Replacement Lemma and its proof

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Abstract

This note proves the Replacement Lemma that we talked about in class on Tuesday 2/19/2002.

Lemma 0.1 (Replacement) If

1. \( D \triangleright \Gamma \vdash E[e] : t \), such that the hole in \( E \) occurs at position \( p \)
2. \( D' \triangleright \Gamma \vdash e : t' \)
3. \( D' \) is a subderivation of \( D \) occurring at position \( p \) and
4. \( \Gamma \vdash e' : t' \)
then, \( \Gamma \vdash E[e'] : t \).

It is crucial that \( D' \) be a subderivation of \( D \). For otherwise, the hypotheses

1. \( \Gamma \vdash E[e] : t \),
2. \( \Gamma \vdash e : t' \) and
3. \( \Gamma \vdash e' : t' \)
do not imply \( \Gamma \vdash E[e'] : t \). Consider the counter-example \( E = \Box, e = \text{DivZero} \), \( e' = 5, t = \text{bool} \) and \( t' = \text{int} \). The judgements
1. $\Gamma \vdash \Box [\text{DivZero}] : \text{bool}$,

2. $\Gamma \vdash \text{DivZero} : \text{int}$ and

3. $\Gamma \vdash 5 : \text{int}$

are all true but imply $\Gamma \vdash \Box [5] : \text{int}$, which is false.

Also, it is necessary that the position of $D'$ in $D$ and the position of $e$ in $E$ be the same. Otherwise, we have the counter-example $E[\text{DivZero}]$, where $E = \text{if } \Box \text{ then DivZero else 1}$. If

1. $D \triangleright \emptyset \vdash e : \text{int}$

2. $D_1 \triangleright \emptyset \vdash \text{DivZero} : \text{bool}$ and

3. $D_2 \triangleright \emptyset \vdash \text{DivZero} : \text{int}$

then both $D_1$ and $D_2$ are subderivations of $D$. If the restriction about the sub-derivation $D'$ being at position $p$ were removed, then choosing $D'$ to be $D_2$ means that the propositions

1. $D \triangleright \emptyset \vdash E[\text{DivZero}] : \text{int}$

2. $D_2 \triangleright \emptyset \vdash \text{DivZero} : \text{int}$

3. $D_2$ is a subderivation of $D$ and

4. $\emptyset \vdash 5 : \text{int}$

are all true, but imply the false judgement $\emptyset \vdash E[5] : \text{int}$.

**Proof** (of Replacement Lemma)

By induction on $D$. For the base cases, $D$ has exactly one node. Therefore, $E = \Box$, $D' = D$, and $t = t'$ and the result follows.

For the inductive cases, we have the following subcases depending on $E$:

1. $E = \Box$. This implies $D = D'$ and $t = t'$ and this is similar to the case above.
2. \( E = +(E_1, e_2) \): By the Inversion Lemma, \( t = \text{int} \), and there are derivations \( D_1 \) and \( D_2 \) such that

\[
D = \text{AOP} \quad \frac{D_1 \triangleright \Gamma \vdash E_1[e] : \text{int} \quad D_2 \triangleright \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash +(E_1[e], e) : \text{int}}
\]

It follows that \( D' \) is a subderivation of \( D_1 \). Clearly, \( D_1 \) is a proper subderivation of \( D \). Thus, by the induction hypothesis, \( \Gamma \vdash E_1[e'] : \text{int} \). Again, by the Inversion Lemma, \( \Gamma \vdash e_2 : \text{int} \). The result follows from the application of the AOP rule.

The other cases for \( E \) are similar and omitted.