BOOK REVIEW OF

N. RESCHER'S

MANY-VALUED LOGIC

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NICHOLAS RESCHER. Many-valued logic. McGraw-Hill Book Company, New York, etc., 1969, xv+359 pp.

There are exactly three books devoted to the subject of multiple-valued logic and which may be regarded as precursors to this book. They are: (1) Many-valued logics by Rosser and Turquette (XX 45); (2) Philosophical problems of many-valued logic by Zinov'ěv (XXVIII 225, XXIX 213); and (3) An introduction to many-valued logics by Robert Ackermann (London and New York 1967). While these three books pursue special topics within the field of multiple-valued logic, the present book is devoted to presenting a comprehensive view of this field as a whole. The primary concern is with the "semantical and meaning-related side of logic." The exposition relies primarily on the use of truth-tables. This provides a definiteness to the discussion which can be easily understood by the non-specialist, yet still allows for some discussion of syntactical and algebraic aspects. While this latter side of logic is thus hardly neglected, one must look to other sources, such as Rasiowa's book reviewed above, for further treatment.

The first chapter (pp. 1-16) is historical in nature. It begins with the well-known discussion by Aristotle in Chapter 9 of his treatise De interpretatione concerning future contingent propositions (an example is given later in this review). It is the notion that such propositions are neither true (T) nor false (F) which can lead to the idea of a third indeterminate value (I). (See the development by Łukasiewicz, 1868). This historical conspectus moves through the times and work of Duns Scotus and William of Ockham to those of Hugh MacColl, Peirce, and Vasil'év. The work of contributors in this century--Brouwer, Łukasiewicz, Post, etc.--forms the basis for the survey in Chapter 2.

The material of Chapter 2 (pp. 17-212) may be divided into subject groupings as follows: (1) structural observations (pp. 55-65, 154-165); (2) characterizing conditions (pp. 102-106, 122-153); (3) specific systems (pp. 17-54, 76-90, 96-101, 120-121, 166-205); (4) generalization or extension of systems (pp. 66-75, 91-95); and (5) applications (pp. 206-212). These will be treated separately in this review.

A number of structural features are noted for truth-tables which define many-valued connectives. Foremost is the feature of being normal, in the sense that the truth-table does not depart from the classical case when only the two values T and F are involved. Among other key features are those of being uniform, decisive, regular (in the sense of Kleene), or categorical. All of these provide some degree of insight into essential characteristics of many-valued logics.

Truth-tables can also be specified through requirements that certain axioms be satisfied. Further insight can be gained by investigating this with respect to basic connectives and certain pivotal axioms or laws. There is an exhaustive discussion of various kinds of negation, which includes Łukasiewicz negation and intuitionist negation, among others. The "intuitionist" negation is so called from Łukasiewicz VII 35(5); though originally due to Heyting it is not the negation of Heyting's intuitionistic propositional calculus, but it is in a sense a version of it.) There is also a discussion for three alethic unary operators, but this occurs later in a section on modal structures, without any attempt to make duality connections with the discussion on negation. The reader may easily observe from the three-valued cases provided in the book that the possibility of p (denoted by ◇p) is equivalent to the double intuitionist negation of p, that the necessity of p (□p) is equivalent to the double dual-intuitionist negation of p, and so forth. The discussions for conjunction and disjunction are presented contiguously, and do exploit the analogues which obtain because of duality. The example of Bochvar's system is sufficient to show that conjunction or disjunction need not yield the maximum or minimum of the values of the arguments, respectively. (In Bochvar's internal system the truth-tables for these operations are normal but do not yield the value T or F if any of the arguments has a value which is neither T nor F.) An example is given of a logic which does not possess many of the features of conjunction and disjunction which might reasonably be expected. Reasonable expectations for features of conjunction and disjunction which are not challenged in this section are those of normality and the properties of idempotency, commutativity, and associativity. This approach is then continued for implication (and afterwards for equivalence), building on a like listing of reasonable expectations for features of implication given by Salomaa in <u>On many-valued systems of logic</u> (XXV 291). This list includes, of course, reflexivity and transitivity conditions, among others. Two of these listed conditions will be illustrated in the immediately following paragraphs, as part of the review of the specific systems discussed by the author. The pivotal laws or axioms discussed have had prominent impact since the time of Aristotle. A listing of the various versions of these laws--for the law of contradiction that $x \wedge \neg x$ is logically false, that $\neg(x \wedge \neg x)$ is logically true, etc.; for the law of excluded middle that $x \vee \neg x$ is logically true, that x is true or $\neg x$ is true, etc.--leads to discussions of the consequent reflections of these conditions on the truth-tables for basic connectives (viz., for negation alone, it being assumed that conjunction and disjunction are not involved at the heart of the matter). While the discussions for these two pivotal rules are presented contiguously, there is no drawing of any dual connections, so that further investigations into related expressions $\neg x \vee \neg\neg x$, $\neg x \wedge \neg\neg x$, $\neg(\neg x \wedge \neg\neg x)$, etc., or into the involvement of conjunction and disjunction, are left to the reader. There is also discussion concerning the reflection of other issues, such as axiomatizability, completeness, and consistency, on the truth-tables for basic connectives.

In connection with specific multiple-valued logic systems, assume first that the values of the conjunction and disjunction of x and y are given by the greatest lower bound and least upper bound, respectively, of the values of x and y. Assume further that these values are linearly ordered so that these become min(x,y) and max(x,y). For the case n=3, the three values are denoted by F, I, T, as already mentioned. Concentration on this case usually lays a foundation for the extension of a particular system to cases where n>3, though such extension is not always evident, nor need it be unique. The truth-tables for all basic connectives are assumed to be normal.

Consider the requirement on implication which results from exactly two of the many conditions given by Rescher, namely that the value of $p \rightarrow q$ be T if and only if the value of p is less than or equal to the value of q. While this is a relatively strong requirement, which is mentioned but not explored by Rescher, nevertheless three major systems are included under all the conditions just cited. Moreover, these appear as three of the four possible solutions for these constraints when n=3, corresponding, that is, to the four possible truth-tables for implication under this strong requirement (there are $(n-2)(n+1)/2^{n-1}$ such truth-tables for general n). Implications for the three major systems are the three-valued Łukasiewicz implication, a three-valued implication of Heyting (3852,XXI 367), and a three-valued implication related to Lewis's systems of strict implication.

(This latter implication appears among Rescher's "standard" systems with Łukasiewicz negation--intuitionist negation could have been used, and the introduction of alethic operations is especially appropriate once the connection with modal logic is noted.) The new fourth implication shares with the other three tables tautologies such as $p \to p$ and $((p \to q) \to r) \to (((q \to p) \to r) \to r)$--observe that the latter does not hold for $n > 3$ in this class of tables while the former obviously does--but differs from each of the three others because $(p \to q) \to ((q \to r) \to (p \to r))$ does not hold.

Relaxing only the above strong requirement on implication, by allowing the value of $I \to I$ to be I rather than T, yields the Kleene and Post three-valued systems. The Kleene system uses Łukasiewicz negation, while the Post system uses cyclic succession, that is, $p' = p+1 \pmod n$ when the values p are construed as integers.

Relaxing next the requirement that the conjunction or disjunction of x and y be given by min(x,y) or max(x,y) admits the bulk of the remaining systems: those of Bochvar, Sobocinski, T-split, and F-split. In Bochvar's internal system, any connective has a fixed value which is neither T nor F whenever any one of the arguments is neither T nor F. In the external system of Bochvar, each of these fixed values from the corresponding internal system is resolved, one way or another, into one of the two values T or F, so that the resulting tables have entries whose range is two-valued over T, F (such tables afford a classic example of decisive tables--the word dicisive seems etymologically better, with tricisive for the tables of Bochvar's internal system).

All of these last four systems again have the above distinguishing three-valued implication feature for I→I, although I is designated T in the T-split system and antidesignated F in the F-split system. For n=3, the Sobocinski and Bochvar internal systems use Łukasiewicz negation, the F-split and Bochvar internal systems use dual-intuitionist negation, while the T-split system uses intuitionist negation.

There are other kindred systems. Both probabilistic systems stemming from Reichenbach, Zawirski, and Carnap and the fuzzy (formalized vagueness) systems stemming from Zadeh and Bellman have the feature of overlapping with these systems through the use of probabilistic or fuzzy values.

The last step is to drop the assumption that the values are linearly ordered. This leads to the important product logics (primary cases are the four-valued 2x2, six-valued 2x3, and nine-valued 3x3 logics, the first of which is discussed as an example); to the n-tuple interpretation and logics stemming from Post, which are (n+1)-valued logics determined by certain n-tuples from the product $\prod_{j=1}^{n} \mathcal{L}_j$ where $\mathcal{L}_j = \mathcal{L}$ for each j; and to the quasi-truth functional systems, where any entry of a truth-table may be ambiguous, but is restricted to a particular subset of the possible logic values. Other major systems which might have been discussed are those of Moisil described in his <u>Notes sur les logiques non-chrysippiennes</u> (XIII 50). These systems withhold all intermediate constants and model the Łukasiewicz systems when n<5. The unary operations first introduced by Moisil were earlier suggested tions which range, more or less, from the abstract at one end to the physical at the other. Logic applications concern the so-called semantical paradoxes such as the liar (viz., this sentence is false); and non-designating singular terms (e.g. that sentence which is between two of these consecutive printed lines is true). Philosophy applications concern future contingency (e.g. tomorrow at noon you will write a true sentence between two of these consecutive printed lines), also the pivotal laws or axioms already mentioned above, and the issue of relativism and conventionalism in logic, which forms the subject matter of Chapter 3. The physical applications are, however, limited to a treatment on quantum theory (referring to work on three-valued systems in Reichenbach's X97, pp. 142-148 and 160-166, and work on non-distributive systems by Birkhoff and von Neumann in II 44) and a mention of switching theory (which does not distinguish between theories for electrical circuit and logic design, nor between theories for subprogram and logic-module structure). While some complementary support may be derived from the introduction in Chapter I, the meaning and semantic emphases of this book are adequately served to the degree that actual applications which have potential for illumination, are fully discussed. Rescher skimps such applications from the field of computer science and from other fields such as neural science, linguistics, and psychological dynamics.

The reviewer suggests the following as a simple unifying example. Consider that function in neural or computer science which becomes true or activated for all values i satisfying $i \geq j$, but becomes false or unactivated otherwise. Thus in an n-valued logic there might be n-1 corresponding operations $D_j(i)$, excluding the constant operation D_F. The value j corresponds equally well to a level of threshold detection within a neuron, a model for the psychology of affirming, a multivibrator circuit, or a differential comparator implemented in either hardware or software. In all cases, the output corresponds to the truth or falsity of the proposition, "The value of i is greater than or equal to j." Any D_j unary operation may be regarded as a designation operation in the sense that i is designated if and only if $D_j(i)$ is T, with some fixed $j = j_1$ being the least designated value. There is a clear advantage in dealing with $D_j(i)$, whose range is two-valued, rather than with i, whose range must be n-valued. Further, the class of alethic operations includes the D_i operations, with D_T corresponding to □, etc.

Chapter 3 (pp. 213-235) is devoted to the philosophical issues of absolutism and relativism as they relate to logical systems—the analogy is made to corresponding issues relating to geometries and the selection of frames of references for physical systems (pp. 217-220). The analogy between logic and grammars is drawn next (pp. 222-224). This leads to the balanced conclusion (pp. 234-235) that the selection between alternate logic systems must be made practically (i.e. on an ad hoc basis, which depends on locale and milieu), with the realization that there are "certain minimal but universal regulative standards [which] must be met at the metasystematic level." As psychologists view the unconscious selection of dreams to be overdetermined, Rescher views the conscious selection of logic systems to be underdetermined. The author's illustrative analogy using m simultaneous linear equations in n unknowns actually works well for both situations (i.e., the case $n<m$ for the former, and $m<n$ for the latter).

Chapter 4 (pp. 236-330) contains an extensive bibliography covering work to 1965. This bibliography is arranged in threefold order as follows: (1) chronologically, (2) author-lexicographically, and (3) topically. The bibliography is comprehensive and large, with just a small number of omissions occurring mostly near the year 1965.

The appendix (pp. 332-350) provides a summary of the survey given in Chapter 2. The reader is cautioned that there are a few logic systems which appear in one of these two places, but not in the other. The major systems are all fully summarized and adequately cross-referenced.

The overall effect of this book is a highly integrative one, bringing together many diverse systems into a coherent whole. The use of clear truth-tables allows the book to be read easily by those without previous background in multiple-valued logic. Even the specialist will benefit, for the diversity of systems which are brought forward is likely to bring some of these systems into new or different perspective. GEORGE EPSTEIN