LECTURE 2:

COMPLETENESS OF PROPOSITIONAL DYNAMIC LOGIC

Completeness, finite model property, decidability

Completeness Theorem for PDL

If φ is a PDL formula, and $\nvdash_{PDL} \varphi$, then there is a PD structure $\mathcal{K} = (S, [\![\cdot]\!])$ and some state $s_0 \in S$ such that $\mathcal{K}, s_0 \not\models \varphi$.

Moreover, S is finite, with $\leq 2^n$ states, where n is the size of φ.

Corollary: Decidability

There is an effective procedure which decides, given a PDL formula φ,
whether it is valid or not.

Proof. Given φ of size n,
check whether φ is true in all PD structures with up to $\leq 2^n$ states.

The Leibniz (filtration) method.

- Identify objects with their descriptions
- Hence: take descriptions to be your objects

- Used in Henkin's proof of completeness of FO Logic:
  description = maximal and witnessed set of formulas
- Many other completeness proofs.

- For PDL "descriptions" can be taken to be particular finite set of formulas.
- Regularity + [layered alternation between formulas and programs]
  make this possible.

The Fisher-Ladner Closure

- Formula $\varphi \mapsto C(\varphi)$ (finite set of formulas)
- Truth of $C(\varphi)$ in various states
  determine the truth of φ in any given state

The Fishher-Ladner productions:

- $(\neg\psi) \Rightarrow \psi$.
- $(\psi \wedge \chi) \Rightarrow \psi, \chi$.
  Similarly for other conenctives, if used.
- $[\alpha]\psi \Rightarrow \psi$.
- $[\alpha;\beta]\psi \Rightarrow [\alpha][\beta]\varphi$.
- $[\alpha \cup \beta]\psi \Rightarrow [\alpha]\varphi$ and $[\beta]\varphi$.
- $[\alpha^*]\psi \Rightarrow [\alpha][\alpha^*]\psi$.
- $[?\chi]\psi \Rightarrow \chi$.

- Write $\varphi \Longrightarrow_{FL} \psi$ for $\Longrightarrow^*$.
- $C(\varphi) =_{\mathrm{df}} \{\psi \mid \varphi \Longrightarrow_{FL} \psi\}$

Example of closure

- For $\varphi \quad \equiv \quad [(a \cup b)^*]((\neg p) \wedge [a;b][?r]q)$

$$
\begin{aligned}
C(\varphi) \quad = \quad \{ \ & \varphi, \ \ [a \cup b]\varphi, \ \ [a]\varphi, \ \ [b]\varphi, \\
& \neg p \wedge [a;b][?r]q, \ \ \neg p, \ \ [a;b][?r]q, \\
& p, \ \ [a][b][?r]q, \ \ [b][?r]q, \ \ [?r]q, \\
& r, \ \ q \ \ \}
\end{aligned}
$$

Size of the closure

- Is $C(\varphi)$ finite?
  Closure conditions increase formula-size!
- Is $C(\varphi)$ sub-exponential?
  Closure conditions replace a formula by two!

THEOREM. $C(\varphi)$ has at most as many formulas as the length of φ.

Factoring closure conditions

- Closing $[\alpha]\varphi$ can be factored:
  some dealing with α alone   some dealing with φ alone
- Use propositional identifier $\bullet$ as "placeholder" for φ in $[\alpha]\varphi$.
- $\psi \triangleleft \chi$: result of simultaneous substitution of χ for $\bullet$ in ψ.
- $\Psi \triangleleft \chi =_{\mathrm{df}} \{\psi \triangleleft \chi \mid \psi \in \Psi\}$.

LEMMA. For $[\alpha^*]\varphi \Longrightarrow^n \psi$, 1otf:

1. ψ is $[\alpha^*]\varphi$
2. ψ is of the form $B[\alpha^*]\varphi$ (B a block of boxes),   and $[\alpha]\bullet \Longrightarrow^* B\bullet$
3. $\varphi \Longrightarrow^* \psi$.

(Easy Induction on n.)

Corollary. $C([\alpha^*]\varphi) = \{[\alpha^*]\varphi\} \cup C([\alpha]\bullet) \triangleleft ([\alpha^*]\varphi) \cup C(\varphi)$

The factoring lemma

$$C([\alpha]\varphi) = C([\alpha]\bullet) \lhd \varphi \;\cup\; C(\varphi)$$

Proof. Induction on α.

- $C([a]\varphi)$

$$\begin{aligned}
&= \{[a]\varphi\} \;\cup\; C(\varphi) && \text{by the definition of } \Longrightarrow_{FL} \\
&= \{\varphi, [a]\varphi\} \;\cup\; C(\varphi) && \text{since } \varphi \in C(\varphi) \\
&= \{\bullet, [a]\bullet\} \lhd \varphi \;\cup\; C(\varphi) \\
&= C([a]\bullet) \lhd \varphi \;\cup\; C(\varphi)
\end{aligned}$$

- $C([\alpha;\beta]\bullet)$

$$
\begin{array}{lll}
= & \{[\alpha;\beta]\bullet\} \cup C([\alpha][\beta]\bullet) \cup \{\bullet\} & \text{by the definition of } \Longrightarrow_{FL} \\
= & \{[\alpha;\beta]\bullet\} \cup C([\alpha]\bullet) \lhd ([\beta]\bullet) \cup C([\beta]\bullet) \cup \{\bullet\} & \text{IH for } \alpha \\
= & \{[\alpha;\beta]\bullet\} \cup C([\alpha]\bullet) \lhd ([\beta]\bullet) \cup C([\beta]\bullet) \cup \{\bullet\} & \text{IH for } \beta
\end{array}
$$

Calculating now for $[\alpha;\beta]\varphi$,

$C([\alpha;\beta]\varphi)$

$$
\begin{array}{lll}
= & \{[\alpha;\beta]\varphi\} \cup C([\alpha][\beta]\varphi) \cup C(\varphi) & \text{by the definition of } \Longrightarrow_{FL} \\
= & \{[\alpha;\beta]\varphi\} \cup C([\alpha]\bullet) \lhd ([\beta]\varphi) \cup C([\beta]\varphi) \cup C(\varphi) & \text{IH for } \alpha \\
= & \{[\alpha;\beta]\varphi\} \cup C([\alpha]\bullet) \lhd ([\beta]\varphi) \cup C([\beta]\bullet) \lhd \varphi \cup C(\varphi) & \text{IH for } \beta \\
= & (\{[\alpha;\beta]\bullet\} \cup C([\alpha]\bullet) \lhd ([\beta]\bullet) \cup C([\beta]\bullet)) \lhd \varphi & \\
 & \qquad \cup C(\varphi) & \\
= & C([\alpha;\beta]\bullet) \lhd \varphi \cup C(\varphi) & \text{by our previous calculation}
\end{array}
$$

- $C([\alpha \cup \beta]\varphi)$

$$\begin{array}{rll} = & \{[\alpha \cup \beta]\varphi\} \,\cup\, C([\alpha]\varphi) \,\cup\, C([\beta]\varphi) \,\cup\, C(\varphi) & \text{by the definition of } \Longrightarrow_{FL} \\ = & \{[\alpha \cup \beta]\varphi\} \,\cup\, C([\alpha]\bullet) \lhd \varphi \,\cup\, C([\beta]\bullet) \lhd \varphi & \\ & \qquad \cup\, C(\varphi) & \text{IH} \\ = & (\{[\alpha \cup \beta]\bullet\} \,\cup\, C([\alpha]\bullet) \,\cup\, C([\beta]\bullet) \lhd \varphi & \\ & \qquad \cup\, C(\varphi) & \text{by the definition of } \lhd \\ = & C([\alpha \cup \beta]\bullet) \lhd \varphi \,\cup\, C(\varphi) & \end{array}$$

- $C([\alpha^*]\varphi)$

$$\begin{array}{rll} = & \{[\alpha^*]\varphi\} \,\cup\, C([\alpha]\bullet) \lhd ([\alpha^*]\varphi) \,\cup\, C(\varphi) & \text{by Corollary} \\ = & \{[\alpha^*]\varphi\} \,\cup\, C([\alpha]\bullet) \lhd ([\alpha^*]\varphi) \,\cup\, \{\varphi\} \,\cup\, C(\varphi) & \text{since } \varphi \in C(\varphi) \\ = & C([\alpha^*]\bullet) \lhd \varphi \,\cup\, C(\varphi) & \end{array}$$

since, by the Corollary above,

$$C([\alpha^*]\bullet) = \{[\alpha^*]\bullet\} \,\cup\, C([\alpha]\bullet) \lhd ([\alpha^*]\bullet \,\cup\, \{\bullet\}$$

- $C([?\chi]\varphi)$

$$\begin{array}{rll} = & \{[?\chi]\varphi\} \,\cup\, C(\chi) \,\cup\, C(\varphi) & \text{by the definition of } \Longrightarrow_{FL} \\ = & (\{[?\chi]\bullet\} \,\cup\, C(\chi)) \lhd \varphi \,\cup\, C(\varphi) & \text{since } \chi \text{ is } \bullet\text{-free} \\ = & (C([?\chi]\bullet) \lhd \varphi \,\cup\, C(\varphi) & \end{array}$$

$\dashv$

Linear size of closures

- $\#D$ = number of elements in D.
- $|\sigma|$ = size of σ (= number of nodes in syntax tree) for σ a formula or program.

Proposition.

1. $\#C(\varphi) \leq |\varphi|$.
2. $\#C([\alpha]\bullet) \leq |\alpha|$.

Proof. Simultaneous induction on programs and formulas.

- $\#C(\mathbf{p}) = \#(\{\mathbf{p}\}) = 1 = |\mathbf{p}|$.
- $\#C([\mathbf{a}]\bullet) = \#\{[\mathbf{a}]\bullet, \bullet\} = 2 = |[\mathbf{a}]\bullet|$
- $C(\varphi \wedge \psi)$

$$
\begin{aligned}
&= \{\varphi \wedge \psi\} \cup \#(C(\varphi) \cup C(\psi)) && \\
&\leq 1 + \#C(\varphi) + \#C(\psi) && \\
&\leq 1 + |\varphi| + |\psi| && \text{IH} \\
&= |\varphi \wedge \psi| &&
\end{aligned}
$$

- $\#C(\neg\varphi)$

$$\begin{aligned} &= \#(\{\neg\varphi\} \ \cup \ C(\varphi) \\ &= 1 + \#C(\varphi) \\ &\leq 1 + |\varphi| && \text{IH} \\ &= |\neg\varphi| \end{aligned}$$

- $\#C([\alpha]\varphi)$

$$\begin{aligned} &= \#(C([\alpha]\bullet) \lhd \varphi \ \cup \ C(\varphi))\text{by Lemma} \\ &\leq \#C([\alpha]\bullet) + \#C(\varphi) \\ &\leq |\alpha| + |\varphi| && \text{IH} \\ &\leq |[\alpha]\varphi| \end{aligned}$$

- $\#C([\alpha;\beta]\bullet)$

$$\begin{aligned}
&= \#(\{[\alpha;\beta]\bullet\} \ \cup \ C([\alpha][\beta]\bullet) \ \cup \ \{\bullet\}) && \text{by the definition of } \Longrightarrow_{FL} \\
&= \#(\{[\alpha;\beta]\bullet\} \ \cup \ C([\alpha][\beta]\bullet)) && \text{since } \bullet \in C([\alpha][\beta]\bullet) \\
&\leq 1 + \#(C([\alpha]\bullet) \lhd ([\beta]\bullet) \ \cup \ C([\beta]\bullet)) && \text{by Lemma} \\
&\leq 1 + \#C([\alpha]\bullet) \lhd ([\beta]\bullet) + \#C([\beta]\bullet)) && \\
&= 1 + \#C([\alpha]\bullet) + \#C([\beta]\bullet)) && \\
&\leq 1 + |\alpha| + |\beta| && \text{IH} \\
&= |\alpha;\beta| &&
\end{aligned}$$

- $\#C([\alpha \cup \beta]\bullet)$

$$\begin{aligned}
&= \#(\{[\alpha \cup \beta]\bullet\} \ \cup \ C([\alpha]\bullet) \ \cup \ C([\beta]\bullet) \ \cup \ \{\bullet\}) && \text{by the definition of } \Longrightarrow_{FL} \\
&= \#(\{[\alpha \cup \beta]\bullet\} \ \cup \ C([\alpha]\bullet) \ \cup \ C([\beta]\bullet)) && \\
&\leq 1 + \#C([\alpha]\bullet) + \#C([\alpha]\bullet) && \\
&\leq 1 + |\alpha| + |\beta| && \text{IH} \\
&= |\alpha \cup \beta| &&
\end{aligned}$$

- $\#C([?\chi]\bullet)$

$$\begin{aligned}
&= \#(C(\chi) \ \cup \ \{\bullet\}) && \text{by the definition of } \Longrightarrow_{FL} \\
&\leq \#C(\chi) + 1 && \\
&\leq |\chi| + 1 && \text{IH} \\
&= |?\chi| &&
\end{aligned}$$

Definition of the counter-model

- Suppose $\not\vdash \varphi$
- Construct structure $\mathcal{K}_\varphi = (S, [\![\cdot]\!]_\mathcal{K})$
  - $S =$ boolean combinations over $C(\varphi)$.
  - $[\![p]\!](s)$ iff $p \in s$.
  - $s \xrightarrow{a} t$ iff for all $[a]\psi \in C(\varphi)$,
    if $t \vdash \neg\psi$ then $s \vdash \neg[a]\psi$.
- I.e., $s \xrightarrow{a} t$ when, for every relevant assertion ψ, s rejects a promise that ψ will hold after a, whenever t rejects ψ.
- Note: States need not be consistent under PDL inference rules.

Formulas and states

LEMMA.

1. For $G \subseteq C(\varphi)$,
$$\vdash (\bigvee_{u \in G} u) \longleftrightarrow (\bigwedge_{v \in S-G} \neg v)$$
2. For $\boldsymbol{\psi} \in C(\varphi)$,
$$\vdash \boldsymbol{\psi} \leftrightarrow \bigvee_{\boldsymbol{\psi} \in s \in S} s$$
3. For $\boldsymbol{\psi} \in C(\varphi)$,
$$\vdash \boldsymbol{\psi} \leftrightarrow \bigwedge_{\psi \notin s \in S} \neg s$$

Proof of (1)

By dfn of boolean combinations, $u \rightarrow \neg v$ for any distinct $u, v \in S$.
So

$$(\bigvee_{u \in G} u) \rightarrow (\bigwedge_{v \in S-G} \neg v)$$

Conversely, by Prop Logic,

$$\vdash (\bigvee_{u \in G} u) \ \vee \ (\bigvee_{v \in S-G} v)$$

So

$$\vdash \neg(\bigvee_{v \in S-G} v) \ \rightarrow \ (\bigvee G)$$

i.e.

$$\vdash (\bigwedge_{v \in S-G} \neg v) \ \rightarrow \ (\bigvee G)$$

Proof of remaining dualities

- $$\begin{array}{rcll} \psi & \leftrightarrow & (\psi \wedge \bigvee_{s \in S} s) & \text{by (1)} \\ & \leftrightarrow & \bigvee_{s \in S} (\psi \wedge s) & \\ & \leftrightarrow & \bigvee_{\psi \in s} s & \psi \text{ is absorbed in these disjuncts,} \\ & & \text{and contradicts the others} & \end{array}$$

- Set $G = \{s \in S \mid \boldsymbol{\psi} \in S\}$. Then

$$\begin{array}{rcll} \boldsymbol{\psi} & \leftrightarrow & \bigvee_{s \in G} s & \text{by (2)} \\ & \leftrightarrow & \bigwedge_{\psi \notin s} \neg s & \text{by (1)} \end{array}$$

Matching proofs and truths

MAIN LEMMA.

For all ψ and α occuring in $C(\varphi)$,

1. If $s \models \psi$ then $s \vdash \psi$.
2. If $s \not\models \psi$ then $s \vdash \neg\psi$.
3. If $s \xrightarrow{\alpha} t$ then for all $[\alpha]\xi$ in $C(\varphi)$, if $t \vdash \neg\xi$ then $s \vdash \neg[\alpha]\xi$.
4. If $s \not\xrightarrow{\alpha} t$, then $s \vdash [\alpha]\neg t$.

PROOF OF COMPLETENESS.

- Given φ, construct $\mathcal{K}_\varphi$.
- If $\not\vdash \varphi$, then $(\bigvee_{s\in S} s) \not\vdash \varphi$, by Lemma.
- By propositional logic, exists $s \in S$ for which $s \not\vdash \varphi$.
- By part (1) of Proposition above, $\mathcal{K}_\varphi, s \not\models \varphi$.

The truth/proof cases

Proof of Main Lemma: simultaneous structural induction on ψ and α.

- $s \models \mathbf{p}$

$$\begin{aligned} &\Longrightarrow \mathbf{p} \in s \text{ by definition of } \mathcal{K}_\varphi \\ &\Longrightarrow s \vdash \mathbf{p} \end{aligned}$$

- $s \not\models \mathbf{p}$

$$\begin{aligned} &\Longrightarrow \mathbf{p} \notin s && \text{by definition of } \mathcal{K}_\varphi \\ &\Longrightarrow \neg\mathbf{p} \in s && \text{since } s \text{ is a boolean combination of } C(\varphi) \\ &\Longrightarrow s \vdash \neg\mathbf{p} \end{aligned}$$

- $s \models \neg\psi$

$$\begin{aligned} &\Longrightarrow s \not\models \psi \\ &\Longrightarrow s \vdash \neg\psi \ \text{ IH} \end{aligned}$$

- $s \not\models \neg\psi$

$$\begin{aligned} &\Longrightarrow s \models \psi \\ &\Longrightarrow s \vdash \psi && \text{IH} \\ &\Longrightarrow s \vdash \neg\neg\psi \end{aligned}$$

- $s \models \psi \wedge \chi$

$$\begin{aligned} &\Longrightarrow \ s \models \psi \quad \text{and} \quad s \models \chi \\ &\Longrightarrow \ s \vdash \psi \quad \text{and} \quad s \vdash \chi \quad \text{IH} \\ &\Longrightarrow \ s \vdash \psi \wedge \chi \end{aligned}$$

- $s\neg \models \psi \wedge \chi$

$$\begin{aligned} &\Longrightarrow \ s\neg \models \psi \quad \text{or} \quad s\neg \models \chi \\ &\Longrightarrow \ s \vdash \neg\psi \quad \text{or} \quad s \vdash \neg\chi \quad \text{IH} \\ &\Longrightarrow \ s \vdash \neg(\psi \wedge \chi) \end{aligned}$$

- $s \models [\alpha]\psi$

$$
\begin{array}{lll}
\Longrightarrow & \text{for all states } t, s \not\xrightarrow{\alpha} t \quad \text{or} \quad t \models \psi & \\
\Longrightarrow & \text{for all states } t, s \vdash [\alpha]\neg t \quad \text{or} \quad t \vdash \psi & \text{IH} \\
\Longrightarrow & \text{for all states } t, s \vdash [\alpha](t \rightarrow \psi) & \text{by PDL rules} \\
\Longrightarrow & s \vdash \bigwedge_{t \in S} [\alpha](t \rightarrow \psi) & \text{by prop logic} \\
\Longrightarrow & s \vdash [\alpha] \bigwedge_{t \in S} (t \rightarrow \psi) & \text{by PDL} \\
\Longrightarrow & s \vdash [\alpha]((\bigvee_{t \in S} t) \rightarrow \psi) & \text{by prop logic and box distribution} \\
\Longrightarrow & s \vdash [\alpha](\top \rightarrow \psi) & \text{by Lemma} \\
\Longrightarrow & s \vdash [\alpha]\psi &
\end{array}
$$

- $s \not\models [\alpha]\psi$

$$
\begin{array}{lll}
\Longrightarrow & \text{for some state } t, s \xrightarrow{\alpha} t \quad \text{and} \quad t \not\models \psi & \\
\Longrightarrow & \text{for some state } t, s \xrightarrow{\alpha} t \quad \text{and} \quad t \vdash \neg\psi & \text{IH(2)} \\
\Longrightarrow & \text{for some state } t, s \vdash \neg[\alpha]\psi & \text{IH(3)}
\end{array}
$$

Case of atomic transitions

- For α atomic, (3) holds by the definition of $\mathcal{K}_\varphi$.
- $s \not\xrightarrow{a} t$

$\Longrightarrow$ for some $[a]\xi$ in $C(\varphi)$, $t \vdash \neg\xi$ and $s \not\vdash \neg[a]\xi$

$\xi \vdash \neg t$ and $(\neg[a]\xi) \notin s$

$\xi \vdash \neg t$ and $[a]\xi \in s$ since s is a state

$s \vdash [a]\xi$and $[a]\xi \vdash [a]\neg t$ by PDL rules

$s \vdash [a]\neg t$

Case of composed transitions

- $s \xrightarrow{\alpha;\beta} t$

$$
\begin{array}{ll}
\Longrightarrow & \text{for some state } u, \quad s \xrightarrow{\alpha} u \xrightarrow{\beta} t \\
\Longrightarrow & \text{for all formula } [\alpha;\beta]\xi \text{ in } C(\varphi), \\
& \quad t \vdash \neg\xi \\
& \quad u \vdash \neg[\beta]\xi \qquad \text{by IH, since } [\beta]\xi \text{ is in } C(\varphi) \\
& \quad s \vdash \neg[\alpha][\beta]\xi \qquad \text{by IH, since } [\alpha][\beta]\xi \text{ is in } C(\varphi) \\
& \quad s \vdash \neg[\alpha;\beta]\xi \qquad \text{by PDL rules}
\end{array}
$$

- $s \not\xrightarrow{\alpha;\beta} t$

$$
\begin{array}{lll}
\Longrightarrow & \text{for all states } u, \quad s \not\xrightarrow{\alpha} u \quad \text{or} \quad u \not\xrightarrow{\beta} t & \\
\Longrightarrow & \text{for all states } u, \quad s \vdash [\alpha]\neg u \quad \text{or} \quad u \vdash [\beta]\neg t & \text{IH} \\
\Longrightarrow & \text{for all states } u, \quad s \vdash [\alpha](u \rightarrow [\beta]\neg t) & \text{by PDL} \\
\Longrightarrow & s \vdash [\alpha][\beta]\neg t & \text{as above} \\
\Longrightarrow & s \vdash [\alpha;\beta]\neg t &
\end{array}
$$

Case of union transitions

- $s \xrightarrow{\alpha\cup\beta} t$

$$
\begin{array}{lll}
\Longrightarrow & s \xrightarrow{\alpha} t \quad \text{or} \quad s \xrightarrow{\beta} t & \\
\Longrightarrow & \text{for all formula } [\alpha]\xi \text{ in } C(\varphi), & \\
 & \quad t \vdash \neg\xi \quad \text{implies} \quad s \vdash \neg[\alpha]\xi & \\
 & \quad \text{or for all formula } [\beta]\xi \text{ in } C(\varphi), & \\
 & t \vdash \neg\xi \quad \text{implies} \quad s \vdash \neg[\alpha]\xi & \text{IH} \\
\Longrightarrow & \text{for all formula } [\alpha\cup\beta]\xi \text{ in } C(\varphi), & \\
 & t \vdash \neg\xi \quad \text{implies} \quad s \vdash \neg[\alpha]\xi \quad \text{or} \quad s \vdash \neg[\beta]\xi & \text{definition of } C(\varphi) \\
\Longrightarrow & \text{for all formula } [\alpha\cup\beta]\xi \text{ in } C(\varphi), & \\
 & t \vdash \neg\xi \quad \text{implies} \quad s \vdash \neg[\alpha\cup\beta]\xi & \text{PDL rules}
\end{array}
$$

- $s \not\xrightarrow{\alpha\cup\beta} t$

$$
\begin{array}{lll}
\Longrightarrow & s \not\xrightarrow{\alpha} t \quad \text{and} \quad s \not\xrightarrow{\beta} t & \\
\Longrightarrow & s \vdash [\alpha]\neg t \quad \text{and} \quad s \vdash [\beta]\neg t & \text{IH} \\
\Longrightarrow & s \vdash [\alpha\cup\beta]\neg t & \text{by PDL rules}
\end{array}
$$

Case of iterated transitions

- $s \xrightarrow{\alpha^*} t$

$\Longrightarrow$ $s = u_1 \xrightarrow{\alpha} u_2 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} u_k = t$ for some states $u_1, \ldots, u_k$

$\Longrightarrow$ for all formulas $[\alpha^*]\xi$ in $C(\varphi)$ (whence $\xi, [\alpha][\alpha^*]\alpha$ in $C(\varphi)$)

$t \vdash \neg\xi \quad \Longrightarrow t \vdash \neg[\alpha^*]\xi$	by PDL rules
$\Longrightarrow u_{k-1} \vdash \neg[\alpha][\alpha^*]\xi$	IH
$\Longrightarrow u_{k-1} \vdash \neg[\alpha^*]\xi$	by PDL
$\cdots$	induction on k
$\Longrightarrow s \vdash \neg[\alpha^*]\xi$	

- Suppose $s \not\xrightarrow{\alpha^*} t$.

  Let $G = \{u \in S \mid s \xrightarrow{\alpha^*} u\}$.

  For $u \in G$, $v \notin G$, we have $u \not\xrightarrow{\alpha} v$, and so (by IH) $u \vdash [\alpha]\neg v$. Thus

$$\begin{aligned} \vdash \textstyle\bigvee_{u \in G} u \; &\rightarrow \textstyle\bigwedge_{v \notin G}[\alpha]\neg v \\ &\rightarrow [\alpha] \textstyle\bigwedge_{v \notin G} \neg v \quad \text{by PDL} \\ &\rightarrow [\alpha] \textstyle\bigvee_{v \in G} v \quad \text{by prop logic} \end{aligned}$$

  So by the Invariance Rule of PDL,

$$\vdash \bigvee_{u \in G} u \; \rightarrow \; [\alpha^*] \bigvee_{v \in G} v$$

  Reversing the argument above, this implies that $u \vdash [\alpha^*]\neg v$ for all $u \in G$, $v \notin G$. In particular, $s \vdash [\alpha^*]\neg t$.

Case of test transitions

- $s \xrightarrow{?\chi} t$

$$\begin{array}{lll}
\Longrightarrow & s = t \quad \text{and} \quad s \models \chi & \\
\Longrightarrow & s = t \quad \text{and} \quad s \vdash \chi & \text{IH} \\
\Longrightarrow & \text{for all formulas } [?\chi]\xi \text{ in } C(\varphi) & (\text{whence } \chi, \xi \in C(\varphi)) \\
& \quad t \vdash \neg\xi \quad \Longrightarrow s \vdash \neg(\chi \rightarrow \xi) & \\
& \quad\quad \Longrightarrow s \vdash \neg[?\chi]\xi &
\end{array}$$

- $s \not\xrightarrow{?\chi} t$

$$\begin{array}{lll}
\Longrightarrow & s \neq t \quad \text{or} \quad s \not\models \chi & \\
\Longrightarrow & s \vdash \neg t \quad \text{or} \quad s \vdash \neg\chi & \text{IH} \\
\Longrightarrow & s \vdash t \rightarrow \neg\chi & \\
\Longrightarrow & s \vdash \chi \rightarrow \neg t & \text{prop logic} \\
\Longrightarrow & s \vdash [?\chi]\neg t &
\end{array}$$

Independent fun

Construct the structure $\mathcal{K}_\varphi$ for the following formulas φ.

1. $p \rightarrow [a]p$.
2. $p \rightarrow \langle a \rangle p$
3. $(p \rightarrow [a]q) \vee (q \rightarrow [a]p)$

Size and Decidability

Size of $\mathcal{K}_\varphi$:

- $O(n)$ formulas in FL closure.
- $O(*^n)$ states.
- $O(*^{*^n})$ PD structures to check.
- Decision in double exponential time.

THEOREM (Pratt)
PDL is decidable in exponential time.

(Complete proof in lecture notes.)

Failure of strong completeness

Strong completeness: For set of formulas Γ, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Consequence: Joint Satisfiability (Compactness):
if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable, then so is Γ.

THEOREM. PDL for regular programs does not have the Joint Satisfiability Property.

Proof. Let

$$\Phi = \{\, \mathbf{p},\ [a]\mathbf{p},\ [a;a]\mathbf{p},\ [a;a;a]\mathbf{p}, \ldots,\ [a^i]\mathbf{p},\ \ldots\}$$

and

$$\Gamma = \Phi \cup \{\neg[a^*]\mathbf{p}\}$$

If $\Gamma_0 \subset \Gamma$ is finite, then it is satisfiable.
But Γ is not.