DISTRIBUTIVE LATTICES
OF ORDER n *

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Abstract

The purpose of this work is to investigate distributive lattices which are chain based of order n and subsumed with respect to certain sublattices containing their centers.

The particular ideas presented here are new. The general ideas first appeared around 1970 in the work of G. Rousseau, and later in the works of G. Epstein and A. Horn.

Introduction and Background

As Boolean algebra is the algebra of Boolean 2-valued switching theory, so are the P1 algebras (1=0,1,2, or 3) the corresponding algebras for multiple-valued switching theory. The first formulation of P2-algebras was by R. Rosenbloom in 1942.1 It was more complicated than the formulation of P3-algebras given just below. The first formulation of P0-algebras was by T. Traczyk in 1963.2 The first formulation of P1 and P2-algebras was by G. Epstein and A. Horn in 1974.3 The class of P3-algebras of order 2 is precisely the class of Boolean algebras, and connections of P3-algebras with multiple-valued switching theory may be obtained through appropriate generalizations of known results for Boolean algebras and 2-valued switching theory.

Let A be a distributive lattice with least element 0, greatest element 1, and center of complemented elements B. The complement of x in B is denoted by x.

The dual of A is denoted by A. For any two elements x, y in A, x and y are used for the operation join, x ∨ y, or for the operation meet. A finite sequence

0 ≤ e0 ≤ e1 ≤ ... ≤ en-1 ≤ 1

is called a chain base of A if A is generated by B ∨ {e0, ..., en-1}. If A has a chain base then A is called a P0-algebra. The order of A is the smallest number of elements in a chain base of A. The largest x in A (if it exists) such that x ∨ y is denoted by x ∨ y or by x ∨ y. The lattice is called a Heyting algebra if x ∨ y exists for all x, y in A. When x ∨ y exists, it is called the pseudo-complement of x. An element x is called dense when x ∨ 0 = 0. The lattice A is called pseudo-complemented if x ∨ 0 exists for all x in A. If A is a P0-algebra and if e0, ..., en-1 is a chain base of A, then A is a Heyting algebra satisfying the identity (x ∨ y) ∨ (y ∨ x) = 1. Moreover, there exists a unique strictly increasing Heyting chain base 0 = f0 < ... < fn-1 = 1 such that fn+1 = fn+1 for all i ≥ 0. In this case A is a P1-algebra.

Let x be the greatest b in A such that b ≤ x. A P2-algebra is a P1-algebra if and only if it is a direct product of Pn algebras of maximum order n. The class of P2-algebras of order n can be characterized as an equational class of

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algebras $A; V, A; e_0, e_1, e_2, \ldots, e_{n-1}$. A $P_j$-Algebra is a $P_2$-algebra $A; e_0, e_1, \ldots, e_{n-1}$ such that $e_{n-2} = 0$. The order of a $P_j$-algebra $A$ is one more than the number of prime ideals in a chain of prime ideals of maximum length. The class of all $P_j$-algebras can be characterized by simple properties of the set of their prime ideals; the class of all $P_2$-algebras can also be so characterized [5, Section 7].

When a lattice of multiple-valued switching functions is defined over a finite set of variables and contains a finite number of elements, it is of interest to note the following: (1) Every $P_0$-algebra with a finite number of elements is a $P_2$-algebra; (2) Every $P_2$-algebra with a finite number of elements is a direct product of chains of finite length; (3) Every $P_j$-algebra with a finite number of elements is a direct product of chains of finite and equal length [6], [7, Theorem 4.11].

In this development the central role of $P_2$, the complemented elements of $A$, is quite clear. The arrow connective $\text{x} \quad \downarrow \text{y}$ stands for the greatest complemented element $b \in B$ satisfying $x \wedge b = y$, and is related to the strict implication of C.I. Lewis or Casimir Lewy as described by G. Epstein and A. Horn in 1976. In the general case, $\text{x} \quad \downarrow \text{y}$ stands for the greatest element $q$ in $Q$ satisfying $xq \leq y$, where $Q$ is a sublattice of $A$ containing $0, 1$. The lattice $A$ is said to be subresiduated with respect to $Q$ if $x \quad \downarrow \text{y}$ exists for each $x, y$ in $A$. When $Q = A$, $A$ is commonly called an Heyting algebra.

Consider a propositional calculus $R$ with symbols AND, $\lor$ OR $\rightarrow$ and $\neg$ for conjunction, disjunction, implication and negation, and possibly with a symbol $\bot$ for affirmation or necessitation. By assigning to the propositional variables values in a subresiduated lattice, a valuation $v(a)$ is obtained for each formula $a$ by the rules $v(a \land b) = v(a) \wedge v(b)$, $v(a \lor b) = v(a) \vee v(b)$, $v(\neg a) = v(a) \rightarrow 0$, $v(0) = 1$, $v(1) = 1$. A formula $a$ in $R$ is said to be valid in $A$ if $v(a) = 1$ for every assignment in $A$. A logic is said to be characterized by a class $K$ of subresiduated lattices if it consists of those formulas which are valid in every member of $K$.

This framework provides a unified method of classifying several known calculi and lends naturally to new calculi which are of some algebraic and philosophical interest. For example, it is well known that the intuitionistic propositional calculus is characterized by the class of all Heyting algebras. In the technical description below the major concern will be with $Q$ an extension of the center $B$, satisfying $B \preceq Q$, where $B$ will be subresiduated with respect to $B$ and generated by $Q \cup \{c_0, \ldots, c_{n-1}\}$, with the $c_i$ forming a Heyting chain base. This will be a proper extension of $P_j$-lattices into distributive lattices of order $n$. The corresponding propositional calculus will be between $R_4$ and $R_5$ (that is, between the low versions of the Lewis systems $S_4$ and $S_5$) [6, Sections 2 and 3].

**Technical Description**

Consider distributive lattices with chain $0 = c_0 \leq c_1 \leq \cdots \leq c_{n-2} \leq c_{n-1} = 1$ such that for each $c_i$, $c_{i+1}$ is the least dense element in the interval $[c_i, 1]$. In other words, $c_{i+1}$ is the least $d$ satisfying $d \preceq c_i$ and $d \neq 0$ for no $c_i$. If such a lattice is a Heyting algebra, then this chain is called a Heyting chain with, for each $i$, $c_i \downarrow c_{i+1} = 1$. A lattice is said to be of order $n$ if it contains such a chain having exactly $n$ terms. These notions are abstracted from Section 3 of [5].

It is mentioned here that there is an alternate definition of lattices of order $n$ which has been considered—viz., as lattices in which the maximum length of a chain of properly ascending prime ideals in $n-1$-dimensional [5, Section 7] or [2]. However such an approach does not in general guarantee the existence of elements $c_1, c_2, \ldots, c_n$. In particular, a prime ideal characterization for the $P_2$-lattices, which were first introduced by T. Tarski in [9], is still unknown [5, pp. 83-84]. It is these complexities which motivate the approach given in the previous paragraph.

Let $A$ be a lattice of order $n$ where for each $x$ in $A$, $x = \bigvee_{i=0}^{n-1} c_{i+1} \downarrow c_i$ in $C$, a fixed sublattice of $A$ containing $0, 1$. The lattice $A$ is said to be chain based with respect to $C$. The notation used here is also in Section 1 of
[5]. The case where \( C \) is the center of complemented elements of \( A \) is defined in Section 2 of [5], with the remainder of that paper devoted to consequent developments for this case (viz., to the \( P_i \)-lattices, \( i = 0,1,2,3 \), where here \( P_3 \)-lattices stand for Post algebras [5, pp. 73-74]).

A basic question concerns the nature of \( C \) for arbitrary \( A \). To illustrate, while the lattice \( A_1 \) has center \( C_1 = \{0,b,5,1\} \), which is clearly sufficient for \( A_1 \) to be chain based with respect to \( C_1 \), the lattice \( A_2 \) has center \( C_2 = \{0,1\} \), and this is clearly insufficient for \( A_2 \) to be chain based with respect to \( C_2 \). While it is furthermore clear that any lattice \( A \) is trivially chain based with respect to itself, it is evident from the last example that a lattice \( A \) may be chain based with respect to a proper sublattice of \( A \). For the lattice \( A_2 \) is chain based with respect to the proper sublattice \( \{0,b,c,e_2,1\} \). This is immediate from the fact that \( A_1 \) is chain based with respect to \( C_1 \).

Accordingly, the notion of a lattice center will be extended to that of a lattice exocenter. This will be accomplished as follows. Let \( b \) in \( A \), \( c \) in \( A \) satisfy the 4 conditions:

1. \( c \) is the greatest element \( x \) in \( A \) such that \( bx \)
2. \( c \) is the least element \( x \) in \( A \) such that \( bx \)
3. \( b \) is the greatest element \( x \) in \( A \) such that \( cx \)
4. \( b \) is the least element \( x \) in \( A \) such that \( cx \)

These conditions are satisfied by any complemented pair \( b,c \) of the center of \( A \), as illustrated in \( A_1 \), but also by the pair \( b,c \) illustrated in \( A_2 \). This may be regarded as a generalization of the notion of a complemented pair. The properties for the definition of the exocenter of a distributive lattice \( A \) as being the sublattice of \( A \) generated by all pairs \( b,c \) satisfying (1), (ii), (iii), (iv). In particular, the exocenter of \( A_1 \) is \( C_1 \) and the exocenter of \( A_2 \) is \( \{0,b,c,e_2,1\} \). By definition, the exocenter of \( A \) always contains the center of \( A \). When \( A \) is a double Heyting algebra (that is, both \( A \) and its dual \( A^\dual \) are Heyting algebras) with \( y \leq x \) denoting the greatest \( x \) such that \( yx \); and \( y \not\leq x \) denoting the least \( x \) such that \( yx = x \); the above 4 conditions simply become \( b \leq c = c \leq b \) and \( c \leq b = b \leq c \).

Moreover, \( C \) denotes the exocenter of a distributive lattice \( A \).

It is seen at once that the distributive lattices \( A \) which are chain based with respect to the exocenter \( C \) are a proper extension of the distributive lattices \( A \) which are chain based with respect to the center \( B \) (the so-called \( P_3 \)-lattices, \( i = 0,1,2,3 \)). The exocenter is thus an appropriate extension of the center - the resulting class of chain based distributive lattices is broader than the class of \( P_3 \)-lattices. This applies particularly to finite distributive lattices, each of which obviously has a Heyting chain and so is of order \( n \) for some finite \( n \).

The above aimed at this last mentioned development for finite distributive lattices, yet there are additional logical or algebraic relationships involving the exocenter, similar to those which involve the center and appear in [4,6]. These relationships provide additional logical and algebraic insight and provide tools for further development of chain based lattices. The development for lattices which are characterized by lattices subresiduated with respect to their excenters clearly falls between the case for lattices subresiduated with respect to themselves (Heyting algebras) and the case for lattices subresiduated with respect to their centers [5, Section 4].

Following [6], let the lattice \( A \) be subresiduated with respect to a sublattice \( Q \) containing \( 0,1 \). That is, for each \( y \) in \( A \), \( z \) in \( A \), there is an element \( x \) in \( Q \) such that \( yx \leq z \) if and only if \( qx \leq z \). This \( x \) is denoted by \( y \triangleleft z \).

Suppose now that \( A \) is chain based and \( \not\leq z \). Then it is shown that \( A \) is a Heyting algebra, with

\[ y \triangleleft z = \bigwedge_{i=1}^{n} e_i (ye_i \not\leq z) . \]
Lattices subresiduated with respect to general $\mathcal{Q}$ and their corresponding logics are characterized in Sections 1 and 2 of [6]. They need not be Heyting algebras, nor even pseudo-complemented. Details concerning these logics are given in the appendix at the end of this paper. In particular, the pivotal axiom $Y(x \circ (x \rightarrow x))$ is motivated from Theorem 22 of [6]. An algebraic discussion is just below.

Let $A$ be subresiduated with respect to $A_1$, with $y \circ x (x \rightarrow y)$ for all $x, y$ in $A_1$; let the dual statement hold for $A_1$. Then $A$ is a double Heyting algebra and $\mathcal{Q}$ contains the exocenter of $A$. For let $b, c$ satisfy (i) and (ii).

From $b(b \circ c) \leq c \leq b(b \circ c)$

it follows easily that $(b \circ c) = c \circ (b \circ c)$, and similarly using (i) and (iv) for $b$.

It follows that $x \circ y = y \circ x$, which follows from Theorem 22 of [6], which states the equivalence of $x \circ y (x \rightarrow y)$ with $x \circ y = y \circ x$. There is a dual statement for $A_1$. Thus $x \circ y$ is the pseudo-complement of $x$, with a dual statement for $A_1$.

It remains to explore the further connections to lattices which are chain based with respect to their exocenters. This has been done for lattices which are chain based with respect to their centers in [3] and in Section 4 of [5]. Note the special role of center $B$ relative to general $\mathcal{Q}$ in [6]. Details may be found in the appendix at the end of this paper.

Finally, there is a direct suggestion from the development of chain based lattices [5, Section 4] concerning a canonical representation for each element $x$ in $A$. Namely, that if $A$ is subresiduated and chain based with respect to $C$, $x = \bigvee \{e_1(x) \mid x \in A\}$

This is also suggested by the work of G. Rousseau in [8]. To illustrate, this last representation holds not only for each $x$ in $A_1$, but also for each $x$ in $A_2$. This is easily verified in the figures shown above. Because $P_2$-lattices can be characterized as a class of algebras

\begin{align*}
\langle l, v, \lambda, B, e_0, \ldots, e_{n-1} \rangle & \quad [5, \text{Section 5}], \\
\text{it is natural to seek a similar result for} \langle l, v, \lambda, C, e_0, \ldots, e_{n-1} \rangle.
\end{align*}

Summary

The notions of complement and center have been generalized to provide a proper extension of $P_2$-lattices into distributive lattices of order $n$. The case when these lattices are subresiduated lead to new logics of some algebraic and philosophical interest.

Appendix

This appendix gives propositional calculus which lie between the logic system $\mathcal{R}$ characterized by the class of all subresiduated lattices, and the logic system characterized by the class of all lattices which are subresiduated with respect to their centers (the so-called $\beta$-algebras of [4]). For case of presentation, each such system is obtained by adding to $\mathcal{R}$ a single axiom, and each such addition is properly stronger than the one before. At the end of this appendix, three additional systems are given which do not fall precisely within this scheme, but are deserving of inclusion. All of the systems in this appendix are given in order to be illustrative - there is no attempt to be exhaustive.

A propositional calculus is a set of axioms together with rules of inference from which provable formulas are obtained. The rules of inference for all of the logics below are:

(a) Substitution; that is, any formula $P_i$ may be substituted for propositional variable $X_j$ in a formula $F$, provided only that all occurrences of $X_j$ in that formula $F$ are replaced by $P_i$.

(b) Detachment; that is, if $P_i$ is a provable formula and $P_i \rightarrow F_j$ is a provable formula, then $F_j$ is a provable formula.

The axioms for $\mathcal{R}$ are:

\begin{align*}
(1) & \quad X_1 \rightarrow X_1 \\
(2) & \quad (X_1 \rightarrow X_2) \rightarrow (X_2 \rightarrow (X_1 \rightarrow X_2)) \\
(3) & \quad (X_1 \rightarrow (X_2 \rightarrow X_3)) \rightarrow ((X_1 \rightarrow X_2) \rightarrow (X_1 \rightarrow X_3)) \\
(4) & \quad (X_1 \rightarrow X_2) \rightarrow X_1 \\
(5) & \quad (X_1 \rightarrow X_2) \rightarrow X_2 \\
(6) & \quad (X_1 \rightarrow X_2) \rightarrow (X_1 \rightarrow X_3) \rightarrow (X_2 \rightarrow X_3)) \\
(7) & \quad X_1 \rightarrow (X_1 \rightarrow X_2) \\
(8) & \quad X_2 \rightarrow (X_1 \rightarrow X_2) \\
(9) & \quad (X_1 \rightarrow X_2) \rightarrow (X_2 \rightarrow X_3) \rightarrow (X_1 \rightarrow X_3) \\
(10) & \quad (X_1 \rightarrow X_2) \rightarrow (X_2 \rightarrow X_3) \rightarrow (X_1 \rightarrow X_3) \rightarrow (X_1 \rightarrow X_2) \\
& \quad \text{OR} (X_1 \rightarrow X_3) \\
\end{align*}
(A11) \neg X_1 \rightarrow (X_1 \rightarrow X_2)
(A12) (X_1 \rightarrow \neg X_1) \rightarrow X_1
(A13) (\neg X_1 \rightarrow (X_1 \rightarrow X_1))
(A14) (\neg X_1 \rightarrow X_1) \rightarrow \neg X_1

This system is quite weak with respect to the symbol \(\neg\). Consider, for example, 
\(A = \{0, b, c, 1\}\), \(Q = \{0, b, 1\}\) as shown below:

Here \(b = \neg a \rightarrow \neg c \rightarrow \neg e = 1\). Thus the additional axiom

\(\neg \neg X_1 \rightarrow X_1\)

(A15)

is a proper extension of R4, but not sufficient for requiring \(Q\) to contain all the pseudo-complemented elements of \(A\). This is shown in the example just below, where

\(A = \{0, b, c, e, 1\}\), \(Q = \{0, b, e, 1\}\).

In this example \(\neg a \rightarrow \neg X_1 \rightarrow X_1\) for all \(a\) in \(A\) but \(Q\) does not contain the pseudo-complemented element \(e\).

Consider the replacement of (A15) with

\(X_1 \rightarrow (X_1 \lor (X_1 \rightarrow X_2))\).

It has been shown that \(Q\) now contains all the pseudo-complemented elements of \(A\), so that this is properly stronger than the previous system. Furthermore an intuitionist implicative connective \(\rightarrow\) may be introduced by letting \(X_1 \rightarrow X_2\) be an abbreviation for \(X_1 \lor (X_1 \rightarrow X_2)\).

Next (A15)' is replaced with

\(\neg X_1 \lor \neg \neg X_1\).

(A15)'

It is transparent from (A15)' that \(Q\) must contain only complemented elements of \(A\). This logic, called R5, is characterized by the class of lattices subresiduated with respect to all Boolean subalgebras of its centers.

Lastly, (A15)'' is replaced with

\(\neg X_1 \lor \neg \neg X_1 \lor \neg (X_1 \rightarrow X_2) \lor \neg (X_1 \land X_2)\).

(A15)''

It is not at all transparent that this last system is properly stronger than the previous system, R5, or that this logic is characterized by the class of all lattices subresiduated with respect to their centers (these were called B-algebras in [4], and the logic developed in [6]).

There are three remaining systems to be mentioned.

First, consider the addition of the axiom

\([\neg X_1 \lor X_2] \lor (X_1 \rightarrow X_1)\)

to R4. This system is called R4.3, and transparently requires any elements

\(a, b, c, d, e\) in \(Q\) to satisfy \(a \lor b \lor c \lor d \lor e = 1\).

Thus this logic is characterized by the class of all lattices subresiduated with respect to sublattices \(Q\) which satisfy this identity. It is sufficient, in fact, for these sublattices \(Q\) to be linearly ordered [6, pp.203-204].

Second, consider the addition of the axiom

\(X_1 \rightarrow X_2 \lor (X_2 \rightarrow X_1)\)

to R4. This system is stronger than the one just discussed, and stronger than R4' + (A15)''.

For

\(\neg \neg X_1 \lor (X_1 \rightarrow X_2)\)

follows easily by substitution, while (A15)' follows straightforwardly from Theorem 3.1(vi) of [4] and Theorem 32 of [6].

Third, consider the addition of two axioms to R4; namely this last axiom (Ax)

\(X_1 \lor X_2 \lor (X_2 \lor X_1)\)

together with the axiom (A10)

\(\neg X_1 \lor \neg \neg X_1 \lor \neg (X_1 \lor X_2) \lor \neg (X_1 \land X_2)\).

In [4], B-algebras satisfying the identity

\((x \lor y) \lor (x \land y)) = 1\)

were called P-algebras. This last logic system is called PC.

It is characterized by the class of all P-algebras.

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References


