Last Time: Growth of functions & Ranking Runtime Functions.
Today: Ranking Runtime Functions continued, and Celebrity Problem.

\[
i = 8; \quad \lim_{n \to \infty} \frac{n^k}{c^n} = \lim_{n \to \infty} \frac{kn^{k-1}}{\log c \cdot c^n} = \ldots = \lim_{n \to \infty} \frac{k!}{(\log c)^k \cdot c^n} = 0
\]

\[
i = 9; \quad \lim_{n \to \infty} \frac{c^n}{d^n} = \lim_{n \to \infty} (\frac{c}{d})^n = 0
\]

\[
i = 10; \quad \lim_{n \to \infty} \frac{d^n}{n!} = \lim_{n \to \infty} \frac{d^n}{1 \cdot 2 \cdot \ldots \cdot (2d) \cdot (2d+1) \ldots n} \leq \lim_{n \to \infty} \frac{d^n}{(2d)! \cdot (2d)^{n-2d}} = 0
\]

* So, n! grows faster than any exponential. *

Recurrence:

\[
\int_{a-1}^{b} f(x) \, dx \leq \sum_{i=a}^{b+1} f(i) \leq \int_{a}^{b+1} f(x) \, dx
\]

\[
T(n) = a \cdot T\left(\frac{n}{b}\right) + \sum_{i=0}^{n+1} f(i)
\]

\[
f(i) = i, \quad \sum_{i=0}^{n+1} \frac{x}{2} = \frac{(n+1)^2}{2} - 0 = \Theta(n^2)
\]
THE CELEBRITY PROBLEM:

The celebrity problem is to identify the celebrity amongst n people. Celebrity is a person who is well known to everyone but doesn’t know anyone. Suppose we have a room with a number of people {1, 2, 3… n} and our task is to find out the celebrity by asking all only one question that is whether one knows the other person.

Solution I:

Draw a Matrix with a number of people {1, 2, 3… n} in rows and columns as shown below. Then go on asking that question. If answer comes YES put a cross; if not then leave blank.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>*</td>
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<td>n</td>
<td>*</td>
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<td></td>
</tr>
</tbody>
</table>

Once all columns are filled, the column that has greatest density with its corresponding row having smallest density indicates a Celebrity. There are n(n-1)/2 pairs of persons, so potentially we need to ask n(n-1) questions.

And hence we get, \( T(n) = \Theta(n^2) \)

Solution II: (Think Recursively)

- Randomly pick 2 people, A and B.
  Ask if A knows B.
  if YES, then A cannot be a celebrity;
  if NO, then B cannot be a celebrity.
  Then go on eliminating non-celebrities. Thus with each one question we can eliminate one of two people.
As shown above, take out person A, and inductively solve celebrity problem for (n-1) people. Just need to check that the person A knows the celebrity, and the celebrity does not know him.

There are 2 possible outcomes:
- the celebrity is among first n-1;
- the celebrity is the person A; (the celebrity cannot be A since A has been ruled out)
- no celebrity at all.

Recurrence:

\[
T(n) = \begin{cases} 
T(n-1) + 3 & \text{if } n > 2 \\
2 & \text{if } n = 2 
\end{cases}
\]

Solution III:

Pick the pairs of people, ask one question to each pair and eliminate one of two in a pair. Repeat until one person remains.

So, there will be n/2 pairs and n/2 questions.

Now go back, and ask 2(n-1) questions, comparing this celebrity with (n-1) losers.

Recurrence:

\[
T(n) \leq \begin{cases} 
T(n/2) + \frac{3n}{2} & \text{if } n > 2 \\
2 & \text{if } n = 2 
\end{cases}
\]

So, if a = 1

b = 2

\(f(n) = n\)

\(f(b) = 2 > a\)

\(T(n) \leq O(n)\)