Exact Formulas for the Buddy System

Paul W. Purdom Jr.

Cynthia A. Brown

Computer Science Department

Indiana University

Bloomington, Indiana 47401

TECHNICAL REPORT NO. 76

EXACT FORMULAS FOR THE BUDDY SYSTEM

PAUL W. PURDOM JR.

CYNTHIA A. BROWN

NOVEMBER, 1978

Abstract.

The statistical pairing of blocks of memory on the bottom level of the buddy system is studied. It is shown that the probability that k pairs of cells have one cell in use and that ℓ pairs of cells have both cells in use is given by a rational function of ρ, the ratio of the request rate for cells to the decay rate for cells in use. The values of the denominator, the leading and trailing coefficients of the numerator, and a common factor of the coefficients of the numerator are given.

1. Introduction

The buddy system of storage allocation, devised by Knowlton [1], divides memory into nested blocks of size 2^k. Under typical conditions the algorithm is fast and has low external fragmentation. Unless the size distribution of requests closely matches the size of blocks provided, the internal fragmentation is large (up to 1/3 of memory not available). A good discussion of the algorithm is given by Knuth [2].

The performance of the buddy system with Poisson inter-request times and exponentially distributed full block life times was studied by Purdom and Stigler [3]. Using simulations, they showed that the performance of the buddy system could, in most cases, be closely approximated by considering the situation where all requests were of one size. (Further simulations were done by Purdom, Stigler, and Cheam [4] and by Standish [5]). The analysis of Purdom and Stigler showed that both the speed and external fragmentation were determined by the probability that memory was in a state where no pairs of cells were half full (one cell available and the other in use). Finally a fit to numerical calculations of this item showed that it was about $0.53\rho^{-\frac{1}{2}}$ for large ρ, where ρ is the ratio of the request rate to the decay rate. This provides most of the practical information that one may need about the buddy system.

Since having a major result depend on a numerical fit is unsatisfying, in this paper we study those aspects of the performance of the buddy system for which exact answers can be found. The results include showing that the probability, $p_{k\ell}$, that the memory is in a state with k pairs of cells half full and ℓ pairs full is a rational function of ρ. The value of the denominator, the leading and trailing coefficients of the numerator, and a common factor of the coefficients of the numerator are given, as well as some relations among the probabilities. These partial results give a good indication of the nature of the formulas for $p_{k\ell}$. Formulas for $p_{k\ell}$ are, however, primarily of theoretical interest, because the $p_{k\ell}$ can be calculated more rapidly by an algorithm we present (it was referred to in [3]).

We do not give any non-trivial asymptotic formulas that are valid for the practical case $1 \ll \rho \ll 2n$, where n is the number of pairs of cells in the system. It would be particularly useful to find a proof that $\sum_{\ell} p_{0\ell} \approx 0.53\rho^{-\frac{1}{2}}$, which was established empirically in [3]. (We omit the limits on summations when all non-zero values of the summand are included in the sum).

As shown in [3], the probability $p_{k\ell}$ obeys the equation

$$(\rho+k+2\ell)p_{k\ell} = (k+1)p_{k+1,\ell}+2(\ell+1)p_{k-1,\ell+1}+\rho p_{k+1,\ell-1} + [\rho p_{0\ell}]_{\text{if } k=1 \text{ and } \ell\neq n} + [\rho p_{0n}]_{\text{if } \ell=n \text{ and } k=0} \tag{1}$$

for $k,\ell\geq 0$, with boundary conditions $p_{k\ell}=0$ if $k<0$, $\ell<0$, or $k+\ell>n$. The bracketed terms arise from the convention that requests which arrive when all $2n$ cells are full are ignored.

2. Preliminary Results

In this section we consider those properties of $p_{k\ell}$ that are simple to derive and to calculate. Many of these results are straightforward and so will be treated briefly. Consider the probability that i cells are in use, $p_i = \sum_j p_{i-2j,j}$.

It obeys the equation

$$(\rho+i)p_i = [\rho p_{i-1}]_{\text{if } i\neq 2n+1} + (i+1)p_{i+1} + [\rho p_i]_{\text{if } i=2n}$$

which has the solution

$$p_i = \sum_j p_{i-2j,j} = \frac{(2n)!\rho^i}{i!N} \text{ for } 0\leq i\leq 2n, \tag{2}$$

where

$$N = \sum_{0\leq j\leq 2n} \frac{(2n)!\rho^i}{i!} .$$

This gives

$$p_{00} = \frac{(2n)!}{N},\ p_{01} = \frac{(2n)!\rho}{N},\ p_{1,n-1} = \frac{2n\rho^{2n-1}}{N},\ \text{and } p_{0n} = \frac{\rho^{2n}}{N} \tag{3}$$

since for $i = 0, 1, 2n-1,$ and $2n$, the sum for p_i has only one term.

Another interesting relation is given by

$$\rho p_{0\ell} = \sum_k k p_{k,\ell-k+1} \quad . \tag{4}$$

The left side is the rate at which the system shifts from states with the number of pairs of cells not empty equal to ℓ to states with the number of such cells equal to $\ell+1$, while the right side is the rate for the reverse transition.

Now define p_i to be the solution of the equation

$$(\rho+n+i)p_i = 2(i+1)p_{i+1} + \rho p_{i-1} \tag{5}$$

For $i \le n-1$, p_i is proportional to $p_{n-i,i}$, since for $i \le n-2$, p_i and $p_{n-i,i}$ obey the same difference equation. Define the generating function $g(z) = \sum_i p_i z^i$. Then $g(z)$ obeys the equation $(\rho+n)g + z\frac{dg}{dz} = 2\frac{dg}{dz} + \rho z g$. The solution is $g(z) = p_0 e^{\rho z}(1-\frac{z}{2})^{\rho-n} = p_0 \sum_{i\ge 0} (\sum_j \binom{\rho-n}{j} \frac{(-1)^j 2^{-j}}{(i-j)!} \rho^{i-j}) z^i$. Since $p_{1,n-1} = \frac{2n\rho^{2n-1}}{N}$

$$p_{n-i,i} = \frac{n!\,2^{n-i}\rho^{2n-1}R_{0i}}{i!\,D_n} \quad \text{for} \quad i \le n-1 \tag{6}$$

where

$$R_{mi} = \begin{cases} 1 & \text{for} \quad m<0 \\ \sum_j \binom{\rho-n+m}{j} \frac{i!}{(i-j)!} (-1)^j 2^{i-j} \rho^{i-j} & \\ = \sum_\ell \left(\sum_j \sum_k \begin{bmatrix} i-j \\ k \end{bmatrix} \binom{i}{j} \binom{k}{\ell-j}\right) (-1)^{j+\ell} 2^j (n-m)^{j+k-\ell} \rho^\ell & \text{for } 0 \le m \le n \end{cases} ,$$

$$D_\ell = N \prod_{i \ge \ell} F_i \quad , \text{and}$$

$$F_i = R_{n-i,i-1} \quad .$$

(Stirling numbers of the first kind are represented by $\left[{j \atop k}\right]$, using the sign conventions of Knuth [6].)

Equations (2) and (6) give

$$p_{0,n-1} = \frac{2n\rho^{2n-2}}{D_n}\Big((2n-1)F_n - 2(n-1)\rho R_{0,n-2}\Big) \quad \text{and}$$

$$p_{1,n-2} = \frac{4n(n-1)\rho^{2n-3}}{D_n}\Big((2n-1)F_n - 2(n-2)\rho^2 R_{0,n-3}\Big) \quad . \qquad (7)$$

3. An Algorithm for the Probabilities $p_{k\ell}$

The following algorithm provides an efficient way to calculate the $p_{k\ell}$.

1. Let $p_{2,n-1} \leftarrow 0$, $p_{0n} \leftarrow \frac{\rho^{2n}}{N}$, $p_{1,n-1} \leftarrow \frac{2n\rho^{2n-1}}{N}$, $p_{00} \leftarrow \frac{1}{N}$, $r_{-1} \leftarrow 0$, $r_0 \leftarrow 1$, $s_{-1} \leftarrow 0$, and $s_0 \leftarrow 0$.
2. For $0 \le i \le n-2$ set $r_{i+1} \leftarrow \frac{1}{2(i+1)}[(\rho+n+i)r_i - \rho r_{i-1}]$. Then set $f \leftarrow \frac{p_{1,n-1}}{r_{n-1}}$.
3. For $0 \le i \le n-2$ set $p_{n-i,i} \leftarrow f r_i$.

Now do steps four through six for $1 \le m \le n-1$.

4. Set $p_{0,n-m} \leftarrow \frac{1}{\rho}[(\rho+2n-2m+1)p_{1,n-m} - 2p_{2,n-m} - 2(n-m+1)p_{0,n-m+1} - \rho p_{2,n-m-1}]$ and $p_{1,n-m-1} \leftarrow \frac{1}{\rho}[(\rho+2n-2m)p_{0,n-m} - p_{1,n-m}]$.
5. For $0 \le i \le n-m-2$ set

$$r_{i+1} \leftarrow \frac{1}{2(i+1)}[(\rho+n-m+i)r_i - \rho r_{i-1}] \quad \text{and}$$

$$s_{i+1} \leftarrow \frac{1}{2(i+1)}[(\rho+n-m+i)s_i - \rho s_{i-1} - (n-m-i+1)p_{n-m-i+1,i}] \quad .$$

Then set $f \leftarrow \frac{p_{1,n-m-1} - s_{n-m-1}}{r_{n-m-1}}$

6. For $0 \le i \le n-m-2$ set $p_{n-m-i,i} \leftarrow f r_i + s_i$.

Step 1 of the algorithm comes directly from equation (3). Step 4 comes from equation (1). Steps 2 and 3 and also steps 5 and 6 depend on the fact that the solution of a linear difference equation is equal to a particular solution plus some multiple of the solution of the corresponding homogeneous equation.

Assuming unit time for arithmetic operations, the algorithm can compute the numerical value for all $p_{k\ell}$ in time $\Theta(n^2)$, where $\Theta(f(n))$ indicates bounded above and below by constants times $f(n)$ (see [7]). Any particular $p_{k\ell}$ can be computed in time $\Theta(n(n-k-\ell))$. It will be shown below that the formula for $p_{k\ell}$ also has $\Theta(n(n-k-\ell))$ terms (assuming there are no common factors between the numerator and denominator), except for p_{00} and p_{01} . Therefore the algorithm provides the fastest way known to compute the value of $p_{k\ell}$. If the algorithm uses rational functions for variables, it can compute formulas for $p_{k\ell}$ in time $\Theta(n^4)$.

4. General form of the probabilities.

We will now show that

$$p_{0\ell} = a_\ell \, \rho^{2\ell} \, \frac{P_{0\ell}}{D_{\ell+1}}$$

$$p_{1\ell} = a_\ell \, \rho^{2\ell+1} \, \frac{P_{1\ell}}{D_{\ell+2}} \quad , \text{ and}$$

$$p_{k\ell} = \frac{a_{k+\ell-1}(k+\ell-1)!^{2^{k-1}}}{\ell!} \rho^{2\ell+2k-1} \, \frac{P_{k\ell}}{D_{k+\ell}} \quad \text{for } k\geq 2 \quad , \qquad (8)$$

where a_ℓ is an integer and $p_{k\ell}$ is a polynomial in ρ with integer coefficients. Notice that N, R_{mi}, and D_ℓ are also polynomials in ρ with integer coefficients. More will be said about a_i later.

We will also show that

$$r_i = \frac{1}{i!2^i} R_{mi} \quad \text{and}$$

$$s_i = a_{n-m} \frac{(n-m)!2^{n-m-i}}{i!} \rho^{2n-2m-1} \frac{S_{mi}}{D_{n-m+1}} ,$$

where S_{mi} is a polynomial in ρ with integer coefficients.

The variable r_i has the above value because it satisfies equation (5) with boundary conditions $r_{-1} = 0$ and $r_0 = 1$. To apply the formula for r_i to step 2 of the algorithm use $m = 0$.

The claims about $p_{k\ell}$ and s_i are proved by induction as follows. The algorithm calculates all the values of $p_{k\ell}$ and s_i. At each step it uses only previously computed values. If the claims are true for the initial values, and are preserved by the steps used to calculate subsequent values, then they are true for all $p_{k\ell}$ and s_i. The algorithm obviously halts.

We will now show that the claims are true at each step provided the a_ℓ are chosen correctly. One choice for a_ℓ is 1. We are interested, however, in how large a_ℓ can be. Step 1 implies $a_{n+1}P_{2,n-1} = 0$, $a_nP_{0n} = 1$, $a_{n-1}P_{1,n-1} = 2n$, and $a_0P_{00} = \frac{D_1}{N}$. Step 2 implies $f = \frac{n!2^n\rho^{2n-1}}{N\,R_{0,n-1}} = \frac{n!2^n\rho^{2n-1}}{D_n}$.

Step 3 implies that $a_{n-1}P_{n-i,i} = 2nR_{0i}$. Therefore for $m = 0$, $k+\ell = n$, the claims are true with $a_n = 1$ and $a_{n-1} = 2n$. Larger values cannot be used for a_n or a_{n-1} if the claims are to hold. (Notice that the coefficient of the leading term of R_{mi} is one).

Step 4 implies $a_{n-m}P_{0,n-m} = a_{n-m}(\rho+2n-2m+1)F_{n-m+1}P_{1,n-m}$ $-a_{n-m+1}4\rho^2(n-m+1)F_{n-m+1}P_{2,n-m} - a_{n-m+1}2(n-m+1)\rho F_{n-m+1}P_{0,n-m+1}$ $-a_{n-m}2\rho(n-m)P_{2,n-m-1}$ and $a_{n-m-1}P_{1,n-m-1} = a_{n-m}(\rho+2n-2m)P_{0,n-m}$ $-a_{n-m}\rho F_{n-m+1}P_{1,n-m}$. Step 5 implies $a_{n-m}S_{m,i+1} =$ $a_{n-m}(\rho+n-m+i)S_{mi} - a_{n-m}2iS_{m,i-1} - a_{n-m}\rho^2(n-m-i+1)P_{n-m-i+1,i}$ and $f = \frac{(n-m-1)!2^{n-m-1}\rho^{2n-2m-1}}{D_{n-m}}(a_{n-m-1}P_{1,n-m-1}-a_{n-m}2(n-m)S_{m,n-m-1})$.

Step 6 implies that $a_{n-m-1}P_{n-m-i,i} = a_{n-m-1}R_{mi}P_{1,n-m-1}$ $+ a_{n-m}2(n-m)(F_{n-m}S_{mi} - R_{mi}S_{m,n-m-1})$.

This shows that $a_iP_{k\ell}$, where $i = k+\ell-1+\delta_{k0}$, is a polynominal in ρ with integer coefficients and that

$p_{k\ell}$ has the indicated form. If one wants $P_{k\ell}$ to have integer coefficients then a_i must be a divisor of the gcd of the coefficients of the polynominals $a_i P_{k\ell}$, where k and ℓ are selected so that $i=k+\ell-1+\delta_{k0}$. One choice that always works is $a_i=1$. The above shows that for each a_i there is an integer $\alpha_i \geq 1$ such that a_i can be selected to be $a_i=\alpha_i a_{i+1}$.

It is interesting to consider whether the α_i can be greater than one. We have already shown that α_{n-1} can be a multiple of $2n$. The formula for $p_{1,n-2}$ indicates that the g.c.d. of the coefficients of $a_{n-2}P_{1,n-2}$ is a multiple of $4n(n-1)3^{\delta_{2,n \bmod 3}}$. All steps in the algorithm except the last part of step 4 are consistent with α_i being a multiple of $2(i+1)$. Therefore α_{n-2} can be a multiple of $2(n-1)$. It appears that one can always have $\alpha_i=2(i+1)\alpha_{i+1}$, but we have no proof.

To summarize, we have proven that the $p_{k\ell}$ have the indicated form with $a_i=4n(n-1)$ for $i \leq n-2$, $a_{n-1}=2n$, and $a_n=1$. The details of the proof indicate how to compute $P_{k\ell}$ using polynomial arithmetic.

5. The Two Dimensional Generating Function

Consider the two dimensional generating function $g(x,y) = \sum_{k,\ell} p_{k\ell} \; x^k y^\ell$,which is a polynomial. Multiplying equation (1) by $x^k y^\ell$ and summing over k and ℓ (while paying careful attention to terms near the $k=0$ boundary) gives the following equation which $g(x,y)$ must satisfy:

$$(x-1)\frac{\partial g}{\partial x} + 2(y-x)\frac{\partial g}{\partial y} = \rho(\frac{y}{x} -1)g + \rho(x-\frac{y}{x})\sum_m y^m p_{0m}$$

$$+\rho(1-x)y^n p_{0n} \quad . \qquad (9)$$

Using standard techniques for solving first order partial differential equations, we can eliminate the dependence on y by solving the equation $\frac{dy}{dx} + \frac{2y}{1-x} = \frac{2x}{1-x}$ and replacing y in equation (9) by the solution $y = Cx^2 - 2(C-1)x + C-1$, where C is the constant of integration. We now have the following equation which $g(x)$ must satisfy:

$$\frac{dg}{dx} = (\rho C-\rho\frac{(C-1)}{x})g + (C-1)(1-\frac{1}{x}) \sum_m (Cx^2-2(C-1)x+C-1)^m p_{0m}$$

$$-\rho(Cx^2-2(C-1)x+C-1)^n p_{0n} \quad . \qquad (10)$$

Defining $g(x) = \sum_{e\geq 0} d_e x^e$ and expanding the above equation in a power series gives the recurrence equations

$$d_0 = \sum_m (C-1)^m p_{0m}$$

$$[e+1+\rho(C-1)]d_{e+1} =$$

$$\rho C d_e - \rho\sum_{m,b} \binom{m+1}{e-b+1}\binom{e-b+1}{b}(-2)^{e-2b}\left(\frac{2m-e+1}{m+1}\right) C^b (C-1)^{m-b+1} p_{0m}$$

$$-\rho \sum_b \binom{n}{e-b}\binom{e-b}{b}(-2)^{e-2b}C^b(C-1)^{n-b}p_{0n} \quad \text{for } e \geq 0 \quad . \qquad (11)$$

This is a linear first order difference equation with solution

$$d_e = \frac{(\rho C)^e}{\prod_{0\le f<e}[\rho(C-1)+e-f]} \sum_m (C-1)^m p_{0m} - \rho \sum_{1\le f\le e} \frac{(\rho C)^{f-1}}{\prod_{0\le g<f}[\rho(C-1)+e-g]}$$

$$\left[\sum_{m,b} \binom{m+1}{e-b-f+1}\binom{e-b-f+1}{b}(-2)^{e-f-2b}\,\frac{2m-e+f+1}{m+1}C^b(C-1)^{m-b+1}p_{om}\right.$$

$$\left. + \sum_b \binom{n}{e-b-f}\binom{e-b-f}{b}(-2)^{e-f-2b}\,C^b(C-1)^{n-b}p_{0n}\right] \qquad (12)$$

Since $g(x)$ is a polynomial of degree $2n$, d_{2n+1} is equal to zero for all C. This gives nontrivial equations for the p_{0m}. Although they could be solved to find the values of the p_{0m}, this would require more work than to find the p_{0m} with the algorithm given earlier.

Replacing the C in equation (12) by its value in terms of x and y, $C = \frac{y-2x+1}{(1-x)^2}$, gives a formula for $g(x,y)$. The expression for $g(x,y)$ can be expanded in a power series in x and y by using

$$\frac{1}{\prod_{1\le i\le n}(i+x)} = \sum_{1\le i\le n} \frac{(-1)^{i-1}}{(i-1)!(n-i)!(i+x)} \quad \text{for} \quad n\ge 1 \qquad (13)$$

and the binomial theorem. The coefficient of $x^k y^\ell$ is $p_{k\ell}$. This long calculation gives

$$p_{k\ell} = \sum_e \sum_{h\ge 0} \rho^{e+h} \sum_g \binom{e-1}{g-1} \frac{1}{(e-1)!\,g^{h+1}} \sum_{m,i} \binom{h+m}{i}\binom{e}{e+h+m-i-\ell}$$

$$\sum_j \binom{-2(e+h+m)}{j}\binom{e+h+m-i-\ell}{k-e-j-2i}(-1)^{e+g+h+i+k+1}2^{k-e-j-2i}p_{0m}$$

$$+\sum_{e,f}\sum_{h\ge 0}\rho^{f+h}\sum_g\binom{f-1}{e-g}\frac{1}{(f-1)!\,g^{h+1}}\Bigg[\sum_{m,b}\binom{m+1}{e-b-f+1}\binom{e-b-f+1}{b}$$

$$\frac{2m+f-e+1}{m+1}\sum_i\binom{m+h-b+1}{i}\binom{b+f-1}{f+h+m-i-\ell}\sum_j\binom{-2(f+h+\ell)}{j}$$

$$\binom{f+h+m-i-\ell}{k-e-j-2i}(-1)^{e+g+h+i+k}2^{k-f-j-2b-2i}p_{0m}$$

$$+\sum_b\binom{n}{e-b-f}\binom{e-b-f}{b}\sum_i\binom{n+h-b}{i}\binom{b+f-1}{f+h+n-i-\ell-1}$$

$$\sum_j\binom{2(1-f-h-n)}{j}\binom{f+h+n-i-\ell-1}{k-e-j-2i}(-1)^{e+g+h+i+k}\,2^{k-f-j-2b-2i}\,p_{0n}\Bigg]$$

$$+\sum_m\binom{m}{\ell}\binom{-2m}{k+2\ell-2m}(-1)^{k+\ell+m}\,p_{0m} \qquad (14)$$

For small k this gives

$$p_{0\ell} = p_{0\ell}\ ,$$

$$p_{1\ell} = \sum_{m\le\ell}(2m+\rho)(-1)^{\ell-m}\rho^{\ell-m}p_{0m} + [(-1)^{m-n+1}\rho^{m-n+1}p_{0n}]_{\ell\ge n}\ ,\ \text{and}$$

$$p_{2\ell} = \sum_{m\le\ell}(-1)^{\ell-m}\rho^{\ell-m}p_{0m}[4\ell m-2m-m(2m-3)(\tfrac{1}{2})^{\ell-m}$$

$$(2\ell+2m-1-(3m-2)(\tfrac{1}{2})^{\ell-m+1})\rho + (1-(\tfrac{1}{2})^{\ell-m+1})\rho^2] - (\ell+1)p_{0,\ell+1}$$

$$+[(-1)^{\ell-n}\rho^{\ell-n+1}p_{0n}(-2\ell+n2^{n-\ell}+(-1+2^{n-\ell-1})\rho)]_{\text{if }\ell\ge n}$$

$$+[(-1)^{\ell-n}\rho^{\ell-n+1}p_{0n}(1-2^{n-\ell})]_{\text{if }\ell\ge n+1}\ , \qquad (15)$$

but the formulas become rapidly more complex as k increases.

and

$$r_{k\ell m} = \frac{2^{k-1}(\ell+k-1)!(k+2\ell-1)!}{\ell!(2k+2\ell-2)!} r_{1,\ell+k-1,m} - \sum_{0\le j\le k-2} \frac{(k+2\ell-1)!(j+k+2\ell-1)}{(j+k+2\ell+1)!} r_{k-j,\ell+j,m-1}$$

$$+ \sum_{0\le j\le k-2} \frac{(k+2\ell-1)!(k+1)}{(j+k+2\ell)!} r_{k-j+1,\ell+j,m-2} + \frac{1}{k+2\ell} r_{k+1,\ell-1,m-1}$$

$$- \frac{(k+2\ell-1)!}{(2k+2\ell-1)!} r_{2,k+\ell-2,m-1} \quad . \qquad (19)$$

For small m these reduce to

$$q_{0\ell 0} = \frac{1}{(2\ell)!} \quad , \quad q_{1\ell 0} = \frac{1}{(2\ell+1)!} \quad , \quad q_{k\ell 0} = 0 \quad \text{for} \quad k\ge 2 \quad ,$$

$$q_{0\ell 1} = -\frac{1}{(2\ell+1)!} \quad , \quad q_{1\ell 1} = 0 \quad , \quad q_{2\ell 1} = \frac{1}{(2\ell+3)!} \quad , \quad q_{k\ell 1} = 0 \quad \text{for} \quad k\ge 3 \quad ,$$

$$q_{002} = 0 \quad , \quad q_{0\ell 2} = \frac{1}{2\ell(2\ell-1)(2\ell+1)!} \quad \text{for} \quad \ell\ge 1 \quad , \quad q_{1\ell 2} = \frac{-2\ell}{(2\ell+1)(2\ell+3)!} \quad ,$$

$$q_{k,\ell,k-1} = \frac{(k+\ell-1)!(k+2\ell-1)!2^{k-1}}{\ell!(2k+2\ell-2)!(2k+2\ell-1)!} \quad \text{for} \quad k\ge 1, \quad \text{and}$$

$$q_{k\ell k} = \frac{(k+\ell-1)!(k+2\ell-1)!2^{k}}{\ell!(2k+2\ell-2)!(2k+2\ell-1)!} \left[-\frac{(k+\ell)(k-1)}{(k+2\ell)(2k+2\ell-1)} + \frac{\ell}{(k+2\ell)(k+2\ell-1)} \right.$$

$$\left. - \frac{(k+\ell-1)}{(2k+2\ell-1)(2k+2\ell-2)} + \frac{1}{2}\left(H_{2k+2\ell-2} - H_{k+2\ell-1}\right) \right] \quad \text{for} \quad k\ge 2 \quad , \qquad (20)$$

where $H_i = \sum_{1\le j\le i} \frac{1}{j}$. Again the higher terms become rapidly more complex.

These methods were also used to compute the leading terms of the power series

$$N\sum_{\ell} p_{0\ell} = 1+\tfrac{1}{2}\rho^2-\tfrac{1}{6}\rho^3+\tfrac{1}{8}\rho^4-\tfrac{19}{360}\rho^5+\tfrac{11}{432}\rho^6 \ldots \quad . \qquad (21)$$

7. Decreasing Power Series

One can develop a decreasing power series for $Np_{k\ell}$ by defining $s_{k\ell m}$ by $Np_{k\ell} = \sum_{m\ge 0} s_{k\ell m}\rho^{k+2\ell-m}$.

Equation (2), together with the fact that $p_{k\ell} \geq 0$, implies that negative values of m do not need to be considered. The equation for $s_{k\ell m}$ is

$$s_{k,\ell,m+1} + (k+2\ell)s_{k\ell m} = (k+1)s_{k+1,\ell,m+1} + 2(\ell+1)s_{k-1,\ell+1,m+1}$$
$$+s_{k+1,\ell-1,m} + [s_{k-1,\ell,m}]_{\text{if } k=1} + [s_{k,\ell,m+1}]_{\text{if } \ell=n \text{ and } k=0} \tag{22}$$

where terms with $k+\ell>n$ have been omitted. This equation can be solved for successive small values of m by using a two dimensional generating function.

For $m=0$ and 1 the solutions are

$$s_{k\ell 0} = \sum_j \binom{n}{j}\binom{j}{\ell}\frac{(2n-2j)!}{(k+2\ell-2j)!}(-1)^{j+\ell} \quad \text{and}$$

$$s_{k\ell 1} = 2\sum_j \binom{n}{j}\binom{j}{\ell}\frac{(2n-2j)!}{(k+2\ell-2j)!}j[(2n-2k+1)(H_{2n-2j+1}-H_{k+2\ell-2j}-1)+k+2\ell-2j](-1)^{j+\ell}$$
$$-\sum_j \sum_{\lfloor\frac{k}{2}\rfloor+\ell-j\leq i\leq n-j+1} \binom{n}{i+j}\binom{i+j}{j-1}\binom{j}{\ell}\frac{(2n-2j-2i)!(2i+1)!}{(k+2\ell-2j)!}(-1)^{j+\ell} . \tag{23}$$

8. Rational Expressions

Equations (8), (20), and (22) can be used to find the leading and trailing terms of $a_\ell P_{0\ell}$, $a_\ell P_{1\ell}$, and $a_{k+\ell-1} P_{k\ell}$, using formulas for division of power series [8]. The results, using f_i for $\frac{(i-1)!}{(2i-2)!}\sum_j \left[{i-1 \atop j}\right] j i^{j-1}$ and g_i for $\prod_{i\le j\le n} \frac{(2j-2)!}{(j-1)!}$, are

$$
\begin{aligned}
\frac{a_\ell P_{0\ell}}{(2n)!} &= \frac{g_{\ell+1}}{(2\ell)!} + \frac{g_{\ell+1}}{(2\ell+1)!}\left[(2\ell+1)\left(n-\ell-\sum_{\ell+1\le i\le n} f_i\right)-1\right]\rho \\
&+ \frac{g_{\ell+1}}{(2\ell+1)!}\left\{\frac{1}{2\ell(2\ell-1)} - \left[n-\ell-\sum_{\ell+1\le i\le n} f_i\right] + (2\ell+1)\left[n(n-\ell) - \frac{n^2-\ell^2}{2}\right.\right. \\
&-(n-\ell-1)\sum_{\ell+1\le i\le n} f_i + \tfrac{1}{2}\left(\sum_{\ell+1\le i\le n} f_i\right)^2 - \tfrac{1}{2}\sum_{\ell+1\le i\le n} f_i^2 \\
&-\tfrac{1}{2}\sum_{\ell+1\le i\le n}\frac{1}{2i-3} - \sum_{\ell+1\le i\le n}\sum_a \left[{i-2 \atop a}\right]\frac{(i-1)!}{(2i-3)!}\, a i^{a-1} \\
&\left.\left.+\sum_{\ell+1\le i\le n}\sum_a \left[{i-1 \atop a}\right]\binom{a}{2}\frac{(i-1)!}{(2i-2)!}\, i^{a-2}\right]\right\}\rho^2 + \dots \\
&+\binom{n}{\ell}\left\{(2n-2\ell)!\left[\binom{n+1}{3}+\binom{n}{2}-\binom{\ell+1}{3}-\binom{\ell}{2} + 2\ell(2n-2\ell+1)(H_{2n-2\ell+1}-1)\right]\right. \\
&\left.-\frac{\ell}{n-\ell+1}\sum_{0\le i\le n-\ell-1}\binom{n-\ell+1}{i+1}(2n-2\ell-2i)!\,(2i+1)!\right\}\rho^{\frac{n(n-1)-\ell(\ell-1)}{2}-1} \\
&+\binom{n}{\ell}(2n-2\ell)!\,\rho^{\frac{n(n-1)-\ell(\ell-1)}{2}} \quad \text{for } \ell\ge 1 ,
\end{aligned}
$$

$$\frac{a_{\ell}P_{1\ell}}{(2n)!} = \frac{g_{\ell+2}}{(2\ell+1)!} + \frac{g_{\ell+2}}{(2\ell+1)!}\Big(n-\ell-1-\sum_{\ell+2\le i\le n} f_i\Big)\rho$$

$$+ \frac{g_{\ell+2}}{(2\ell+1)!}\Bigg\{\frac{-\ell}{(\ell+1)(2\ell+1)(2\ell+3)} + n(n-\ell-1) - \frac{n^2-(\ell+1)^2}{2}$$

$$-(n-\ell-2)\sum_{\ell+2\le i\le n} f_i + \tfrac{1}{2}\Big(\sum_{\ell+2\le i\le n} f_i\Big)^2 - \tfrac{1}{2}\sum_{\ell+2\le i\le n} f_i^{\,2} - \tfrac{1}{2}\sum_{\ell+2\le i\le n}\frac{1}{2i-3}$$

$$-\sum_{\ell+2\le i\le n}\sum_{a}\begin{bmatrix} i-2\\ a\end{bmatrix}\frac{(i-1)!}{(2i-3)!}a i^{a-1}$$

$$+\sum_{\ell+2\le i\le n}\sum_{a}\begin{bmatrix} i-1\\ a\end{bmatrix}\binom{a}{2}\frac{(i-1)!}{(2i-2)!}i^{a-2}\Bigg\}\rho^2 \;+\dots$$

$$+\binom{n}{\ell}\Bigg\{(2n-2\ell)!\Bigg[\binom{n+1}{3}+\binom{n}{2}-\binom{\ell+2}{3}-\binom{\ell+1}{2}$$

$$+2\ell(2n-2\ell+1)(H_{2n-2\ell+1}-2)+2\ell\Bigg] -$$

$$-\frac{\ell}{n-\ell+1}\sum_{0\le i\le n-\ell-1}\binom{n-\ell+1}{i+1}(2n-2i-2\ell)!(2i+1)!\Bigg\}\rho^{\frac{n(n-1)-\ell(\ell+1)}{2}-1}$$

$$+\binom{n}{\ell}(2n-2\ell)!\,\rho^{\frac{n(n-1)-\ell(\ell+1)}{2}}\quad\text{, and}$$

$$\frac{a_{k+\ell-1}(k+\ell-1)!\,2^{k-1}}{\ell!(2n)!}P_{k\ell} = g_{k+\ell}\frac{2^{k-1}(k+\ell-1)!(k+2\ell-1)!}{\ell!(2k+2\ell-2)!(2k+2\ell-1)!}$$

$$+g_{k+\ell}\frac{2^{k-1}(k+\ell-1)!(k+2\ell-1)!}{\ell!(2k+2\ell-2)!(2k+2\ell-1)!}\Bigg\{n-k-\ell+1-\sum_{k+\ell\le i\le n} f_i + \frac{2\ell}{(k+2\ell)(k+2\ell-1)}$$

$$-\frac{2(k+\ell)(k-1)}{(k+2\ell)(2k+2\ell-1)} - \frac{2(k+\ell-1)}{(2k+2\ell-1)(2k+2\ell-2)} + H_{2k+2\ell-2} - H_{k+2\ell-1}\Bigg\}\rho \;+\dots$$

$$+\binom{n}{\ell}\Bigg\{\sum_j\binom{n-\ell}{j}\frac{(2n-2\ell-2j)!}{(k-2j)!}(-1)^j\Bigg\{\binom{n+1}{3}+\binom{n}{2}-\binom{k+\ell}{3}-\binom{k+\ell-1}{2}$$

$$+2(j+\ell)[(2n-2\ell-2j+1)(H_{2n-2j-2\ell+1}-H_{k-2j}-1)+k-2j]\Bigg\}$$

$$-\sum_j\binom{n-\ell}{j}\frac{(j+\ell)(-1)^j}{(k-2j)!(n-j-\ell+1)}\sum_{\lfloor\frac{k}{2}\rfloor-j\le i\le n-j-\ell-1}\binom{n-j-\ell+1}{i+1}(2n-2i-2j-2\ell)!$$

$$(2i+1)!\Bigg\}\;\rho^{\frac{n(n-1)-(k+\ell-1)(k+\ell-2)}{2}-k}$$

$$+\binom{n}{\ell}\sum_j\binom{n-\ell}{j}\frac{(2n-2j-2\ell)!}{(k-2j)!}(-1)^j\,\rho^{\frac{n(n-1)-(k+\ell-1)(k+\ell-2}{2}-k+1}$$

for $k\ge 2$. (24)

One can also solve for $p_{k\ell}$ when n is small. The results are given in Table 1.

9. Conclusion

A large amount of partial information about the probability that the bottom level of the buddy system has ℓ pairs of cells which are both in use and k pairs in which just one is in use has been collected. The value can be expressed as a rational function with a known denominator. The leading and trailing terms of the numerator are given. The formulas can be used to calculate $p_{k\ell}$ when ρ is very large ($\rho >> 2n$) or very small ($\rho << 1$), but do not help for the important case $1 << \rho << 2n$. They are useful for computing $\sum_{\ell} p_{0\ell}$ only when p_{00} or p_{0n} is a good approximation to the entire sum. The results of Purdom and Stigler [3] indicate that there are interesting asymptotic results for the case $1 << \rho << 2n$, but this study suggests they will not be easy to find.

Certain cases do have simple formulas. These include small n, small powers of ρ, $k+\ell$ near n, and certain sums of the probabilities.

References

1. Knowlton, Kenneth C. A fast storage allocator. Comm. ACM 8 (1965), pp. 623-625.

2. Knuth, Donald E. The Art of Computer Programming, Vol. 1, Addison-Wesley, Reading, Mass. 1973, pp. 442-445.

3. Purdom, Paul W., Jr. and Stigler, Stephen M. Statistical Properties of the Buddy System. J. ACM 17 (1970), pp. 683-697.

4. Purdom, Paul W., Jr., Stigler, Stephen M., and Cheam, Tat-Ong. Statistical Investigations of Three Storage Allocation Algorithms. Bit 11 (1971), pp. 187-195.

5. Standish, T. A., Fast-Fit: A Dynamic Memory Allocation Technique, University of California at Irvine, 1978.

6. Knuth, Donald E., ibid, p. 65-66.

7. Knuth, Donald E., Big Omicron and Big Omega and Big Theta, SIGACT News, Vol. 8, No. 2 (1976) p. 18.

8. Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, U. S. Government Printing Office, 1964, p. 15.

TABLE I

Formulas for $\frac{N\,p_{k\ell}}{(2n)!}$ when $1\le n\le 4$.

n=1	n=2	n=3
$p_{00}=1$ $\quad p_{10}=\rho$ $p_{01}=\frac{\rho^2}{2}$	$p_{00}=1$ $\quad p_{10}=\rho$ $\quad p_{20}=\frac{2\rho^3}{3!(2+\rho)}$ $p_{01}=\frac{\rho^2(6+\rho)}{3!(2+\rho)}$ $\quad p_{11}=\frac{\rho^3}{3!}$ $p_{02}=\frac{\rho^4}{4!}$	$p_{00}=1$ $\quad p_{10}=\rho$ $\quad p_{20}=\frac{4\rho^3(60+25\rho+6\rho^2)}{5!(2+\rho)(12+5\rho+\rho^2)}$ $\quad p_{30}=\frac{8\rho^5}{5!(12+5\rho+\rho^2)}$ $p_{01}=\frac{4\rho^2(360+210\rho+55\rho^2+3\rho^3)}{5!(2+\rho)(12+5\rho+\rho^2)}$ $\quad p_{11}=\frac{4\rho^3(60+25\rho+3\rho^2)}{5!(12+5\rho+\rho^2)}$ $\quad p_{21}=\frac{4\rho^5(3+\rho)}{5!(12+5\rho+\rho^2)}$ $p_{02}=\frac{\rho^4(60+13\rho+\rho^2)}{5!(12+5\rho+\rho^2)}$ $\quad p_{12}=\frac{\rho^5}{5!}$ $p_{03}=\frac{\rho^6}{6!}$

n=4

$p_{00}=1$ $\qquad p_{10}=\rho$

$$p_{01}=\frac{24\rho^2(15\rho^6+550\rho^5+5901\rho^4+35980\rho^3+136500\rho^2+292320\rho+302400)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)(\rho+2)}$$

$$p_{11}=\frac{24\rho^3(15\rho^5+340\rho^4+2961\rho^3+14350\rho^2+40320\rho+50400)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)}$$

$$p_{02}=\frac{6\rho^4(3\rho^5+110\rho^4+1513\rho^3+8806\rho^2+30240\rho+50400)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)}$$

$$p_{12}=\frac{6\rho^5(3\rho^3+47\rho^2+322\rho+840)}{7!(\rho^3+9\rho^2+46\rho+120)}$$

$$p_{03}=\frac{\rho^6(\rho^3+21\rho^2+202\rho+840)}{7!(\rho^3+9\rho^2+46\rho+120)}$$

$$p_{13}=\rho^7/7!$$

$$p_{04}=\rho^8/8!$$

$$p_{20}=\frac{48\rho^3(45\rho^5+565\rho^4+3927\rho^3+16870\rho^2+40320\rho+50400)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)(\rho+2)}$$

$$p_{30}=\frac{48\rho^5(10\rho^3+75\rho^2+322\rho+840)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)}$$

$$p_{40}=\frac{48\rho^7}{7!(\rho^3+9\rho^2+46\rho+120)}$$

$$p_{21}=\frac{24\rho^5(6\rho^4+85\rho^3+499\rho^2+1806\rho+2520)}{7!(\rho^3+9\rho^2+46\rho+120)(\rho^2+5\rho+12)}$$

$$p_{31}=\frac{24\rho^7(\rho+4)}{7!(\rho^3+9\rho^2+46\rho+120)}$$

$$p_{22}=\frac{6\rho^7(\rho^2+7\rho+20)}{7!(\rho^3+9\rho^2+46\rho+120)}$$