is made explicit here in order to lay groundwork for what follows---in the Boolean case shown in Figure 1 or 2, the operator $\Box$ becomes the identity operator. The unary operator $\sim$ denotes the operation of pseudo-complementation which in the Boolean case becomes the operation of complementation.

The usual Venn diagram for the two variables p,q is shown in Figure 2.

The first Venn diagram in Figure 1 shows a bipartite division into two regions $\Box p$, $\sim p$, and suggests for consideration $2^{2^1}=4$ possible disjunctions of these. The second Venn diagram in Figure 2 shows the consequent division into 4 regions, denoted by the 4 conjunctions $\Box p \& \sim q$, $\Box p \& \Box q$, $\sim p \& \Box q$, $\sim p \& \sim q$.

This suggests for consideration $2^{2^2}=16$ possible disjunctions ofthese 4 conjunctives. Consideration of these full 16 possibilities by Post and Wittgenstein in this century was preceeded by a long history of work in which fewer than 16 possibilities were considered. We are about to explore the corresponding situation for the general case of decisive Venn diagrams, with emphasis on the case when the diagram contains 3 regions.

Consider the decisive Venn diagram for the single variable p with 3 regions, as shown in Figure 3. Here the annule marked $\sim(\sim p \vee \Box p)$ pictures a region which is neither in the interior or exterior of the disk. This allows for a middle term between the terms $\sim p$, $\Box p$ to give a law of included middle[4,6]:

$$\sim p \vee \sim(\sim p \vee \Box p) \vee \Box p.$$

A simplifying case occurs with the 3-valued case, where there is exactly one extra value $p=\underline{e}$ for which $\sim(\sim p \vee \Box p)$ evaluates to $\underline{t}$(true), as there is exactly one value $p=\underline{f}$(false) for which $\sim p$ evaluates to $\underline{t}$, and one value $p=\underline{t}$ for which $\Box p$ evaluates to $\underline{t}$. The presence of additional extra values $\underline{e_1}$, $\underline{e_2}$, $\cdots$ does not alter this interpretation, as the above law of included middle still holds (further details for such systems may be found in [2,8,9,13]).

The resulting decisive Venn diagram for two variables p,q is now obtained in similar fashion, and is shown in Figure 4. Since there are 3 terms in the above law of included middle, these now generate $3^2=9$ conjunctions---namely, $\sim p \& \sim q$, $\sim p \& [\sim(\sim q \vee \Box q)]$, $\sim p \& \Box q$, $[\sim(\sim p \vee \Box p)] \& \sim q$, $[\sim(\sim p \vee \Box p)] \& [\sim(\sim q \vee \Box q)]$, $[\sim(\sim p \vee \Box p)] \& \Box q$, $\Box p \& \sim q$, $\Box p \& [\sim(\sim q \vee \Box q)]$, $\Box p \& \Box q$. These are shown in Figure 4, and it is seen that the number of possible disjuntions of these 9 conjunctives is $2^{3^2}=512$.

There is an immediate and striking contrast between the 16 possible disjunctions which may be generated from Figure 2 and the 512 possible disjunctions which may be generated from Figure 4. The difference between these numbers is very large. On one hand, the number of additional possible disjunctions from Figure 4, 496, is so large as to discourage further investigation---recall that the work by Post and Wittgenstein on the full 16 possible disjunctions of Figure 2 did not occur until about 1920. On the other hand, consideration of only 16 possibilites when there may be as many as 496 other possibilities to be taken into account, may have serious consequences in certain critical situations, or when these additional possibilities have ramifications which can not afford to be neglected. This has been observed elsewhere[10], and will be illustrated by example later in this paper.

Before proceeding to examples, it is appropriate to note the generalization of the above to multiple regions and to multiple values.

First, a generalization to n regions may be obtained by picturing n-2 concentric and disjoint annules, using a similar approach. For the case n=3 above, only one annule is pictured, as in Figure 4; for the case n=4, this annule is subdivided into two contiguous concentric annules; for the case n=5, into three contiguous concentric annules, etc. The number of possible disjunctions for 2 variables p,q in this general case is 2^{n^2}. However, for a single variable p, the region ~(~p v □p) still represents the disjunction of all the internal concentric annules, and there are still 512 disjunctions generated from Figure 4 using the single annule ~(~p v □p) as representative of a disjunction of all these concentric annules, doing likewise for the variable q.

Second, it follows from remarks in [1,3] that any function in the n-valued case may be pictured with n-1 copies of these decisive Venn diagrams. Hence, these diagrams are sufficient to characterize n-valued functions. To illustrate, in the 3-valued case where the values are white, gray, black, a 3-valued function would be pictured using 2 diagrams of the kind shown in Figure 4. One of these might be called the diagram for gray, the other might be called the diagram for black. In each of these two decisive diagrams, regions would assume only the 2 values white or black. The value for any particular region, such as □p&~q for example, would be given by a rule of predomination as follows: if the copy of □p&~q assumes the value black in the diagram for black, then the value for the region □p&~q is black; otherwise, if the copy of □p&~q assumes the value black in the diagram for gray, then the value for the region □p&~q is gray; otherwise, the value for the region □p&~q is white. To continue this illustraion, consider the pair of diagrams shown in Figure 5 for the 3-valued implication in the internal system of Bochvar[p. 30,14]. The diagram for black at the right clearly shows that this implication assumes the value black exactly in those regions as would be expected for the normal case of classical 2-valued implication; the diagram for gray at the left shows equally clearly that this implication assumes the value gray everywhere within the annules; the value white is assumed only in the region □p&~q, which also corresponds with the normal case of classical 2-valued implication.

The interest of this paper in 3-valued decisive Venn diagrams stems from the above. The examples which follow are chosen accordingly. Generalizations may be made as indicated above, by making subdivisions of the annule or taking multiple copies of the decisive Venn diagram as the case requires.

A specific illustrative example is afforded by the 3-valued function of 2 variables called decisive implication[5]. The decisive truth table for this function $p \supset q$ using the 3 values $\underline{t}$, $\underline{e}$, $\underline{f}$ is given in Figure 6. For properties of 3-valued decisive implication, see[7,8]. To picture this function using the diagram of Figure 4, it is only necessary to picture those 6 regions as black which correspond to those 6 conjunctives which assume the value $\underline{t}$ in the table of Figure 6. The remaining 3 regions correspond to the other 3 conjunctives which take on the value $\underline{f}$. The result is shown in Figure 7. Since the truth table of Figure 6 is decisive, there is no need for a diagram for gray, and the picture of Figure 7 suffices.

This last example extends easily to general case of n-valued decisive implication. Truth tables for this general case are indicated on page 47 of [14], with corresponding algebraic expressions given by equation (16) of [4](see also [5]).

The main example of this paper is devoted to a practical situation where there are concrete assignments for p, q which are both realistic and typical for this situation. The following situation is taken from the military domain, but might as well have been drawn from other domains (e.g., economics).

For this example, consider a man-computer-radar system which is assumed to be of the long-range type, but may be either a shipboard or land-based system. It is further assumed that some portion of the radar scanning is under control of the computer, and that the computer participates further in the process of automatic target detection and initializing of target tracks. Information about results of these processes are forwarded to other systems or displays, and computer or display operators are responsible for the forwarding of this information, as the situation requires, to supervisors or higher executives in the chain-of-command for further processing and/or decision-making.

This example uses Figure 4 with the following assignments for p=P and q=Q. For P: A track is initialized within radar range for a missile flying at supersonic speed; for Q: A track is initialized within radar range for a flight formation flying at subsonic speed which has a size greater that 50 aircraft.

To simplify this example, first consider the situation at the computer outputs. The status of P and Q in this example is invariably computed by some kind of sequential process in order to determine whether P or Q actually holds. The status of P and Q can not be determined until these sequential processes are completed. An easy illustration is provided by consideration of the moving-density-gate process. In this process, consecutive interrogations are made of the inputs for confirmation or denial of target presence. If the size of the gate is denoted by g, the number of consecutive interrogations, and if d denotes the total number of confirmations within these g consecutive interrogations, then target presence is confirmed when the density d/g becomes sufficiently high. Thus for $g=19$, say, if target presence is confirmed with a density of 12/19 or higher, then whenever there are 12 or more confirmations within any consecutive 19 interrogations, the target presence is confirmed. Target presence is denied in this example as long as $d=0$, so that for values of d in the range $1 \leq d \leq 11$ the target presence is neither confirmed nor denied.

Applying this to P as assigned above: $\sim P$ has the value t for $d=0$; $\sim(\sim P \vee \Box P)$ has the value t for $1 \leq d \leq 11$; $\Box P$ has the value t for $12 \leq d \leq 19$. A similar statement may be made for Q. Further processing may be required for P in order to determine initial track data including a supersonic velocity check; likewise, further processing may be required for Q in order to determine formation size and a subsonic velocity check.

This example has been constructed for this P and this Q to show the criticality of determining the value assumed in the region $\Box P \& \Box Q$. While the confirmation of a single high speed aircraft flying toward the radar system , or the confirmation of a large formation of low speed aircraft flying toward the radar system, might each individually be an accident of a mistake, it is much less likely that the occurrence of both of these simultaneously should be so construed. It may be seen also that the 512 possible disjunctions of Figure 4 now take on specific meanings without ambiguity. For instance, $\sim\Box P$ has the value t for $0 \leq d \leq 11$, $\sim\sim P$ has the value t for $1 \leq d \leq 19$, and $\sim P \vee \Box P$ has the value t for $d=0$ or $12 \leq d \leq 19$. This latter occurs when sequential processing is completed.

A crucial point in this example concerns the time during which $\sim(\sim P \vee \sim P)$ has the value t. This time may be either a fraction of a second or a number of seconds, depending whether target presence can be determined within a single scan of the target or with a number of scans of the target. The number of interrogations is of course proportional to the number of scans of the target by the radar.

A feature of this example has been a restriction of consideration to the computer outputs alone, and consequent rigorous interpretation of $\sim P$, $\sim(\sim P \vee \Box P)$, $\Box P$, with similar treatment for $\sim Q$, $\sim(\sim Q \vee \Box Q)$, $\Box Q$. Moving beyond these computer outputs lends geater force to these observations but also greater uncertainty. Once there are further channels of communication beyond the computer outputs, and especially through human channels such as the display-operator-to-supervisor-to-executive chain-of-command, the time during which $\sim(\sim P \vee \Box P)$ has the value t may increase not only drastically, but in any number of unknown or unexpected ways. To cite just a few of these instances:

- The display-operator drowses because of display fatigue.
- The display-operator, supervisor, or executive takes inappropriate action when $\Box P \& \Box Q$ assumes the value t because of surprise or disbelief.
- The supervisor or executive cannot be contacted or is absent without a back-up.
- There is an undetected component or circuit failure within the computer-display system.

This author has not prepared a computer listing for the 512 disjunctions generated from Figure 4, as has been done by hand for the 16 disjunctions generated from Figure 2. There appears to be no difficulty in doing so, and even some point, if it helps to improve the overall efficiency of our present systems, be they military, economic, etc.

Among other areas of application, brief mention is made here of an experimental situation involving the fields of linguistics and logic, as described in [10]. Work in this area for the 2-valued case has been done by Inhelder and Piaget[11]. A detailed discussion for the 16 disjunctions generated from Figure 2 may be found in [12], with specific application in [p. 102-104,11]. The discussion in [10] extends this beyond the 2-valued case, retaining the same experimental situation in which youngsters describe verbally their reasoning concerning laboratory experiments, but making connections of such verbal statements with different disjunctions from the decisive Venn diagram of Figure 4.

It has not been generally recognized that decisive Venn diagrams are sufficient to characterize multiple-valued functions. The use of these diagrams is straightforward, and without undue complexities.

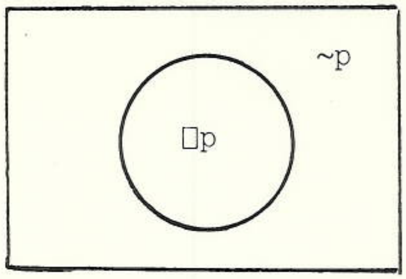

Figure 1-Venn diagram,
one variable p

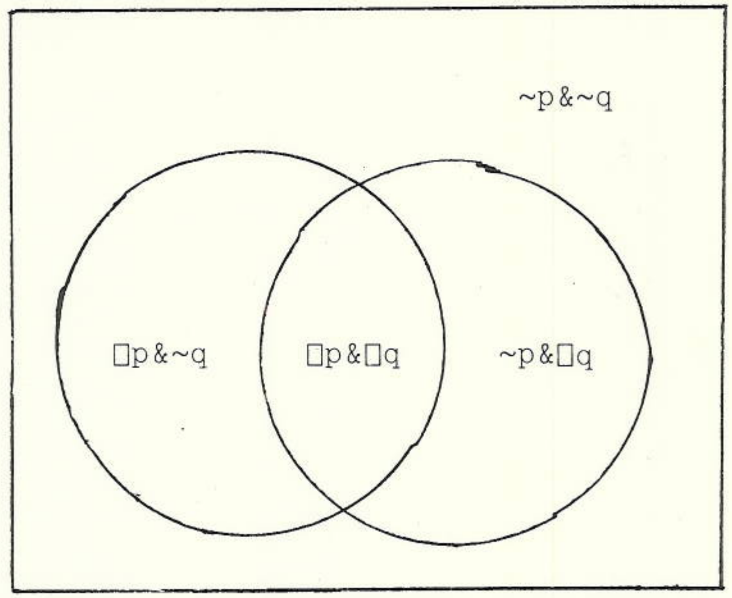

Figure 2-Venn diagram, two variables p, q

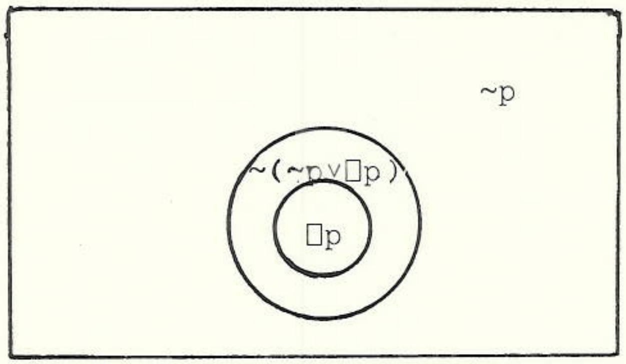

Figure 3-Decisive Venn diagram, one variable p, included middle term

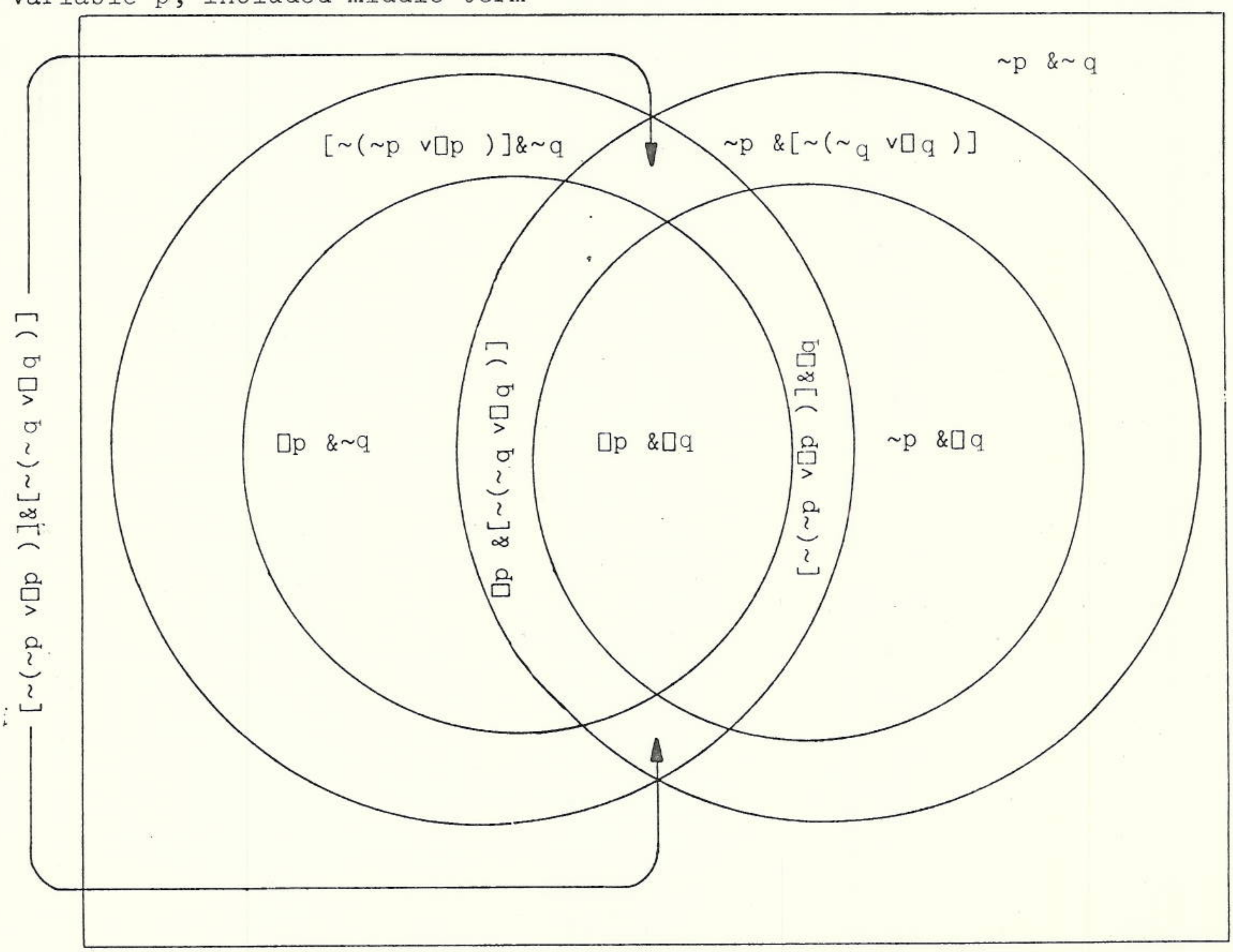

Figure 4-Decisive Venn diagram, two variables p,q, included middle terms

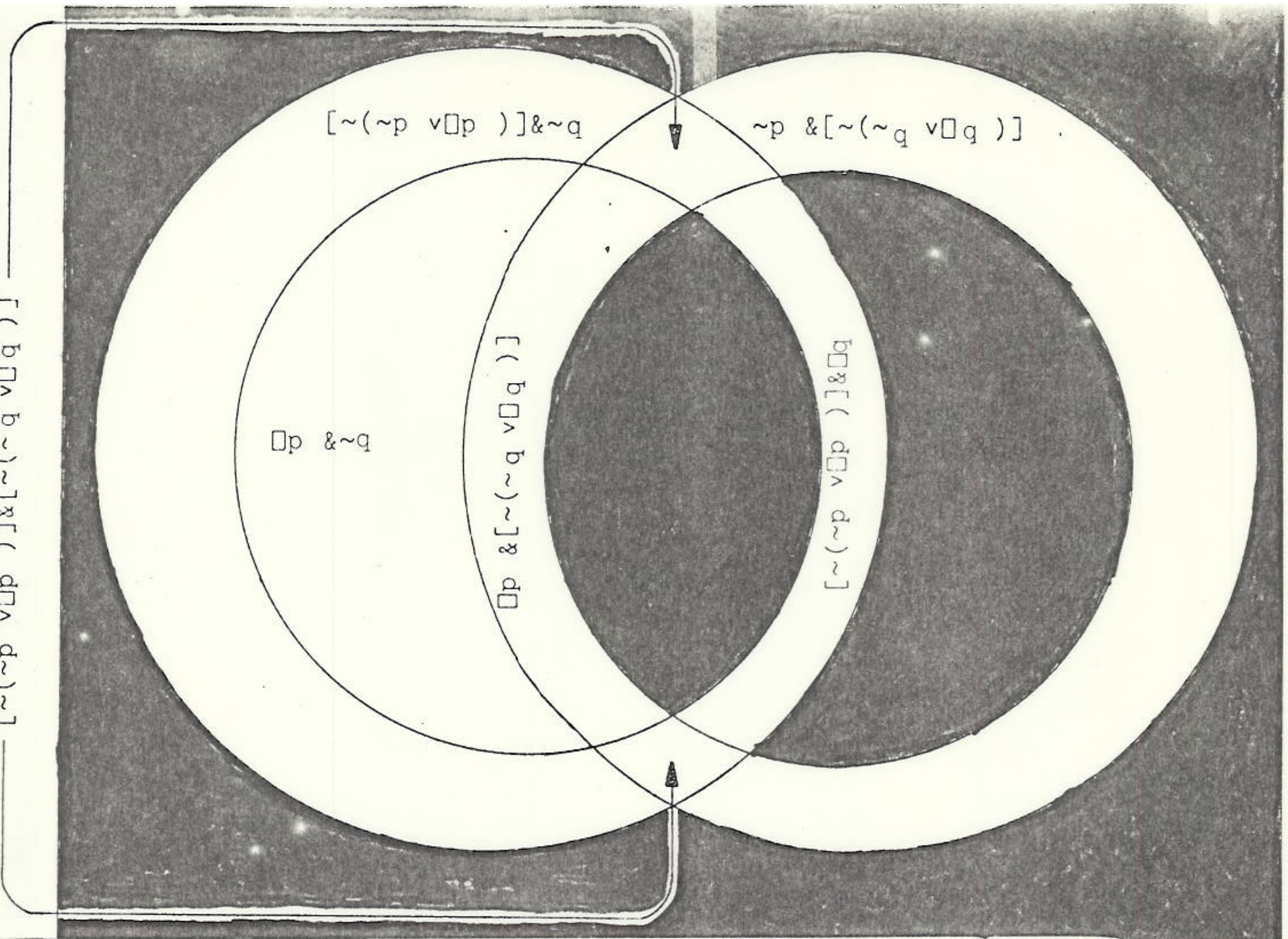

diagram for black

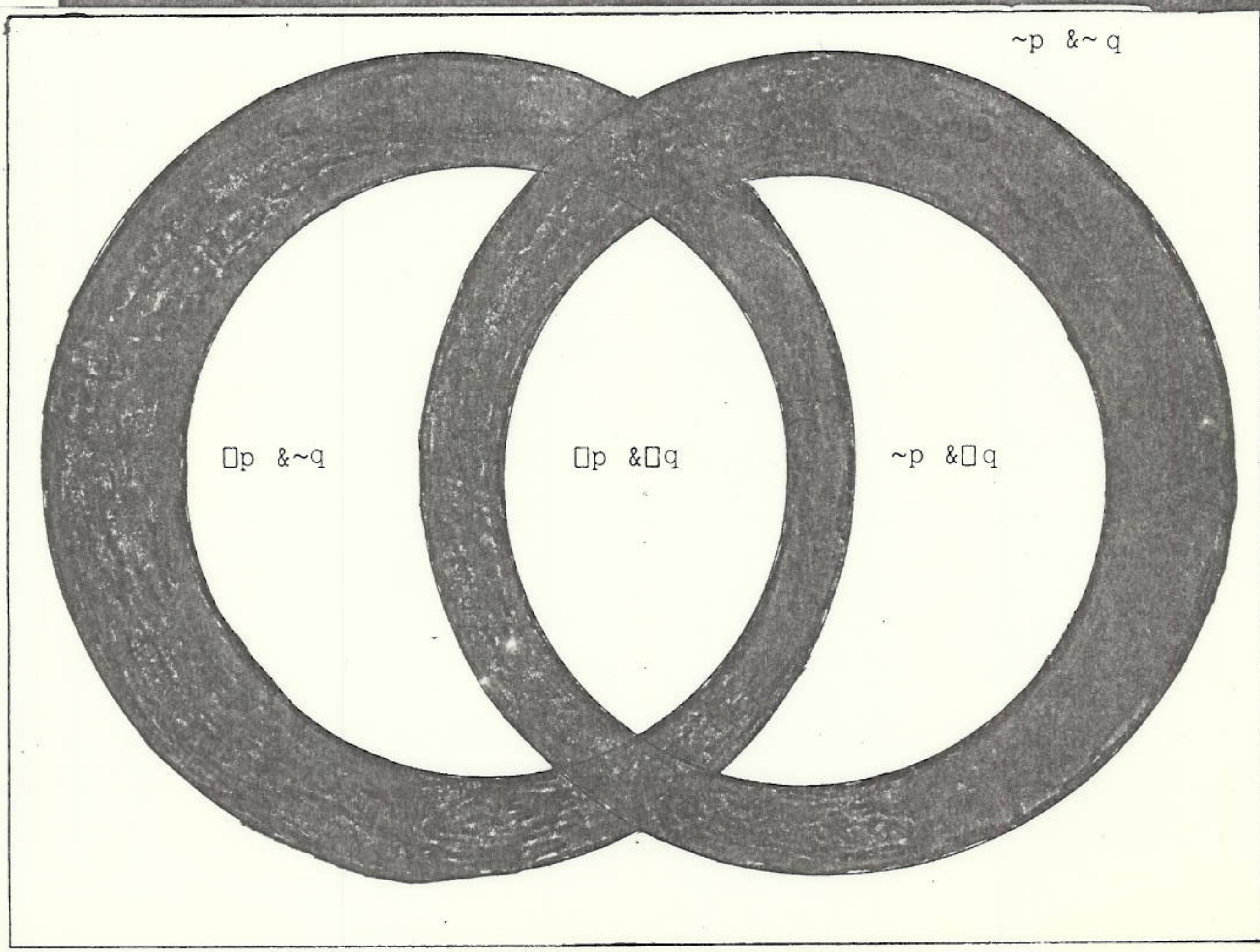

diagram for gray

Figure 5-Decisive Venn diagram pair, 3-valued internal implication of Bochvar

p \ q	f	e	t
f	t	t	t
e	f	t	t
t	f	f	t

Figure 6-Table, 3-valued decisive implication

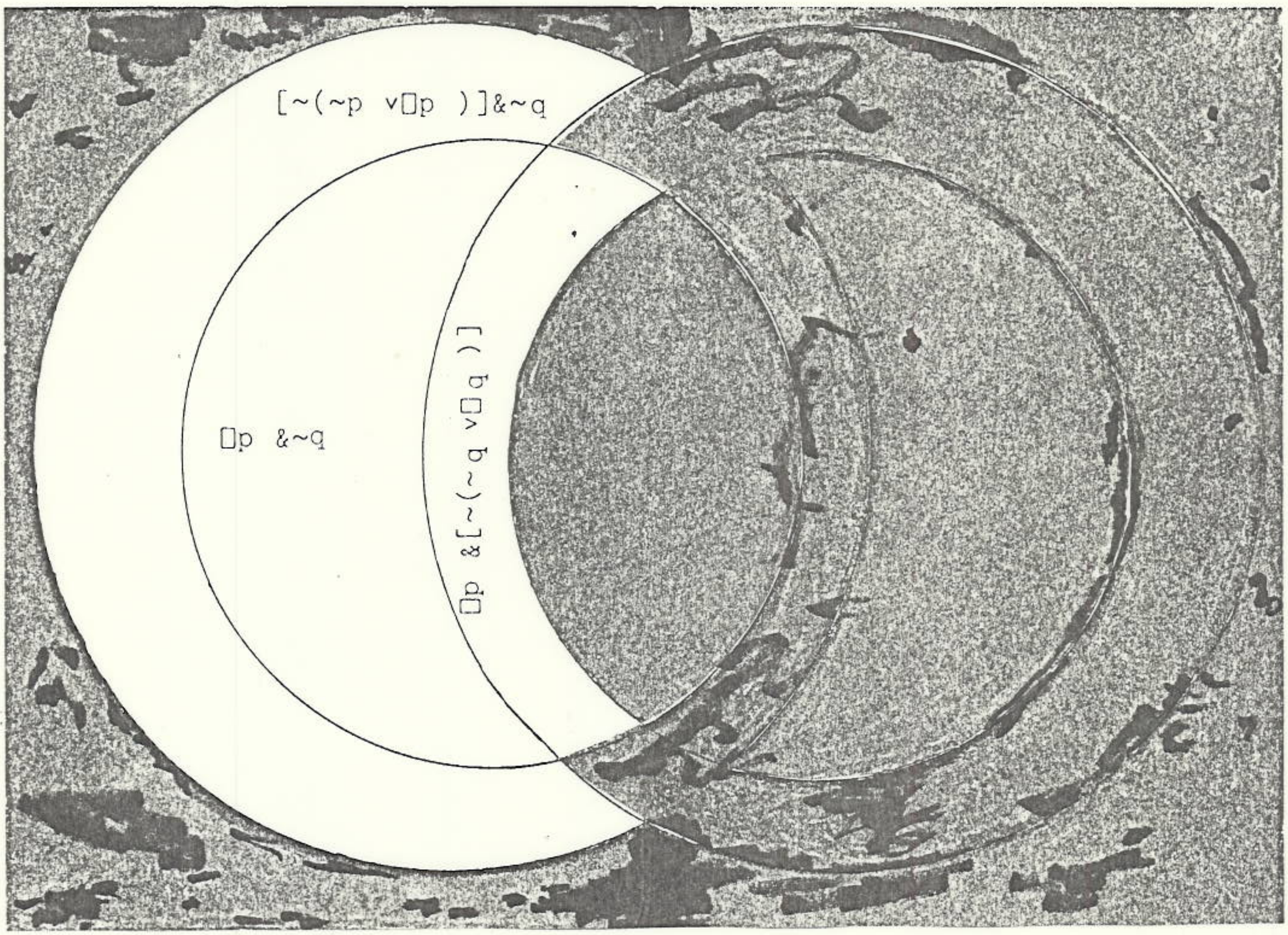

Figure 7-Decisive Venn diagram, 3-valued decisive implication

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