DISTRIBUTIVE LATTICES OF ORDER n *

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Abstract

The purpose of this work is to investigate distributive lattices which are chain based of order n and subresiduated with respect to certain sublattices containing their centers.

The particular ideas presented here are new. The general ideas first appeared around 1970 in the work of G. Rousseau, and later in the works of G. Epstein and A. Horn.

Introduction and Background

As Boolean algebra is the algebra for 2-valued switching theory, so are the P_i algebras (i=0,1,2, or 3) the corresponding algebras for multiple-valued switching theory. The first formulation of P_3-algebras was by P.C. Rosenbloom in 1942.[1] It was more complicated than the formulation of P_3-algebras given just below. The first formulation of P_0-algebras was by T. Traczyk in 1963.[2] The first formulation of P_1 and P_2-algebras was by G. Epstein and A. Horn in 1974.[3] The class of P_3-algebras of order 2 is precisely the class of Boolean algebras, and connections of P_i-algebras with multiple-valued switching theory may be obtained through appropriate generalizations of known results for Boolean algebras and 2-valued switching theory.

Let A be a distributive lattice with least element 0, greatest element 1, and center of complemented elements B. The complement of x in B is denoted by $\bar{x}$. The dual of A is denoted by A^d. For any two elements x, y in A, $x \vee y$ is used for the operation join, $x \wedge y$ or xy for the operation meet. A finite sequence $0 \le e_0 \le e_1 \le \ldots \le e_{n-1} = 1$ is called a chain base of A if A is generated by $B \cup \{e_0, \ldots, e_{n-1}\}$. If A has a chain base then A is called a P_0-algebra. The order of A is the smallest number of elements in a chain base of A. The largest z in A (if z exists) such that $xz \le y$ is denoted by $x \to y$ or by $x \overset{A}{\to} y$. The lattice is called an Heyting algebra if $x \to y$ exists for all x, y in A. When $x \to 0 = \neg x$ exists, it is called the pseudo-complement of x. An element x is called dense when $x \to 0 = 0$. The lattice A is called pseudo-complemented if $x \to 0$ exists for all x in A. If $\langle A; e_0, \ldots, e_{n-1} \rangle$ is a P_0-algebra and if $e_i \to e_j$ exists for all i,j, then A is a Heyting algebra satisfying the identity $(x \to y) \vee (y \to x) = 1$. Moreover, there exists a unique strictly increasing Heyting chain base $0 = f_0 < \ldots < f_{m-1} = 1$ such that $f_{i+1} \to f_i = f_i$ for all $i > 0$. In this case $\langle A; f_0, \ldots, f_{m-1} \rangle$ is called a P_1-algebra. Let $!x$ be the greatest b in B such that $b \le x$. A P_2-algebra is a P_1-algebra $\langle A; e_0, \ldots, e_{n-1} \rangle$ such that $!e_i$ exists for all i. A is a P_2-algebra of order n if and only if it is a direct product of Post algebras of maximum order n. The class of P_2-algebras of order $\le n$ can be characterized as an equational class of

[1]Rosenbloom, P.C. "Post algebras I. Postulates and general theory", Amer. J. Math. 64 (1942) pp.167-188.

[2]Traczyk, T. "Axioms and some properties of Post algebras", Coll. Math. 10 (1963) pp.193-209.

[3]Epstein, G. and Horn, A. "Chain based lattices" Pac. Jour. Math. 55 (1974) pp.65- 84.

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algebras $< A; \vee, \wedge, \rightarrow, !, e_0, \ldots, e_{n-1} >$. A P_3-Algebra is a P_2-algebra $< A; e_0, \ldots, e_{n-1} >$ such that $!e_{n-2} = 0$. The order of a P_0-algebra A is one more than the number of prime ideals in a chain of prime ideals of maximum length. The class of all P_1-algebras can be characterized by simple properties of the set of their prime ideals; the class of all P_2-algebras can also be so characterized [5, Section 7].

When a lattice of multiple-valued switching functions is defined over a finite set of variables and contains a finite number of elements, it is of interest to note the following: (1) Every P_0-algebra with a finite number of elements is a P_2-algebra; (2) Every P_2-algebra with a finite number of elements is a direct product of chains of finite length; (3) Every P_3-algebra with a finite number of elements is a direct product of chains of finite and equal length [3], [5, Theorem 4.11].

In this development the central role of B, the complemented elements of A, is quite clear. The arrow connective $x \overset{B}{\rightarrow} y$ stands for the greatest complemented element b in B satisfying $x \wedge b \leq y$, and is related to the strict implication of C.I. Lewis or Casimir Lewy as described by G.Epstein and A. Horn in 1976.[4] In the general case, $x \overset{Q}{\rightarrow} y$ stands for the greatest element q in Q satisfying $x \wedge q \leq y$, where Q is a sublattice of A containing 0,1. The lattice A is said to be subresiduated with respect to Q if $x \overset{Q}{\rightarrow} y$ exists for each x,y in A. When Q=A, A is commonly called an Heyting algebra.

Consider a propositional calculus R with symbols AND, OR $\supset$ and ~ for conjunction, disjunction, implication and negation, and possibly with a symbol $\Box$ for affirmation or necessitation. By assigning to the propositional variables values in a subresiduated lattice, a valuation $v(\alpha)$ is obtained for each formula α by the rules $v(\alpha \text{ AND } \beta) = v(\alpha) \wedge v(\beta)$, $v(\alpha \text{ OR } \beta) = v(\alpha) \vee v(\beta)$, $v(\alpha \supset \beta) = v(\alpha) \rightarrow v(\beta)$, $v(\sim\alpha) = v(\alpha) \rightarrow 0$ and $v(\Box\alpha) = 1 \rightarrow v(\alpha)$. A formula α in R is said to be valid in A if $v(\alpha) = 1$ for every assignment in A. A logic is said to be characterized by a class K of subresiduated lattices if it consists of those formulas which are valid in every member of K.

[4]Epstein, G. and Horn, A., "Logics which are characterized by subresiduated lattices," Zeitschr. f. math. Logik und Grundlagen d. Math., 22 (1976) pp.199-210.

This framework provides a unified method of classifying several known calculi and leads naturally to new calculi which are of some algebraic and philosophic interest. For example, it is well known that the intuitionist propositional calculus is characterized by the class of all Heyting algebras. In the technical description below the major concern will be with Q an extension of the center B, satisfying $B \subseteq Q \subseteq A$, where A will be subresiduated with respect to Q and generated by $Q \cup \{e_0, \ldots, e_{n-1}\}$, with the e_i forming a Heyting chain base. This will be a proper extension of P_i-lattices into distributive lattices of order n . The corresponding propositional calculus will be between R4 and R5 (that is, between the Lewy versions of the Lewis systems S4 and S5) [6, Sections 2 and 3].

Technical Description

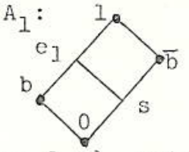
Consider distributive lattices with chain $0 = e_0 < e_1 < \ldots < e_{n-2} < e_{n-1} = 1$ such that for each e_i, e_{i+1} is the least dense element in the interval $[e_i, 1]$. In other words, e_{i+1} is the least d satisfying $e_i \leq d \leq 1$ and $d \wedge y = 0$ for no $y > 0$. If such a lattice is a Heyting algebra, then this chain is called a *Heyting chain* with, for each i, $e_{i+1} \rightarrow e_i = e_i$ and $e_i \rightarrow e_{i+1} = 1$. A lattice is said to be *of order n* if it contains such a chain having exactly n terms. These notions are abstracted from Section 3 of [5].

It is mentioned here that there is an alternate definition of lattices of order n which has been considered--viz., as lattices in which the maximal length of a chain of properly ascending prime ideals in n-1. See for example [5, Section 7] or [2]. However such an approach does not in general guarantee the existence of elements e_i [1,3,5]. In particular, a prime ideal characterization for the P_0-lattices, which were first introduced by T. Traczyk in [9], is still unknown [5, pp.83-84]. It is these complexities which motivate the approach given in the previous paragraph.

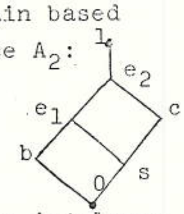
Let A be a lattice of order n where for each x in A, $x = \bigvee_{i=1}^{n-1} c_i e_i$, where each c_i in C, a fixed sublattice of A containing 0, 1. The lattice A is said to be chain based with respect to C. The notation used here is also in Section 1 of

[5]. The case where C is the center of complemented elements of A is defined in Section 2 of [5], with the remainder of that paper devoted to consequent developments for this case (viz., to the P_i-lattices, $i = 0,1,2,3$, where here P_3^1-lattices stand for Post algebras [5, pp. 73-74]).

A basic question concerns the nature of C for arbitrary A. To illustrate, while the lattice A_1:



has center $C_1 = \{0,b,\bar{b},1\}$, which is clearly sufficient for A_1 to be chain based with respect to C_1, the lattice A_2:



has center $C_2 = \{0,1\}$, and this is clearly insufficient for A_2 to be chain based with respect to C_2. While it is furthermore clear that any lattice A is trivially chain based with respect to itself, it is evident from the last example that a lattice A may be chain based with respect to a proper sublattice of A. For the lattice A_2 is chain based with respect to the proper sublattice $\{0,b,c,e_2,1\}$. This is immediate from the fact that A_1 is chain based with respect to C_1.

Accordingly, the notion of a lattice center will be extended to that of a lattice exocenter.* This will be accomplished as follows.

Let b in A, c in A satisfy the 4 conditions:

(i) c is the greatest element x in A such that $bx \le c$

(ii) c is the least element x in A such that $b \vee x \ge c$

(iii) b is the greatest element x in A such that $cx \le b$

(iv) b is the least element x in A such that $c \vee x \ge b$.

These conditions are satisfied by any complemented pair $b,\bar{b}$ of the center of A, as illustrated in A_1, but also by the pair b,c illustrated in A_2. This may be regarded as a generalization of the notion of a complemented pair. This prepares for the definition of the exocenter of a distributive lattice A as being <u>the sublattice of A generated by all pairs b, c satisfying (i), (ii), (iii), (iv)</u>. In particular, the exocenter of A_1 is C_1 and the exocenter of A_2 is $\{0,b,c,e_2,1\}$. By definition, the exocenter of A always contains the center of A. When A is a double Heyting algebra (that is, both A and its dual A^d are Heyting algebras) with $y \to z$ denoting the greatest x such that $yx \le z$ and $y \leftarrow z$ denoting the least x such that $y \vee x \ge z$, the above 4 conditions simply become $b \to c = c = b \leftarrow c$ and $c \to b = b = c \leftarrow b$.

Hereafter C denotes the exocenter of a distributive lattice A.

It is seen at once that the distributive lattices A which are chain based with respect to the exocenter C, are a proper extension of the distributive lattices A which are chain based with respect to the center B (the so-called P_i-lattices, $i = 0,1,2,3$). The exocenter is thus an appropriate extension of the center - the resulting class of chain based distributive lattices are broader than the class of P_i-lattices. This applies particularly to finite distributive lattices, each of which obviously has a Heyting chain and so is of order n for some finite n.

The above aimed at this last mentioned development for finite distributive lattices, yet there are additional logical or algebraic relationships involving the exocenter, similar to those which involve the center and appear in [4,6]. These relationships provide additional algebraic and logical insight, and provide tools for further development of chain based lattices. The development for logics which are characterized by lattices subresiduated with respect to their exocenters clearly falls between the case for lattices subresiduated with respect to themselves (Heyting algebras) and the case for lattices subresiduated with respect to their centers [6, Section 4].

Following [6], let the lattice A be subresiduated with respect to a sublattice Q containing 0,1. That is, for each y in A, z in A, there is an element x in Q such that for all q in Q $yq \le z$ if and only if $q \le x$. This x is denoted by $y \overset{Q}{\to} z$.

Suppose now that A is chain based with respect to Q. It can be shown that A is a Heyting algebra, with

$$y \to z = \bigvee_{i=1}^{n-1} e_i(ye_i \overset{Q}{\to} z).$$

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*Note that the exocenter of A specifies key structural elements of A which visually look like corner points of certain hyperparallelepipeds of A. See for example the rectangle Obe_2c in A_2.

Lattices subresiduated with respect to general Q and their corresponding logics are characterized in Sections 1 and 2 of [6]. They need not be Heyting algebras, nor even pseudo-complemented. Details concerning these logics are given in the appendix at the end of this paper. In particular, the pivotal axiom $Y \supset (X$ OR $(X \supset Y))$ is motivated from Theorem 22 of [6]. An algebraic discussion is just below.

Let A be subresiduated with respect to A, with $y \leq x \vee (x \overset{Q}{\to} y)$ for all x, y in A; let the dual statement hold for A^d. Then A is a double Heyting algebra and Q contains the exocenter of A. For let b,c satisfy (i) and (ii). From

$$b(b \overset{Q}{\leftarrow} c) \leq c \leq b \vee (b \overset{Q}{\to} c)$$

it follows easily that $(b \overset{Q}{\leftarrow} c) = c = (b \overset{Q}{\to} c)$, and similarly using (iii) and (iv) for b. It follows that $x \to y = y \vee x \overset{Q}{\to} y$. This follows from Theorem 22 of [6], which states the equivalence of $y \leq x \vee (x \overset{Q}{\to} y)$ with $x \overset{A}{\to} y = y \vee x \overset{Q}{\to} y$. There is a dual statement for A^d. Thus $x \overset{Q}{\to} 0$ is the pseudo-complement of x, with a dual statement for A^d.

It remains to explore the further connections to lattices which are chain based with respect to their exocenters. This has been done for lattices which are chain based with respect to their centers in [3] and in Section 4 of [5]. Note the special role of center B relative to general Q in [6]. Details may be found in the appendix at the end of this paper.

Finally, there is a direct suggestion from the development of chain based lattices [5, Section 4] concerning a canonical representation for each element x A. Namely, that if A is subresiduated and chain based with respect to C,

$$x = \bigvee_{i=1}^{n-1} e_i(e_i \overset{C}{\to} x) \text{ for each x in A.}$$

This is also suggested by the work of G. Rousseau in [8]. To illustrate, this last representation holds not only for each x in A_1, but also for each x in A_2. This is easily verified in the figures shown above. Because P_2-lattices can be characterized as a class of algebras $\langle L; \vee, \wedge, \overset{B}{\to}, e_0, \ldots, e_{n-1} \rangle$ [5, Section 5], it is natural to seek a similar result for $\langle A; \vee, \wedge, \overset{C}{\to}, e_0, \ldots, e_{n-1} \rangle$.

Summary

The notions of complement and center have been generalized to provide a proper extension of P_1-lattices into distributive lattices of order n. The case when these lattices are subresiduated lead to new logics of some algebraic and philosophical interest.

Appendix

This appendix gives propositional calculi which lie between the logic system R4 characterized by the class of all subresiduated lattices, and the logic system characterized by the class of all lattices which are subresiduated with respect to their centers (the so-called B-algebras of [4]). For ease of presentation, each such system is obtained by adding to R4 a single axiom, and each such addition is properly stronger than the one before. At the end of this appendix, three additional systems are given which do not fall precisely within this scheme, yet are deserving of inclusion. All of the systems in this appendix are given in order to be illustrative - there is no attempt to be exhaustive.

A propositional calculus is a set of axioms together with rules of inference from which provable formulas are obtained. The rules of inference for all of the logics below are:

(a) Substitution; that is, any formula F_i may be substituted for propositional variable X_j in a formula F, provided only that all occurrences of X_j in that formula F are replaced by F_i.

(b) Detachment; that is, if F_i is a provable formula and $F_i \supset F_j$ is a provable formula, then F_j is a provable formula.

The axioms for R4 are:

(A1) $X_1 \supset X_1$

(A2) $(X_1 \supset X_2) \supset (X_3 \supset (X_1 \supset X_2))$

(A3) $(X_1 \supset (X_2 \supset X_3)) \supset ((X_1 \supset X_2) \supset (X_1 \supset X_3))$

(A4) $(X_1$ AND $X_2) \supset X_1$

(A5) $(X_1$ AND $X_2) \supset X_2$

(A6) $(X_1 \supset X_2) \supset ((X_1 \supset X_3) \supset (X_1 \supset (X_2$ AND $X_3)))$

(A7) $X_1 \supset (X_1$ OR $X_2)$

(A8) $X_2 \supset (X_1$ OR $X_2)$

(A9) $(X_1 \supset X_3) \supset ((X_2 \supset X_3) \supset ((X_1$ OR $X_2) \supset X_3))$

(A10) $(X_1$ AND $(X_2$ OR $X_3)) \supset ((X_1$ AND $X_2)$ OR $(X_1$ AND $X_3))$

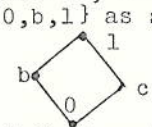
(A11) $\sim X_1 \supset (X_1 \supset X_2)$

(A12) $(X_1 \supset \sim X_1) \supset \sim X_1$

(A13) $\Box X_1 \supset ((X_1 \supset X_1) \supset X_1)$

(A14) $((X_1 \supset X_1) \supset X_1) \supset \Box X_1$

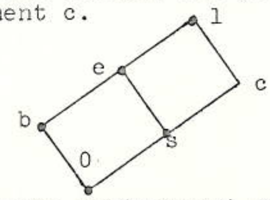
This system is quite weak with respect to the symbol ~. Consider, for example, $A = \{0,b,c,1\}$, $Q = \{0,b,1\}$ as shown below:



Here $b = \neg c < \neg\neg\neg c = 1$. Thus the additional axiom

(A15) $\quad \sim\sim\sim X_1 \supset \sim X_1$

is a proper extension of R4, but not sufficient for requiring Q to contain all the pseudo-complemented elements of A. This is shown in the example just below, where $A = \{0,b,s,e,c,1\}$, $Q = \{0,b,s,e,1\}$. In this example $\neg\neg\neg x = \neg x$ for all x in A but Q does not contain the pseudo-complemented element c.



Consider the replacement of (A15) with

(A15)' $\quad X_2 \supset (X_1 \ \underline{\text{OR}}\ (X_1 \supset X_2))$.

It has been shown that Q now contains all the pseudo-complemented elements of A, so that this is properly stronger than the previous system. Furthermore an intuitionist implication connective $\supset'$ may be introduced by letting $X_1 \supset' X_2$ be an abbreviation for $X_2 \ \underline{\text{OR}}\ (X_1 \supset X_2)$.

Next (A15)' is replaced with

(A15)" $\quad \Box X_1 \ \underline{\text{OR}}\ \sim\Box X_1$.

It is transparent from (A15)" that Q must contain only complemented elements of A. This logic, called R5, is characterized by the class of lattices subresiduated with respect to all Boolean subalgebras of its centers.

Lastly, (A15)" is replaced with

(A15)'''

$\Box X_1 \underline{\text{OR}} \sim\Box X_1 \underline{\text{OR}} \sim\Box(X_1 \underline{\text{OR}}\ X_2)\ \underline{\text{OR}} \sim\sim(X_1 \underline{\text{AND}}\ X_2)$.

It is not at all transparent that this last system is properly stronger than the previous system, R5, or that this logic is characterized by the class of all lattices subresiduated with respect to their centers (these were called B-algebras in [4], and the logic developed in [6]).

There are three remaining systems to be mentioned.

First, consider the addition of the axiom

$$(\Box X_1 \supset \Box X_2)\ \underline{\text{OR}}\ (\Box X_2 \supset \Box X_1)$$

to R4. This system is called R4.3, and transparently requires any elements q_1, q_2 in Q to satisfy $q_1 \overset{Q}{\to} q_2 \vee q_2 \overset{Q}{\to} q_1 = 1$. Thus this logic is characterized by the class of all lattices subresiduated with respect to sublattices Q which satisfy this identity. It is sufficient, in fact, for these sublattices Q to be linearly ordered [6, pp.203-204].

Second, consider the addition of the axiom

(Ax) $\quad (X_1 \supset X_2)\ \underline{\text{OR}}\ (X_2 \supset X_1)$

to R4. This system is stronger than the one just discussed, and stronger than R4 + (A15)'. For

$$(\Box X_1 \supset \Box X_2) \vee (\Box X2 \supset \Box X_1)$$

follows easily by substitution, while (A15)' follows straightforwardly from Theorem 3.1(vi) of [4] and Theorem 22 of [6].

Third, consider the addition of two axioms to R4: namely this last axiom (Ax)

$$(X_1 \supset X_2)\ \underline{\text{OR}}\ (X_2 \supset X_1)$$

together with the axiom (A15)'''

$\Box X_1 \underline{\text{OR}} \sim\Box X_1 \underline{\text{OR}} \sim\Box(X_1 \underline{\text{OR}}\ X_2)\ \underline{\text{OR}} \sim\sim(X_1 \underline{\text{AND}}\ X_2)$.

In [4], B-algebras satisfying the identity $(x \overset{B}{\to} y) \vee (y \overset{B}{\to} x) = 1$ were called P-algebras. This last logic system is called PC.* It is characterized by the class of all P-algebras.

Acknowledgement

The comments from the referees were helpful and appreciated. The appendix just above was provided in response to their requests.

*Variant axiom systems for PC are in pp. 209-210 of [6] and in Epstein, G., "Multiple-valued signal processing with limiting", Conf. Rec. of the 1972 Symp. on the Theory and Applications of Multiple-valued Logic Design", Buffalo, N.Y. (1972) pp.33-45.

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