Tagging as an alternative to object creation*

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Abstract

Based on the observation that graphs play an important role in the representation of databases, an algebra is presented for the manipulation of binary relations, i.e., of directed unlabeled graphs. This so-called Tarski algebra is based on early work by Tarski. The key notion that has been added to it here is tagging, which is needed for providing both enough modeling power and enough querying power. Moreover, tagging can also be seen as a value-based counterpart to object creation in object-oriented data models. We present tagging in a general formal framework that incorporates several specific tagging strategies as a special case. We show that each of these strategies allows for the simulation in the Tarski model of various other database models, in particular of the relational model. Finally, we discuss the genericity of tagging and show that the Tarski algebra augmented with multiple assignments and a while-construct is a computationally complete database language.

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## 1 Introduction

The current trend in database research is towards systems which can support database applications that involve data objects with a complex external and internal organization. This trend is reflected by the emergence of systems which support extendible databases \[6, 27, 31\] and object-oriented databases \[4, 5, 20, 31\]. The database theory community has followed this trend and has devised mathematical database models to both represent and manipulate complex-object databases. These database models can be categorized into two general classes: value-based models (the so-called complex-object database models) and object-identity-based models (the so-called object-oriented database models).

### Figure 1: A persons relational database

<table>
<thead>
<tr>
<th>Parent</th>
<th>Child</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>(p_3)</td>
</tr>
<tr>
<td>(p_1)</td>
<td>(p_4)</td>
</tr>
<tr>
<td>(p_2)</td>
<td>(p_3)</td>
</tr>
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<td>(p_2)</td>
<td>(p_4)</td>
</tr>
<tr>
<td>(p_4)</td>
<td>(p_6)</td>
</tr>
<tr>
<td>(p_5)</td>
<td>(p_6)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Person</th>
<th>Hobby</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>Soccer</td>
</tr>
<tr>
<td>(p_1)</td>
<td>Music</td>
</tr>
<tr>
<td>(p_2)</td>
<td>Dance</td>
</tr>
<tr>
<td>(p_3)</td>
<td>Soccer</td>
</tr>
<tr>
<td>(p_4)</td>
<td>Reading</td>
</tr>
<tr>
<td>(p_4)</td>
<td>Painting</td>
</tr>
<tr>
<td>(p_5)</td>
<td>Basketball</td>
</tr>
<tr>
<td>(p_6)</td>
<td>Music</td>
</tr>
<tr>
<td>(p_6)</td>
<td>Games</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Person</th>
<th>Name</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1)</td>
<td>Frank</td>
<td>50</td>
</tr>
<tr>
<td>(p_2)</td>
<td>Ellen</td>
<td>52</td>
</tr>
<tr>
<td>(p_3)</td>
<td>Eric</td>
<td>30</td>
</tr>
<tr>
<td>(p_4)</td>
<td>Lisa</td>
<td>28</td>
</tr>
<tr>
<td>(p_5)</td>
<td>Brad</td>
<td>31</td>
</tr>
<tr>
<td>(p_6)</td>
<td>Eric</td>
<td>6</td>
</tr>
</tbody>
</table>

To illustrate the difference between these two classes, consider the persons relational database shown in Figure 1. This database could alternatively be represented as a complex-object database as shown in Figure 2. Although transforming a flat relational database into a complex-object database of course requires the creation of complex objects (i.e., the name-age pairs, the children sets, and the hobbies sets in our example), it is important to notice that no new atomic values are introduced.

In contrast, an object-oriented representation requires the introduction of new atomic values in the form of object-identifiers of objects. For example, one possible way of representing the persons database in an object-oriented context is shown in Figure 3. (For simplicity, we used an untyped representation rather than a more traditional, i.e., class-oriented approach.) Each object carries its own value and identity, which in turn is allowed to occur in the value components of other objects. Consider for example the object with object-identifier \(s_1\). This object-identifier uniquely represents a children set. Its value is the set of children of the parent objects identified by \(o_1\) and \(o_2\). Notice how \(s_1\) is shared in the value components of the respective parents.

It would appear from this example that complex-object databases and object-oriented databases are fundamentally different, each offering its own advantages. The main advantage of the complex-object approach is that no new atomic values are introduced. This provides a clean framework in which to extend the theory of relational databases to that of databases with complex objects. On the other hand, the main advantage of the object-oriented approach is its economy of representation. For example, whereas in the object-oriented version...
<table>
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<th>Children</th>
<th>Hobbies</th>
</tr>
</thead>
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<tr>
<td>$p_1$</td>
<td>[name: Frank, age: 50]</td>
<td>Child $p_3$ $p_4$</td>
<td>Hobby Soccer Music</td>
</tr>
<tr>
<td>$p_2$</td>
<td>[name: Ellen, age: 52]</td>
<td>Child $p_3$ $p_4$</td>
<td>Hobby Dance</td>
</tr>
<tr>
<td>$p_3$</td>
<td>[name: Eric, age: 30]</td>
<td>Child $p_6$</td>
<td>Hobby Soccer</td>
</tr>
<tr>
<td>$p_4$</td>
<td>[name: Lisa, age: 28]</td>
<td>Child $p_6$</td>
<td>Hobby Reading Painting</td>
</tr>
<tr>
<td>$p_5$</td>
<td>[name: Brad, age: 31]</td>
<td>Child</td>
<td>Hobby Basketball</td>
</tr>
</tbody>
</table>

Figure 2: The person database represented as a complex-object database
of the person database, there is a unique representation for the children set of $o_1$ and $o_2$, this children set needs to be doubly represented in the complex-object persons database. This economy of representation, however, comes at the expense of being able to extend cleanly the theory of relational databases to that of object-oriented databases.

To bridge the gap between complex-object databases and object-oriented databases, researchers have proposed database models and database languages in which new values can be introduced through object-creation. For example, returning to the persons database, assume that we have derived at some stage in a computation that the sets $\{p_3, p_4\}$ and $\{p_6\}$ are the children-sets of the (unordered) parent pairs $\{p_1, p_2\}$ and $\{p_4, p_5\}$, respectively. Instead of carrying the full representation of this derived information throughout the rest of the computation, we could decide to create four new values, say $\tau_1$, $\tau_2$, $\tau_3$, and $\tau_4$. These values uniquely represent the respective children sets and parent pairs. Using these values, the derived parent-child information can be represented as shown in Figure 4.

In this paper, we will call these new values *tags*. It must be emphasized that tags are different from object-identifiers. Tags can be thought of as succinct representations of complex objects, whereas object-identifiers are uniquely associated with objects whose value can be a complex object.
Indeed, turning back to our example, suppose that \( p_1 \) and \( p_2 \) have a new child, say \( p_7 \). In the complex-object database, this update would be reflected through the addition of \( p_7 \) to the children sets of the respective parents \( p_1 \) and \( p_2 \). In the object-oriented approach, however, \( p_7 \) would simply be added as a new object (say with object identifier \( o_7 \)), and the value of the object \( s_1 \) would be updated to include \( o_7 \). In particular, \( s_1 \) does not have to be replaced throughout the database. In the tag-based approach in contrast, the only reasonable strategy is to replace the tag \( \tau_1 \) of the persons set \( \{ p_3, p_4 \} \) by a new tag, say \( \tau_5 \), representing the set \( \{ p_3, p_4, p_7 \} \). In other words, after this update, \( \tau_1 \) is still the representation of the “old” set \( \{ p_3, p_4 \} \).

For this reason, the tagging technique is essentially value-based. However, at the same time, economy of representation and expression is obtained.

There exist essentially two classes of tag-based database models. The first class, which we call the class of non-deterministic tag-based database models, consists of database models which introduce tags non-deterministically. The non-determinism stems from the idea that it does not matter which new values are actually introduced to represent complex objects. Of course, the consequence of the non-deterministic approach is that there may be several tags, i.e., copies, representing the same complex object. The classic example of a non-deterministic tag-based database model is the the IQL model of Abiteboul and Kanellakis [1]. In the same class are also transactions-based database models of Abiteboul and Vianu [2] and the GOOD model of Gyssens et al. [14, 15]. The second class, which we call the class of deterministic tag-based database models, consists of database models which introduce new tags deterministically through function evaluations of complex objects (this process is closely related to the notion of skolemization in logic). In this class, one finds LDM [23], C-logic [8] and F-logic [19]. In the ILOG model of Hull and Yoshikawa [18], both the non-deterministic and the deterministic approach are combined. In the limit of course, any complex-objects database model can be thought of as deterministic tag-based database model, simply because an object-value can always be thought of as its own tag (such an approach was actually used for theoretical purposes in [13]).

Most tag-based database models have been proposed in a declarative, logic-based context, with the GOOD model and LDM being the exceptions. In contrast with the satisfactory status of declarative formulations of tag-based database models, we feel that there are currently few, if any, satisfactory algebraic formulations. This is unfortunate since algebraic formulations are better platforms from which to build real database systems. In this paper we offer such an algebraic formulation. We will introduce a tag-based database model, the Tarski database model. We called our model, based on binary relations, after the distinguished logician-mathematician Tarski who laid the set-theoretic foundations for our proposal.

Our model is an outgrowth of a mechanism to algebraizate the GOOD model. To understand this, it is necessary to discuss briefly the GOOD database model and its associated data manipulation language. This will be the subject of Section 2. In Section 3, we then formally introduce the Tarski database model. Our main emphasis will be on the data manipulation language, which we called the Tarski algebra. In Section 4, we show how other database models (relational model, nested model, GOOD model) can be simulated using the essentially binary Tarski approach. In particular, we show that the Tarski algebra allows the expression of the relational algebra. In Section 5, we extend the Tarski algebra with a looping construct and show that this yields a computationally complete language. In this section, we
also discuss the notion of genericity in the presence of tags. Finally in Section 6, we mention some ongoing and future research on the Tarski database model, including implementation issues.

2 Towards a tag-based database model

2.1 The Graph-Oriented Object Database model (GOOD)

The Graph-Oriented Object Database model (GOOD) was introduced in [14, 15] as an attempt to construct an object-oriented database model in which both the data representation part and the data manipulation language are graph-based.

In the GOOD model, a database is viewed as a labeled directed graph. For example, in Figure 5 we show a GOOD database for the persons database shown in Figure 1. For simplicity, we omitted the information about age and hobbies. Circle nodes represent printable information, while square nodes can easily be interpreted as tags for complex objects. Ordinary arrows represent functional properties whereas double arrows indicate multivalued properties.

Data manipulation operations in GOOD are specified in term of graph-transformations such as node additions, edge additions, node deletions and edge deletions. For example, in
Figure 7: The effect of the operations in Figure 6 on the persons database

Figure 6 (left) we show a node addition operation the effect of which is to add new parent-pair nodes, i.e., one new such node for each pair of person nodes which have a common child. (In the context of our discussion of tags, think of these new nodes as tags for the respective parent-pairs.) In Figure 6 (middle) we show an edge addition operation. Its effect is to associate the children nodes directly with their corresponding parent-pair nodes. Finally, in Figure 6 (right) we show an edge deletion that removes the original links between parents and children. The effect of the consecutive application of these three operations is displayed in Figure 7.

2.2 GOOD as motivation for a binary tag-based database model

A natural way to represent a GOOD database algebraically is as a collection of mathematical (binary) relations [24]. In one particular such representation there are two relations to represent the nodes in the GOOD database, i.e., one associating node labels to nodes and one associating print labels to printable nodes. Furthermore, there is a separate relation for each edge-label occurring in the GOOD database, i.e., if \( e \) is such an edge label, then \( e \) can be represented by a relation and a pair \((n_1, n_2)\) in this relation denotes that there is an edge in the GOOD database with label \( e \) from node \( n_1 \) to node \( n_2 \). Figure 8 shows this representation for the portion of the persons database represented in GOOD in Figure 5.

Of course, one can easily think of other reasonable ways to represent GOOD databases in terms of mathematical relations.

Once a particular representation has been achieved, one can simulate GOOD graph transformations by set-theoretic operations on mathematical relations.

As it turns out, Tarski [28] had already specified an algebra to manipulate just such binary relations. In fact, Tarski called this algebra a relation algebra in later work. However, he did not use it to manipulate databases, but rather to study set-theoretical concepts. Tarski’s algebra has (only) four fundamental and well-known operators: union, composition, complementation, and inversion. Furthermore, Tarski assumes that there is a special relation
Figure 8: Using binary mathematical relations for representing GOOD instances

fixing the equality relation on the universe of discourse. In a very real sense, mathematicians
and logicians were already doing relational algebra well before Codd introduced his relational
algebra into the context of databases. Of course mathematicians almost invariantly dealt
with infinite relations, while database theoreticians, and certainly database practitioners,
worked with finite relations.

In this paper we approach database theory following Tarski’s rather than Codd’s formal-
isms. We model a database by a set of mathematical relations and manipulate a database
by performing Tarski algebraic operations on these relations. In order to study the notion
of tagging, we add two new operators to Tarski’s algebra. These so-called tagging operators
allow the introduction of (new) tags. Each new tag represents a specific ordered pair in the
database and is derived through a so-called tagging function evaluation. The power of the
tagging technique stems from the fact that these new tag values can be used as values in other
pairs, thus facilitating the succinct representation of complex objects. In combination with
the other operators of the Tarski algebra, the tagging operators allow for complex database
manipulations.

We will furthermore show that tagging can be achieved via tagging functions with special
properties. These properties, we believe, are in fact effectively used in real-world database
systems. So our contribution is two-fold: one, we give a simple algebraic formulation of
tag-based database models, and two, our proposed tagging techniques correspond naturally
to techniques already in use in database systems.

\footnote{We are not the first to do this. Kraegeloh and Lockemann \cite{22} already did so in 1975. The main
difference between their proposal and ours is that they do not consider the additions of new values to their
database. As was demonstrated by Tarski and Givant \cite{29}, the expressive power of the data manipulation
language then is very limited.}
3 The Tarski algebra

3.1 Codd relations and mathematical relations

Throughout this paper, we assume infinitely enumerable sets $U$ and $V$ of attributes and values, respectively.

Using the formalism of [13], a Codd scheme is a finite subset of $U$. Let $\Omega$ be a Codd scheme. A tuple over $\Omega$ is a mapping of $\Omega$ into $V$; a Codd instance over $\Omega$ is a set of tuples over $\Omega$. All Codd instances are considered to be finite unless explicitly stated otherwise. Finally, a Codd relation is a pair $(\Omega, \omega)$ with $\Omega$ a Codd scheme and $\omega$ a Codd instance over $\Omega$.

In contrast to Codd relations, the building blocks of the relational database model [9, 10, 11], mathematical relations are sets of ordered pairs of values. For our purposes, a mathematical relation (or relation, for short) is a subset of $V \times V$. Of course, there is a close connection between a mathematical relation and a Codd relation with two attributes but, unlike the latter, the former does not have a scheme.

In what follows, we will develop tools to manipulate relations in the mathematical sense and show how these techniques apply to various database models. All relations are considered to be finite unless explicitly stated otherwise.

It is often important to know the values actually involved in a given relation $r$. We denote by $r\downarrow$ the smallest subset of $V$ such that $r \subseteq r\downarrow \times r\downarrow$, i.e., $r\downarrow = \{ v \in V \mid \exists w \in V : (v, w) \in r \lor (w, v) \in r \}$. The set $r\downarrow$ will be called the active domain of $r$. Similarly, if $r_1, \ldots, r_m$ are relations, then $(r_1, \ldots, r_m)\downarrow = r_1\downarrow \cup \ldots \cup r_m\downarrow$ is the active domain of the database consisting of the mathematical relations $r_1, \ldots, r_m$.

3.2 The basic algebra on relations

Mathematical relations, because of their natural affinity with graphs, are very important and fundamental data structures in several areas of computer science and, therefore, it is important to have appropriate tools to manipulate them. Because of the close connection between mathematical relations and Codd relations with two attributes, it seems natural to borrow these tools from relational algebra. However, long before Codd devised his relational algebra in 1970 [9], Tarski already had defined an algebra on mathematical relations [28]. The operators defined below are inspired by Tarski’s algebra.

**Definition 3.1** Let $r$ and $s$ be relations. We define the following operations:

- The inverse of $r$, denoted $r^{-1}$, is the relation $\{(v, w) \mid (w, v) \in r\}$.
- The complement of $r$, denoted $\overline{r}$, is the relation $\{(v, w) \mid (v, w) \in r\downarrow \times r\downarrow \land (v, w) \notin r\}$.
- The union of $r$ and $s$, denoted $r \cup s$, is the relation $\{(v, w) \mid (v, w) \in r \lor (v, w) \in s\}$.
- The composition of $r$ and $s$, denoted $r \circ s$, is the relation $\{(u, w) \mid \exists v \in V : (u, v) \in r \land (v, w) \in s\}$. 

A major difference between the operators defined above and their counterparts in Tarski’s algebra is that we define complementation relative to the active domain rather than to the
set $V$ of all values, as Tarski does. Our approach of course aims at avoiding generating infinite relations.

In order to introduce a basic algebra expression (bae) we assume an infinitely enumerable set of (relation) variables $x_1, x_2, x_3, \ldots$. Bae’s are then recursively defined as follows:

1. $\emptyset$ is a bae;
2. each variable $x_i$ is a bae; and
3. an expression obtained by applying the operators of Definition 3.1 to bae’s is again a bae.

In writing down expressions, we assume that unary operators take precedence over binary operators; also, composition takes precedence over set-like operators. If $E(x_1, \ldots, x_m)$ is a bae and $r_1, \ldots, r_m$ are relations, then the interpretation of $E(r_1, \ldots, r_m)$ is defined in the customary way. The set of all bae’s will be called the basic algebra.

### 3.3 Tagging

We will now formally introduce the concept of tagging. Intuitively, we view tagging as attaching new values (tags) to objects of the database with the purpose of identifying them. Tagging is a technique widely used in the field of databases, e.g., in order to manipulate nested relations or to introduce object identifiers in object-oriented databases. In this subsection, we will study the notion of tagging in the context of mathematical relations. Later on, we will discuss tagging in the context of relations, nested relations and object-oriented databases.

In order to introduce tagging, we have to assume that the set of values $V$ consists of an infinite set $D$ of data values and an infinite set $T$ of tag values, i.e., $V = D \cup T$ with $D \cap T = \emptyset$. The set $D$ consists of these values that actually represent “data” of our database, while $T$ is a set of “auxiliary” values, needed to introduce tags.

In the context of mathematical relations, the “objects” we want to tag are, quite straightforwardly, the ordered pairs. Before we can proceed to a formal definition however, we first have to realize that several variations on the idea of tagging are conceivable, most of which are actually around in database practice. To illustrate our point, consider the relations shown in Figure 9.

![Figure 9: Two mathematical relations](image)
A first tagging strategy we wish to discuss will be called *tuple tagging*. In this strategy, a “generic” tag is associated to each ordered pair, e.g., by writing its components between square brackets, as shown in Figure 10. A non-operational variation on tuple tagging has already been considered in [29].

![Figure 10: Tuple tagging](image)

A consequence of this approach is that a same pair always gets the same tag, irrespective of the relation in which it occurs. In Figure 10 for instance, the ordered pair \((a, b)\) obtains the same tag in both \(r\) and \(s\). One could easily conceive situations in which the tag should not only depend on the ordered pair but also on the relation in which the ordered pair occurs. This latter tagging strategy will be called *relation tagging*. The assignment of tags in Figure 11 is consistent with a relation tagging strategy.

![Figure 11: Relation tagging](image)

In the relation tagging strategy it is still possible that a same value is introduced multiple times, e.g., when a same relation needs to be tagged twice in the course of solving a query. The tagging strategy in which no value is introduced more than once as a tag will be called *universal tagging*.

Rather than making a choice between one of these various alternatives, we preferred to introduce a more general notion of tagging which encompasses all three strategies outlined above as special cases.

To define this generalized tagging, let \(V\) be the set of all pairs \((s, t)\) with \(s\) a relation (i.e., a finite subset of \(V \times V\)) and \(t = (v, w)\) an ordered pair over \(V\). \((s)\) will be called a *context*.

A *tagging function* \(\tau\) is a bijection:

\[
\tau : V \rightarrow T
\]

\((T\) being the set of tag values\)) such that \(\tau(s, t)\) does not occur in \(s \cup \{v, w\}\). Intuitively, \(\tau\) associates a tag value to the ordered pair \(t\) that depends on the context \(s\). Moreover, this
tag must be “new”, i.e., it can not be a component of the pair \( t \) and it must not already occur in the context \( s \).

The various tagging strategies discussed above can now easily be obtained by making a particular choice for the context.

If we systematically choose the empty relation \( \emptyset \) as the tagging context, we achieve tuple tagging. If, alternatively, we take for the context the relation in which the pair under consideration occurs, we get relation tagging. Finally, if the context is a representation of the set of all values that up to then did occur at some point in the database under consideration, then we achieve universal tagging. Most of the results we prove in this paper hold independent of the particular tagging strategy chosen. This will become important in Section 5.3, where we discuss the expressive power of the language we are presently defining.

Given a tagging function, we now introduce two tagging operators:

**Definition 3.2** Let \( r \) and \( s \) be relations.

- **The left-tagging of** \( r \) **in the context of** \( s \), denoted \( r_s^\lt \), **is the relation**
  \[
  \left\{ (\tau(s,(v,w)),v) \mid (v,w) \in r \right\}
  \]

- **The right-tagging of** \( r \) **in the context** \( s \), denoted \( r_s^\rg \), **is the relation**
  \[
  \left\{ (\tau(s,(v,w)),w) \mid (v,w) \in r \right\}
  \]

Observe that the relation \( r \) can always be reconstructed from its left- and right-tagging whatever context was used, provided it was the same for both operation. Indeed, left- and right-tagging always satisfy the property:

\[
 r = (r_s^\lt)^{-1} \cdot r_s^\rg
\]

As data values and tag values are disjoint \((D \cap T = \emptyset)\), the actual values of the tags are immaterial; only their equality or difference is of relevance. Therefore, in concrete examples, rather than specifying a tag function which would be quite cumbersome a job, we shall use (consecutive) integers to represent tag values; data values will be represented by lower case letters.

**Example 3.1** In Figure 12, we show the result of left and right tagging a relation in its own context.

The algebra obtained by augmenting the basic algebra with the left- and right-tagging operators will be called the **Tarski algebra**. An expression of the Tarski algebra will be called a **Tarski algebra expression (tae)** and defined in analogy to a bae.

The Tarski algebra is considerably more powerful than the basic algebra. To illustrate this, we show that:

**Proposition 3.1** Let \( r \) be a relation over \( V \). The identity operator

\[
 r^\text{id} = \{(v,w) \in r \downarrow \times r \downarrow \mid v = w \}
\]

**can be expressed in the Tarski algebra.**
\[ r \quad r^a_r \quad r^p_r \]

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<tr>
<td>c</td>
<td>c</td>
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<td>c</td>
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Figure 12: Example of tagging

**Proof:** Let \( s \) be an arbitrary relation. The proposition follows from the equality:

\[
r^{id} = (r^a_s)^{-1} \cdot r^a_s \cup (r^p_s)^{-1} \cdot r^p_s
\]

\( \square \)

Clearly, the identity operator is not expressible in the basic algebra. Several auxiliary operators can be expressed using the identity operator:

**Proposition 3.2** Let \( r \) and \( s \) be relations. The following operations can be expressed in the basic algebra plus the identity operator:

- the diversity operator \( r^{di} = \{(v, w) \in r \downarrow \times r \downarrow \mid v \neq w\} \);
- the relative universe \( r^{un} = r \downarrow \times r \downarrow \);

**Proof:** The proposition follows from the equalities:

\[
\begin{align*}
    r^{di} &= \overline{r^{id}} \\
    r^{un} &= r^{id} \cup r^{di}
\end{align*}
\]

\( \square \)

Another useful operator is the intersection operator. We show that intersection can also be expressed in the Tarski algebra. While the proof is based on the well-known set-theoretic property relating intersection, union, and complementation, some care is needed since, unlike in set theory, our notion of complementation is relative to the set of values occurring in the relation under consideration.

**Proposition 3.3** Let \( r \) and \( s \) be relations over \( V \). The intersection operator \( r \cap s = \{(v, w) \mid (v, w) \in r \land (v, w) \in s\} \) can be expressed in the Tarski algebra.

**Proof:** Let \( (r \cap s)^\equiv \) and \( (r \cap s)^\neq \) be the identical respectively the non-identical tuples of \( r \cap s \). The reader is invited to verify that

\[
(r \cap s)^\neq = r \cup (r \cup s)^{id} \cup s \cup (r \cup s)^{id} \cup (r \cup s)^{id}
\]  (1)
Now let $q$ be an arbitrary relation and let $p = (r \cup s)^c \cdot ((r \cup s)^c)^{-1}$.

Then

$$
(r \cap s)^c = r \cup (r \cup s \cup p)^d \cup s \cup (r \cup s \cup p)^d \cup (r \cup s \cup p)^d
$$

(2)

An expression for $r \cap s$ is now obtained by taking the union of the expressions displayed in (1) and (2). $\Box$

In order to understand better the previous proof it should be noted that whenever $(r \cup s)\downarrow$ is a singleton, $(r \cup s)^d = \emptyset$ due to the relative complementation. Hence $(r \cup s)^d$, contrary to $(r \cup s)\downarrow$, does not always contain all values present in $r$ or $s$, and for that reason the equality

$$(r \cap s)^c = r \cup (r \cup s)^d \cup s \cup (r \cup s)^d \cup (r \cup s)^d$$

does not always hold. If $r$ and $s$ are both the same singleton relations consisting of an identical pair, then the above expression yields the empty relation rather than the singleton relation. Since a tag to a pair by definition cannot be a value of that pair, $(r \cup s \cup p)\downarrow$ contains two values whenever $(r \cup s)\downarrow$ is a singleton. Hence $(r \cup s \cup p)^d$ always contains all values present in $r$, $s$ or $p$, whence equality (2) in the proof holds.

An alternative and much shorter proof follows from the equality

$$
r \cap s = (r^q)^{-1} \cdot s^q
$$

with $q$ an arbitrary relation. The simplicity of the latter equality is due to the fact that associating a tag to a same tuple in a same context, yields the same tag value. We preferred however to exhibit an expression which is not “context-sensitive”, proving that all tagging strategies presented thus far allow the expression of the intersection operator. Notice that although equality (3) can be used to show that intersection is expressible using the tuple tagging strategy (put $q = \emptyset$), it does not show that intersection is also expressible using the relation tagging strategy.

As a corollary to Proposition 3.3, we have:

**Proposition 3.4** Let $r$ and $s$ be relations. The following operations can be expressed in the Tarski algebra:

- the identity selection $r^= = \{(v, w) \in r \mid v = w\};$
- the diversity selection $r^\neq = \{(v, w) \in r \mid v \neq w\};$ and
- the difference $r - s = \{(v, w) \mid (v, w) \in r \land (v, w) \notin s\}$.

**Proof:** The proposition follows from the equalities:

$$
r^= = r^{id} \cap r \\
r^\neq = r^{di} \cap r \\
r - s = r \cap (s \cup r^{id} \cup s \cup r^{di})
$$

$\Box$

As a final example illustrating the power of the Tarski algebra, we formally introduce conditional queries:
Definition 3.3 A conditional query is an expression of the form

$$\begin{align*}
\text{if } E_1(\vec{x}_1) = \emptyset & \\
\text{then } & E_2(\vec{x}_2) \\
\text{else } & E_3(\vec{x}_3)
\end{align*}$$

with $E_1(\vec{x}_1)$, $E_2(\vec{x}_2)$ and $E_3(\vec{x}_3)$ arbitrary Tarski algebra expressions. ($\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ denote finite sequences of (mathematical) relation variables.)

We can show:

Proposition 3.5 The conditional expression $Q$ can be expressed in the Tarski algebra.

Proof: Consider the following expression:

$$E_4(\vec{x}_1, \vec{x}_2, \vec{x}_3) = E_3(\vec{x}_3) \cdot \left(E_1(\vec{x}_1) \cdot (E_1(\vec{x}_1) \cup E_3(\vec{x}_3))^{un}\right)^{id}$$

$$\cup E_2(\vec{x}_2) - E_2(\vec{x}_2) \cdot \left(E_1(\vec{x}_1) \cdot (E_1(\vec{x}_1) \cup E_2(\vec{x}_2))^{un}\right)^{id}$$

Clearly, for any three relation $r_1, r_2, r_3$, $Q(r_1, r_2, r_3) = E_4(r_1, r_2, r_3)$. $\square$

4 Simulating other database models

In this section, we intend to show that binary relations equipped with the Tarski algebra suffice to simulate many other languages. In particular, we shall show that the relational model, the nested model and the GOOD model, together with their respective basic languages, can be expressed in the Tarski model.

The Tarski algebra turns out to be too weak however to express more complete extensions of these languages. Therefore a possible extension of the Tarski algebra dealing with this problem will be discussed in Section 5.

4.1 Relational model

We first turn to the relational model [9, 10, 11], in which a database consists of a set of Codd relations (see Subsection 3.1). A Codd relation can be represented by a set of mathematical binary relations as follows:

Definition 4.1 Let $(\Omega, \omega)$ be a Codd relation over $V$ with $\Omega = \{A_1, \ldots, A_n\}$. A set of mathematical relations $\{r_{A_1}, \ldots, r_{A_n}\}$ is called a representation of $(\Omega, \omega)$ if there exists a one to one mapping $\rho$ from $\omega$ into $T$, called a representation function, such that for all $i = 1, \ldots, n$, $r_{A_i} = \{(\rho(t), t(A_i)) \mid t \in \omega\}$.

Using this encoding of Codd relations into mathematical relations, we can now give a natural meaning to simulating a relational query in the Tarski algebra:
Definition 4.2 Let \( E(x_1, \ldots, x_m) \) be an expression in the relational algebra in which \( x_i, 1 \leq i \leq m \), is a variable representing a Codd-relation of arity \( n_i \). We say that \( E(x_1, \ldots, x_m) \) can be simulated in the Tarski algebra if there exists a tae \( E'(y_1, \ldots, y_{n_1}, \ldots, y_{m_1}, \ldots, y_{mn}) \) such that for all Codd relations \( (\Omega_1, \omega_1), \ldots, (\Omega_m, \omega_m) \) for which \( E((\Omega_1, \omega_1), \ldots, (\Omega_m, \omega_m)) \) exists, and for all mathematical relations \( r_{11}, \ldots, r_{1n_1}, \ldots, r_{m1}, \ldots, r_{mn} \) for which the set \( \{r_{11}, \ldots, r_{mn}\} \) is a representation of \( (\Omega_i, \omega_i) \) for all \( i, 1 \leq i \leq m \), \( E'(r_{11}, \ldots, r_{1n_1}, \ldots, r_{m1}, \ldots, r_{mn}) \) is a representation of \( E((\Omega_1, \omega_1), \ldots, (\Omega_m, \omega_m)) \).

We have:

**Theorem 4.1** The relational algebra can be simulated in the Tarski algebra.

**Proof:** We show how each of the relational algebra operators can be simulated in the Tarski algebra. We shall develop the proof in such a way that it will become obvious that each of the tagging strategies discussed in Subsection 3.3 suffices to achieve the simulations. To this end, \( p \) and \( q \) will denote arbitrary mathematical relations throughout this proof.

In order not to obscure the argument with heavy notation, we shall limit the number of attributes in the relations under consideration. The generalization of our argument, however, will be obvious.

**Projection**

Let \( (\Omega, \omega) \) be a relation with \( \Omega = \{A, B, C\} \) and let \( \{r_A, r_B, r_C\} \) be a representation of \( (\Omega, \omega) \). Consider the projection \( \pi_{\{A,B\}}(\Omega, \omega) \) of \( \Omega, \omega \) onto \( \{A, B\} \). Unfortunately \( \{r_A, r_B\} \) in general is not a representation of \( \pi_{\{A,B\}}(\Omega, \omega) \), simply because some tuples of this projection may be multiply represented. In order to overcome this problem, we “retag” \( r_A \) and \( r_B \) as follows.

First, consider the relation \( r_A' = r_A^{-1} \cdot r_A \). This relation consists of all identical pairs of \( A \)-values occurring in \( \omega \). Now let
\[
s_A = (r_A')_p \cdot r_A^{-1}
\]
Each tag in \( s_A \) corresponds to precisely one tuple of \( \pi_A(\Omega, \omega) \). Moreover, the relation \( s_A \) gives the relationship between the “new” and the “old” tags.

Now, let \( r_B' = s_A \cdot r_B \) and let
\[
s_B = (r_B')_q \cdot r_B^{-1} \cap (r_B')_q \cdot s_A
\]
Each tag in \( s_B \) corresponds to precisely one tuple of \( \pi_{\{A,B\}}(\Omega, \omega) \). Moreover, the relation \( s_B \) gives the relationship between the newly created tags and the original ones. Hence \( \{s_B \cdot r_A, s_B \cdot r_B\} \) is a representation of \( \pi_{\{A,B\}}(\Omega, \omega) \).

**Selection**

Let \( (\Omega, \omega) \) be a relation with \( \Omega = \{A, B, C\} \) and let \( \{r_A, r_B, r_C\} \) be a representation of \( (\Omega, \omega) \). Consider the selection \( \sigma_{A=B}(\Omega, \omega) \) of \( \Omega, \omega \).

Let \( r = (r_A \cdot r_A^{-1})^{id} \) be the set of all identical pairs of “tags” used in the representation of \( (\Omega, \omega) \). Let \( r_{A,B} = r_A \cdot r_B^{-1} \cap r \). The relation \( r_{A,B} \) is the set of all identical pairs of tags corresponding to tuples \( t \) of \( \omega \) with \( t(A) = t(B) \). Hence, \( \{r_{A,B} \cdot r_A, r_{A,B} \cdot r_B, r_{A,B} \cdot r_C\} \) is a representation of \( \sigma_{A=B}(\Omega, \omega) \).
Let \((\Omega, \omega_1)\) and \((\Omega, \omega_2)\) be relations with \(\Omega = \{A,B\}\). Let \(r_{1,A}, r_{1,B}\) and \(r_{2,A}, r_{2,B}\) be representations of \((\Omega, \omega_1)\) and \((\Omega, \omega_2)\), respectively.

The proof for this case is quite similar to the proof for the case of projection.

First, consider the relation \(r'_A = r_{1,A}^{-1} \cdot r_{1,A} \cup r_{2,A}^{-1} \cdot r_{2,A}\) consisting of all identical pairs of \(A\)-values occurring in \(\omega_1\) or \(\omega_2\). Let

\[
\begin{align*}
s_{1,A} &= (r'_A)^p \cdot r_{1,A}^{-1} \\
s_{2,A} &= (r'_A)^p \cdot r_{2,A}^{-1}
\end{align*}
\]

Each tag value introduced above corresponds to precisely one tuple of \(\pi(A)(\omega_1 \cup \omega_2)\). Moreover, \(s_{1,A}\) and \(s_{2,A}\) give the relationship between the “new” and the “old” tags.

Now, let \(r''_B = s_{1,A} \cdot r_{1,B} \cup s_{2,A} \cdot r_{2,B}\) and let

\[
\begin{align*}
s_{1,B} &= (r''_B)^p \cdot r_{1,B}^{-1} \cap (r''_B)^q \cdot s_{1,A} \\
s_{2,B} &= (r''_B)^p \cdot r_{2,B}^{-1} \cap (r''_B)^q \cdot s_{2,A}
\end{align*}
\]

Each tag value introduced above corresponds to precisely one tuple of \((\Omega, \omega_1 \cup \omega_2)\). Moreover, \(s_{1,A}\) and \(s_{2,A}\) give the relationship between the newly created tags and the original ones. Hence \(\{s_{1,B} \cdot r_{1,A} \cup s_{2,B} \cdot r_{2,A}, s_{1,B} \cdot r_{1,B} \cup s_{2,B} \cdot r_{2,B}\}\) is a representation of \((\Omega, \omega_1 \cup \omega_2)\).

**Difference**

Let \((\Omega, \omega_1)\) and \((\Omega, \omega_2)\) be relations with \(\Omega = \{A,B,C\}\). Let \(\{r_{1,A}, r_{1,B}, r_{1,C}\}\) and \(\{r_{2,A}, r_{2,B}, r_{2,C}\}\) be representations of \((\Omega, \omega_1)\) and \((\Omega, \omega_2)\), respectively. Let

\[r = r_{1,A} \cdot r_{2,A}^{-1} \cap r_{1,B} \cdot r_{2,B}^{-1} \cap r_{1,C} \cdot r_{2,C}^{-1}\]

Let \(\rho_1\) and \(\rho_2\) be the representation function used for representing \((\Omega, \omega_1)\) respectively \((\Omega, \omega_2)\). Then \(r = \{(\rho_1(t), \rho_2(t)) | t \in \omega_1 \cap \omega_2\}\). Now let \(s = (r_{1,A} \cdot r_{1,A}^{-1})^{id} - (r \cdot r^{-1})^{id}\). Then \(s = \{(\rho_1(t), \rho_1(t)) | t \in \omega_1 \cap \omega_2\}\). Hence \(\{s \cdot r_A, s \cdot r_B, s \cdot r_C\}\) is a representation of \((\Omega_1, \omega_1 - \omega_2)\).

**Cartesian product**

Finally, let us consider the cartesian product \((\Omega_1 \cup \Omega_2, \omega_1 \times \omega_2)\) of two relations \((\Omega_1, \omega_1)\) and \((\Omega_2, \omega_2)\), \(\Omega_1 \cap \Omega_2 = \emptyset\), with \(\Omega_1 = \{A,B,C\}\) and \(\Omega_2 = \{D,E\}\). Let \(\{r_A, r_B, r_C\}\) be a representation of \((\Omega_1, \omega_1)\) and let \(\{r_D, r_E\}\) be a representation of \((\Omega_2, \omega_2)\). Let \(r_1 = (r_A \cdot r_A^{-1})^{id}\), \(r_2 = (r_D \cdot r_D^{-1})^{id}\) and \(r = (r_1 \cup r_2)^{un}\). Let \(s = r_1 \cdot r_2\). Then \(s = \{(\rho_1(t_1), \rho_2(t_2)) | t_1 \in \omega_1 \land t_2 \in \omega_2\}\). Hence \(\{s_p^a \cdot r_A, s_p^a \cdot r_B, s_p^a \cdot r_C, s_p^a \cdot r_D, s_p^a \cdot r_E\}\) is a representation of \((\Omega_1 \cup \Omega_2, \omega_1 \times \omega_2)\).

\(\square\)

### 4.2 Nested model

In this subsection, we discuss the simulation of the nested model (e.g., [26, 30]) in the Tarski approach. For a brief introduction to the model, we refer to [13, 16], the formalism of which we shall adopt here.
For the sake of concreteness and clarity, we shall use the very simple Codd relation \((\Omega, \omega)\) shown in Figure 13 (left) as a running example. Figure 13 (middle) shows \(\nu_{\{A\}}(\Omega, \omega)\), the result of nesting \((\Omega, \omega)\) over \(\{A\}\) and Figure 13 (right) shows \(\nu_{\{B\}}\nu_{\{A\}}(\Omega, \omega)\), the result of nesting \(\nu_{\{A\}}(\Omega, \omega)\) over \(\{B\}\).

Ways of representing nested relations have already been discussed in the Introduction. We shall therefore concentrate on how these representations can be obtained in the Tarski algebra, starting for instance from the representation of a flat relation.

Therefore, let \(\{r_A, r_B\}\) be some representation of \((\Omega, \omega)\) in the Tarski model. By Theorem 4.1, we know that there exists a representation \(r'_{AB}\) of \(\pi_{\{A\}}(\Omega, \omega)\), as shown in Figure 14 (left). Figure 14 (middle) shows the relation \(r'_B = r'_{AB} \cdot (r_A)^{-1} \cdot r_B\).

Alternatively, the tags in Figure 14 can be interpreted as representing the sets of \(B\)-values occurring in \(\nu_{\{A\}}(\Omega, \omega)\). From this standpoint, the relation in Figure 14 (left) can be seen as
associating $A$-values with tags representing sets of $B$-values while the relation in Figure 14 (middle) identifies these sets.

Unfortunately, and contrary to the representation proposed in the Introduction, the same set of $B$-values can have more than one tag, as is the case in our example. Although we cannot do away with multiply represented sets in the Tarski algebra, we can compute all pairs of tags representing the same set. Turning to our example, let

$$r'_B = r'_B \cdot (r'_B)^{\text{un}} \cdot r'_B - r'_B$$

In $r'_B$, each tag value is associated to all values not belonging to the set of $B$-values identified by that tag. Using the set-theoretical property that two sets are different if and only if the first set and the complement of the second have a nonempty intersection, it follows that $r'_B \cdot (r'_B)^{-1}$ is the set of all pairs of tags representing different sets of $B$-values and that

$$r' = r'_B \cdot (r'_B)^{\text{un}} \cdot (r'_B)^{-1} - r'_B \cdot (r'_B)^{-1}$$

shown in Figure 14 (right) is the set of all pairs of tags representing the same set of $B$-values.

Hence $\{r'_{AB}, r'_B, r'\}$ can be seen as a representation of $\nu\{A\}(\Omega, \omega)$.

<table>
<thead>
<tr>
<th>$r''_{AB}$</th>
<th>$r''_{A}$</th>
<th>$r''_{B}$</th>
<th>$r''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>1 a</td>
<td>1 b</td>
<td>1 1</td>
</tr>
<tr>
<td>2 2</td>
<td>1 d</td>
<td>2 b</td>
<td>2 1</td>
</tr>
<tr>
<td>3 3</td>
<td>2 a</td>
<td>2 c</td>
<td>2 2</td>
</tr>
<tr>
<td></td>
<td>2 d</td>
<td>3 a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 c</td>
<td>3 3</td>
<td></td>
</tr>
</tbody>
</table>

Figure 15: A representation of $\nu\{B\}\nu\{A\}(\Omega, \omega)$

Figure 15 shows a possible representation of $\nu\{B\}\nu\{A\}(\Omega, \omega)$. The tags in $r''_A = r' \cdot r'_{AB}$ can now also be seen as representing the sets of $A$-values occurring in $\nu\{B\}\nu\{A\}(\Omega, \omega)$. $r''_B = r'_B$ still shows the set of $B$-values in $\nu\{B\}\nu\{A\}(\Omega, \omega)$. $r''_{AB} = r''_A \cdot (r''_A)^{-1}$ associates sets of $A$-values and sets of $B$-values and, finally, $r'' = r'$ indicates which pairs of tags identify the same set of $B$-values (and hence also the same set of $A$-values).

To some extent, it is remarkable that the nested model can be simulated in the Tarski model without having to add complex constructs to the latter. However, this fact is not so surprising when seen in the perspective of the main result of [25]. There it has been shown that all nested algebra queries on flat relations returning a flat relation can be expressed in the flat relational algebra as well. In other words, the nest operator and its counterpart the unnest operator do not add anything essential to the language in terms of expressive power and only affect the way in which the data are grouped. In the Tarski algebra, this grouping is achieved using tagging.

### 4.3 GOOD model

In Section 2, we already explained how to represent a GOOD instance [14, 15] in the Tarski model. In particular, Figure 8 represents the GOOD instance shown in Figure 5. In this
subsection, we shall discuss how the three queries dealt with in Subsection 2.1 and shown in Figure 6 can be solved in the Tarski algebra. Due to space limitations, this discussion will be held rather informal. We trust the reader by now has acquired sufficient familiarity with the Tarski algebra to fill in the technical details.

The first query, shown in Figure 6 (left), was a node addition adding a new node for each pair of parents having a common child. Using the representation in Figure 8, this query can be solved as follows in the Tarski algebra. First, the relation $ch \cdot ch^{-1}$ contains the set of all pairs of parent nodes having a common child. For these, new nodes can be “created” by tagging $ch \cdot ch^{-1}$. The left and right tagging automatically provide representations for the newly introduced edge labels “1” and “2”, respectively. A suitable node label for the newly created nodes could be obtained from tagging the relation $(nodelabels^{-1} \cdot nodelabels)^{id}$ as the tag associated to the pair $(P, P)$.

The second query, shown in Figure 6 (middle), was an edge addition associating children directly to parent pairs. In the Tarski representation, a relation representing the edge label $c$ must then be added. This can easily be achieved by composing the relations representing the edge labels “1” and “2” with $ch$, and taking their intersection.

Finally, the third query, shown in Figure 6 (right), was an edge deletion removing all the original parent-children links. This operation can be simulated straightforwardly by removing the relation $ch$.

At this point, it should be emphasized that more elaborate GOOD queries, involving unbounded iteration, cannot be expressed in the Tarski algebra. In order to overcome this problem, we shall discuss ways in the next section to extend the Tarski algebra to a computationally complete language.

5 The extended Tarski algebra

From Theorem 4.1, in which we showed that the Tarski algebra can emulate Codd-relational queries, it follows that the Tarski algebra is at least as expressive as the standard (Codd-) relational algebra. It should be clear however that, as already alluded to at the end of the previous section, there remain queries which can not be expressed in the Tarski algebra. The classical example of such a query is the transitive closure of a binary relation. (If $r$ is a binary relation, then its transitive closure is defined as the smallest relation $s$ containing $r$, such that whenever $(u, v)$ and $(v, w)$ are in $s$, $(u, w)$ is also in $s$.) The usual solution to this problem is to add a looping construct to the language. This will be done in Subsection 5.1.

To examine the expressiveness of this extended Tarski algebra, we need to re-investigate the notion of “generic query”. This will be the subject of Subsection 5.2. In Section 5.3 we will then show that the extended algebra is computationally complete for the class of generic queries defined in Section 5.2, thus emphasizing the combined power of tagging and unbounded looping in the context of mathematical relations.

5.1 While-expressions

In this subsection, we will introduce the extended Tarski algebra obtained by augmenting the Tarski algebra with assignment statements and while-loops. These additional constructs
will be written in a program-like fashion so as to make their semantics obvious.

In the previous section, we saw that in order to simulate a relation or a database in another model, several binary relations are needed. Therefore we shall introduce assignment queries in which several binary relations can be computed in one expression. These assignment queries will then turn out to be key constructs in the definition of an iterative query.

We define:

**Definition 5.1** Let $\vec{u}_1, \ldots, \vec{u}_l$ be finite sequences of relation variables. An assignment query $E(x_1, \ldots, x_m)$ is an expression of the form:

```plaintext
query $E(x_1, \ldots, x_m)$;
begin
  $z_1 := E_1(\vec{u}_1)$;
  $z_2 := E_2(\vec{u}_2)$;
  \vdots
  $z_l := E_l(\vec{u}_l)$;
return $(y_1, \ldots, y_n)$
end
```

where, for $j = 1, \ldots, l$,

1. the variables in $\vec{u}_j$ belong to \{ $x_1, \ldots, x_m, z_1, \ldots, z_{j-1}$ \};
2. $z_j$ may belong to \{ $x_1, \ldots, x_m, z_1, \ldots, z_{j-1}$ \}; and
3. $E_j(\vec{u}_j)$ is an (extended) Tarski algebra expression returning a single relation,

and $y_1, \ldots, y_n$ belong to \{ $x_1, \ldots, x_m, z_1, \ldots, z_l$ \}.

Given a relation $r$, we shall make no distinction between $r$ and the 1-tuple ($r$). In this way, the query remains meaningful if the right-hand side of one of the assignments is itself, e.g., an assignment query returning one relation. Conversely, this assumption also allows us to interpret ordinary Tarski algebra expressions as special cases of assignment queries.

Using assignment queries, we can define iterative queries:

**Definition 5.2** Let $\vec{u}$ and $\vec{v}$ be finite sequences of variables, and let $u_k$ be the last element of $\vec{u}$. An iterative query $E(x_1, \ldots, x_m)$ is an expression of the form:

```plaintext
query $E(x_1, \ldots, x_m)$;
begin
  $\vec{u} := F(x_1, \ldots, x_m)$;
  while $u_k \neq \emptyset$
    do
      $\vec{u} := G(\vec{v})$
    od;
return $(y_1, \ldots, y_n)$
end
```

where $y_1, \ldots, y_m$, and the variables in $\vec{v}$ are in $\vec{u}$ or in \{ $x_1, \ldots, x_m$ \}, and $F$ and $G$ are assignment queries.
**Example 5.1** The following is an iterative query for the transitive closure query mentioned in the introduction to this section:

```plaintext
query E(x);
begin
    y := x;
    z := y \cdot y - y;
    while z \neq \emptyset 
        do
            y := y \cup x \cdot y;
            z := y \cdot y - y;
        od;
    return y
end
```

The smallest set of expressions containing the taes and closed under assignment and iteration will be called the *extended Tarski algebra*; we will denote its expressions as *etae’s*.

We observe that etaes may also contain *conditional statements*. These can indeed be replaced by a number of ordinary assignments as can be easily seen using an argument analogous to the proof of Proposition 3.5.

We claim that adding assignment and iteration to the Tarski algebra makes this query language computationally complete. In order to substantiate this claim, however, we need to review the notion of genericity in the presence of tags.

### 5.2 Generic queries

In this subsection we will discuss the notion of genericity in query environments wherein tags can be introduced. We would like to emphasize that the contents of this section are completely independent of any specific query language. For us, a *query* is merely a (partial) mapping from $R^m$ to $R^n$ for some $m$ and $n$, where $R$ is the set of all (mathematical) relations over $V = D \cup T$.

The notion of genericity was originally introduced by Aho and Ullman [3] and by Chandra and Harel [7] in the context of the relational model and has subsequently become a subject of frequent study in other query environments (e.g., [1, 2, 17]). The principle of all notions of genericity that are around is that a generic query should commute with all permutations on the set of data values. While this principle can immediately be applied to, e.g., the relational model, this is not the case for more elaborate database models where the result of a query may contain objects or values not present in one of the arguments. In these cases, the original permutation must first be extended in some “natural” way to the newly introduced objects or values.

Therefore we shall first discuss how a permutation $\psi$ on the set of data values can naturally be extended to a permutation $\psi^\mathcal{E}$ on $V$. This will be done in an iterative way. Let $D_0 = D$, and let for all $i \geq 0$, $V_i$ be the largest subset of $V^2$ in which only values from $V$ can be.

\[ \text{Recall that } V \text{ is the set of all pairs } (s, t) \text{ with } s \text{ a relation and } t \text{ an ordered pair over } V \text{ (Subsection 3.3).} \]
Theorem 5.1 \textit{The function }$\psi^e$\textit{ has the following properties:}

1. \textit{the restriction of }$\psi^e$\textit{ to }$D$\textit{ equals }$\psi$,

2. \textit{the restriction of }$\psi^e$\textit{ to }$T$\textit{ equals }$\tau \psi^e \tau^{-1}$, and

3. \textit{$\psi^e$ is a permutation on }$V$\textit{.}

\textbf{Proof:} Since Properties 1 and 2 follow directly from the definition of $\psi^e$, we immediately turn to Property 3. Since $V = \bigcup_{i \geq 0} D_i$ and $D_{i-1} \subseteq D_i$ for $i \geq 0$, it suffices to establish by induction that for each $i$, $i \geq 0$, the restriction of $\psi^e$ to $D_i$ is a permutation on $D_i$. Since $D_0 = D$, it follows from Property 1 that $\psi^e$ is permutation on $D_0$. Assume that $\psi^e$ is a permutation on $D_i$, $i \geq 0$. We need to prove that $\psi^e$ is also a permutation on $D_{i+1}$.

First, we show that $\psi^e$ is an injection. Therefore, let $x_1$, $x_2 \in D_{i+1}$ be such that $\psi^e(x_1) = \psi^e(x_2)$. If $x_1$ and $x_2$ are both in $D_i$, then $x_1 = x_2$, by the induction hypothesis. So assume that one of these two values, say $x_1$, is an element of $D_{i+1} - D_i$. Let $(s_1, t_1) \in \mathcal{V}_i$ be such that $\tau(s_1, t_1) = x_1$. Then by Property 2, $\psi^e(x_1) = \tau(\psi^e(s_1), \psi(t_1))$. If $x_2$ would be in $D_0$ then $\psi^e(x_2) = \psi(x_2) = \tau(\psi^e(s_1), \psi(t_1)) = \psi^e(x_1)$, a contradiction, because $\psi(x_2)$ is in $D$. Hence for some $k$, $0 \leq k \leq i$, there exists $(s_2, t_2)$ in $\mathcal{V}_k$ (and hence also in $\mathcal{V}_i$) such that $\tau(s_2, t_2) = x_2$. Then by Property 2, $\psi^e(x_2) = \tau(\psi^e(s_2), \psi(t_2))$. Now, because $\psi^e(x_1) = \psi^e(x_2)$ and $\tau$ is a bijection, it follows that $(\psi^e(s_1), \psi^e(t_1)) = (\psi^e(s_2), \psi^e(t_1))$. Since $s_1$, $s_2$, $t_1$ and $t_2$ take all their values in $D_i$, it follows by induction that $(s_1, t_1) = (s_2, t_2)$ and therefore $x_1 = x_2$. Thus we have established that $\psi^e$ is an injection on $D_{i+1}$.

Let us now finally prove that $\psi^e$ is also a surjection. Let $x \in D_{i+1}$. If $x \in D_i$, it follows from the induction hypothesis that $x$ is in the range of $\psi^e$ restricted to $D_i$ and we are done. Therefore assume that $x \in D_{i+1} - D_i$ and let $(s, t) \in \mathcal{V}_i$ be such that $\tau(s, t) = x$. Since $s$ and $t$ take all their values in $D_i$, it follows from the induction hypothesis that there exist an $s'$ and a $t'$, which take their values in $D_i$, such that $\psi^e(s') = s$ and $\psi^e(t') = t$. By Property 2 it now easily follows that $x = \psi^e(\tau(\gamma', t'))$. Thus $\psi^e$ is a also surjection on $D_{i+1}$, completing the proof. \hfill $\Box$

In Lemma 5.1 we established that any permutation $\psi$ on $D$ can be extended to a permutation $\psi^e$ on $V$ such that this extension permutes with the tagging function $\tau$ on $T$. In the next theorem we will establish that $\psi^e$ is the only extension with this property.

\textbf{Theorem 5.1} Let $\psi$ be a permutation on $D$. Then there exists a unique permutation $\phi$ on $V$ which is an extension of $\psi$ such that the restriction of $\phi$ to $T$ equals $\tau \phi \tau^{-1}$.

\textsuperscript{3}All mappings under consideration are assumed to be extended to sets, ordered pairs and relations in the natural way.
Proof: The existence of the permutation $\phi$ follows from Lemma 5.1. We now show that there is only one such permutation.

Let $\phi_1$ and $\phi_2$ be permutations on $V$ which are both extensions of $\psi$ and which both permute with $\tau$ on $T$. Reconsider the sets $D_i$ ($i \geq 0$). We will prove by induction that $\phi_1$ and $\phi_2$ agree on $D_i$ for each $i \geq 0$. Since $V = \bigcup_{i \geq 0} D_i$ it will then follow that $\phi_1 = \phi_2$.

Since $D_0 = D$, it follows immediately that $\phi_1$ and $\phi_2$ agree on $D_0$. Assume by induction that $\phi_1$ and $\phi_2$ agree on $D_i$ ($i \geq 0$). We need to prove that $\phi_1$ and $\phi_2$ also agree on $D_{i+1}$. Let therefore $x \in D_{i+1} - D_i$. Then there exists $(s,t) \in V_i$ be such that $\tau(s,t) = x$. Since $s$ and $t$ take all their values in $D_i$, it follows from the induction hypothesis that $\phi_1(s,t) = \phi_2(s,t)$ and therefore also that $\tau(\phi_1(s,t)) = \tau(\phi_2(s,t))$. Because both $\phi_1$ and $\phi_2$ permute with $\tau$ and because $x = \tau(s,t)$, it follows that $\phi_1(x) = \phi_2(x)$. Hence, $\phi_1$ and $\phi_2$ also agree on $D_{i+1}$. \qed

Let us denote by $S_\tau$ the set of all permutations on $V$ that permute with $\tau$ on $T$. Theorem 5.1 establishes that there is an injective mapping from the set of permutations over $D$ into $S_\tau$. This mapping sends a permutation $\psi$ on $D$ to its unique extension $\psi^e$ on $V$ which permutes with $\tau$. In the next theorem, we will establish that this mapping is actually a bijection.

Theorem 5.2 Let $\phi$ be a permutation over $V$ which permutes with $\tau$ on $T$. Then there exists a permutation $\psi$ over $D$ such that $\psi^e = \phi$, where $\psi^e$ is the unique extension of $\psi$ the restriction of which to $T$ permutes with $\tau$.

Proof: First we show that the restriction of $\phi$ to $D$ must be a permutation on $D$.

Let $x \in D$ and consider $\phi(x)$. We need to show that $\phi(x)$ is in $D$. Assume that $\phi(x) \notin T$. Since $T = \bigcup_{i \geq 0} D_i - D_0$, it follows that there exists a context $s$ and a pair $t$ taking their values in $V$ such that $\phi(x) = \tau(s,t)$. Thus $x = \phi^{-1}(\tau(s,t))$. Because the restriction of $\phi$ to $T$ permutes with $\tau$, it follows that also $x = \tau(\phi^{-1}(s),\phi^{-1}(t))$ implying that $x \in T$, a contradiction.

Next we need to show that if $x \in T$ then $\phi(x)$ is also in $T$. Since $x \in T$, we can again find a context $s'$ and a pair $t'$ taking their values in $V$ such that $x = \tau(s',t')$. Thus $\phi(x) = \phi(\tau(s',t')) = \tau(\phi(s'),\phi(t'))$ is also in $T$.

Having established that the restriction to $D$ of $\phi$ is a permutation on $D$, let $\psi$ be this restriction. The theorem then immediately follows from Theorem 5.1. \qed

Theorem 5.2 states that there is natural bijection between the set of permutations over $D$ and the set $S_\tau$ of permutations over $V$ that permute with $\tau$ on $T$. This allows us to finally define in a natural way the concept of a $\tau$-generic query in the context of tagging.

Definition 5.3 A query $Q: R^m \rightarrow R^n$ is $\tau$-generic if for each $\phi \in S_\tau$ and for each sequence of relations $r_1, \ldots, r_m$ in $R$ such that $Q(r_1, \ldots, r_m)$ is defined one has that

$$\phi(Q(r_1, \ldots, r_m)) = Q(\phi(r_1, \ldots, r_m))$$

In the following proposition we establish that all the operators of the Tarski algebra define $\tau$-generic queries.
Proposition 5.1 The union, composition, inverse, complement, left-tagging, and right-tagging operators define $\tau$-generic queries.

Proof: We will only prove this proposition for the left-tagging operator. Let $r$ and $s$ be relations and let $\phi \in S_\tau$. We have to establish that

$$\phi(r^\varphi_s) = \phi(r)_{\phi(s)}^\varphi$$

Let $t \in \phi(r^\varphi_s)$. Then there exists a pair $(v, w) \in r$ such that

$$t = \left(\phi\left(\tau(s, (v, w))\right), \phi(v)\right)$$

Since the restriction of $\phi$ to $T$ permutes with $\tau$,

$$t = \left(\tau\left(\phi(s), \phi(v, w)\right), \phi(v)\right)$$

Thus $t \in \phi(r)_{\phi(s)}^\varphi$. A similar argument can be made to show the converse. □

We can now state our first main result.

Theorem 5.3 If $E$ is an extended Tarski expression then $E$ defines a $\tau$-generic query.

Proof: This theorem follows by induction from Proposition 5.1. □

5.3 Computational completeness of the extended Tarski algebra

In this section we will establish that the extended Tarski algebra (relative to a fixed computable tagging function $\tau$) is a computationally complete language for a large class of $\tau$-generic queries.

In the previous subsection, we “decomposed” the set $V$ of all values into a hierarchy of layers $D_i$. In order to define the class of $\tau$-generic queries for which we will show the completeness of the extended Tarski algebra, we need to elaborate a little further on this decomposition into layers.

Definition 5.4 Let $\tau$ be a tagging function. Let $W \subseteq V$. We define recursively:

1. $W_0 = W$,

2. $W_i$, $i \geq 0$, is the largest subset of $V$ in which only values from $W_i$ are used; and

3. for $i > 0$, $W_i = W \cup \tau(W_{i-1})$.

The set $W_\infty = \bigcup_{i \geq 0} W_i$ is called the natural extension of $W$ (with respect to $\tau$).

We now define:
Definition 5.5 Let \( \tau \) be a tagging function. A query \( Q : \mathcal{R}^m \rightarrow \mathcal{R}^n \) is called \( \tau \)-domain-preserving if for all relations \( r_1, \ldots, r_m \),

\[
Q(r_1, \ldots, r_m) \downarrow \subseteq (r_1 \downarrow \cup \ldots \cup r_m \downarrow)_\infty
\]

with \((r_1 \downarrow \cup \ldots \cup r_m \downarrow)_\infty\) the natural extension of \( r_1 \downarrow \cup \ldots \cup r_m \downarrow \) with respect to \( \tau \).

The notion of domain-preserving originates from the fact that the (extended) Tarski algebra only allows for tagging, and not for “de-tagging”. A query such as “find the minimal set of data values necessary to generate all values of a given input relation” is clearly beyond the power of the Tarski operators and must therefore be excluded. Obviously, we have:

Theorem 5.4 If \( E \) is an extended Tarski expression then \( E \) defines a \( \tau \)-domain-preserving query.

We now have the following theorem.

Theorem 5.5 Let \( \tau \) be a computable tagging function. The extended Tarski algebra with tagging function \( \tau \) is complete for the class of computable, \( \tau \)-domain-preserving, \( \tau \)-generic queries.

Due to space limitations, we only give a sketch of the proof of Theorem 5.5. By Theorems 5.3 and 5.4, each extended Tarski expression defines a \( \tau \)-domain-preserving, \( \tau \)-generic query which is obviously computable. So we still need to prove that if \( Q \) is a computable, \( \tau \)-domain-preserving, \( \tau \)-generic query, then there exists an extended Tarski expression \( E_Q \) defining the same function.

Our proof follows that of a theorem by Abiteboul and Vianu ([2], Theorem 3.2.2, p. 212). This theorem states that the language \( \text{det} TL \) is complete with respect to the class of typed computable generic queries on Codd-relational databases. The key ideas of their proof are the following ([2], p. 202):

1. show that \( \text{det} TL \) can simulate counters and thus has full Turing machine capability, and

2. show that \( \text{det} TL \) can be used to compute the “Gödel number” of a Codd-relational database and, conversely, that a Codd-relational database can be computed in \( \text{det} TL \) from its Gödel number.

The computation then proceeds as follows. Let \( I \) be the input database and let \( I \downarrow \) denote the set of domain elements in \( I \). Consider the set \( \Sigma(I) \) of all total orderings on \( I \downarrow \). (Abiteboul and Vianu show that \( \Sigma \) can be computed in \( \text{det} TL \) and represented in a ternary relation.) Then compute (in parallel) for each order \( \sigma \in \Sigma(I) \) the Gödel number \( \Gamma(\sigma, I) \) of the input database \( I \). The genericity of this auxiliary query guarantees that if \( \sigma_1 \) and \( \sigma_2 \) are two orderings on \( I \downarrow \), then \( \Gamma(\sigma_1, I) \) and \( \Gamma(\sigma_2, I) \) represent the same number relative to the respective orderings. (In other words, the Gödel number of \( I \) is independent of the choice of a particular order on its domain \( I \downarrow \).) Next, the simulation of a Turing machine is performed on the Gödel numbers of \( I \) yielding the Gödel numbers of the output database of the query. Due
to the genericity of the simulated query, the Gödel numbers of the resulting Codd-relational database still represent the same number when seen relative to the corresponding orderings. This property allows to decode the Gödel numbers unambiguously to finally yield the actual output database.

We can use the Abiteboul-Vianu proof for Theorem 5.5 if we can show that in the extended Tarski algebra:

1. for an input (mathematical) relational database $I$, the set $\Sigma(I)$ of all orderings over $I \downarrow$ can be computed, and

2. counter machines can be simulated.

Turning to the first item, it should first be mentioned that this claim too would follow from the Abiteboul-Vianu proof if it would only have been the case that the set $\Sigma(I)$ was represented by binary Codd relations. Unfortunately for us, Abiteboul and Vianu coded $\Sigma(I)$ as a ternary relation. To be more specific, let $I$ be a database such that $I \downarrow = \{a, b, c\}$. Then Abiteboul and Vianu represent $\Sigma(I)$ as in Figure 16.

![Figure 16: The ternary relation $\Sigma(I)$](image)

In the ternary relation of Figure 16, the leftmost column ($\Sigma$) gives the names of the six total orderings on $\{a, b, c\}$. Then, for $1 \leq i \leq 6$ and $x, y \in \{a, b, c\}$,
1. a tuple of the form \((\sigma_i, \heartsuit, v)\) indicates that \(v\) is the smallest element in the ordering \(\sigma_i\),

2. a tuple of the form \((\sigma_i, w, \spadesuit)\) indicates that \(w\) is the largest element in the ordering \(\sigma_i\), and

3. a tuple of the form \((\sigma_i, v, w)\) indicates that \(v\) is smaller than \(w\) in the ordering \(\sigma_i\).

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
1 & \sigma_1 & 1 & \heartsuit & 1 & a \\
2 & \sigma_1 & 2 & a & 2 & b \\
3 & \sigma_1 & 3 & b & 3 & c \\
4 & \sigma_1 & 4 & c & 4 & \spadesuit \\
5 & \sigma_2 & 5 & \heartsuit & 5 & a \\
6 & \sigma_2 & 6 & a & 6 & c \\
7 & \sigma_2 & 7 & c & 7 & b \\
8 & \sigma_2 & 8 & b & 8 & \spadesuit \\
9 & \sigma_3 & 9 & \heartsuit & 9 & b \\
10 & \sigma_3 & 10 & b & 10 & a \\
11 & \sigma_3 & 11 & a & 11 & c \\
12 & \sigma_3 & 12 & c & 12 & \spadesuit \\
13 & \sigma_4 & 13 & \heartsuit & 13 & b \\
14 & \sigma_4 & 14 & b & 14 & c \\
15 & \sigma_4 & 15 & c & 15 & a \\
16 & \sigma_4 & 16 & a & 16 & \spadesuit \\
17 & \sigma_5 & 17 & \heartsuit & 17 & c \\
18 & \sigma_5 & 18 & c & 18 & a \\
19 & \sigma_5 & 19 & a & 19 & b \\
20 & \sigma_5 & 20 & b & 20 & \spadesuit \\
21 & \sigma_6 & 21 & \heartsuit & 21 & c \\
22 & \sigma_6 & 22 & c & 22 & b \\
23 & \sigma_6 & 23 & b & 23 & a \\
24 & \sigma_6 & 24 & a & 24 & \spadesuit \\
\end{array}
\]

Figure 17: The three binary ordering relations \(r_\Sigma, r_<, \) and \(r_>\)

Obviously, our strategy is to show that this ternary relation can be computed in three binary relations. Thereto, we can simply employ the representation for a Codd relation outlined in Definition 4.1. In other words, our computation should represent the relation \(\Sigma(I)\) in the three binary relations shown in Figure 17. In this figure, the numbers represent tags for the 24 tuples in \(\Sigma(I)\), the relation \(r_\Sigma\) specifies which tuples belong to which ordering, the relation \(r_<\) codes the “smaller sides” of the orderings, and the relation \(r_>\) codes the “larger sides”.

Notice that in this representation there are the two special constants \(\heartsuit\) and \(\spadesuit\). These constants are not strictly necessary. In our construction of \(\Sigma(I)\), we will in fact dispense with the two special constants and will code a tuple of the form \((\sigma_i, v, \spadesuit)\) by the tuple \((\sigma_i, v, \sigma_i)\) and a tuple of the form \((\sigma_i, \heartsuit, w)\) by the tuple \((\sigma_i, \sigma_i, w)\). Although this decision makes the construction slightly more complicated, it has the benefit of being conceptually cleaner.
Figure 18: The three binary ordering relations $r_\Sigma$, $s_<$, and $s_>$, no longer containing special constants
Before we proceed to show how to compute \( \Sigma(I) \), let us now turn to our second task and show how counter machines can be simulated in the extended Tarski algebra once \( \Sigma(I) \) has been computed.

A counter will be represented by a (binary) relation. We will assume that if the contents of this binary relation is empty, then the counter has the value “0”. Otherwise its value, say “\( j \)”, \( j > 0 \), will be represented by the relation \( \{ (\sigma_i(j), \sigma_i) \mid \sigma_i \in \Sigma(I) \} \), where \( \sigma_i(j) \) denotes the \( j \)th element in the ordering \( \sigma_i \). This representation will be derived from the binary relations representing \( \Sigma(I) \). For example, according to Figure 18, “1” is represented by the relation

\[
\{(a, \sigma_1), (a, \sigma_2), (b, \sigma_4), (b, \sigma_4), (c, \sigma_5), (c, \sigma_6)\}
\]

“2” is represented by the relation

\[
\{(b, \sigma_1), (c, \sigma_2), (a, \sigma_3), (c, \sigma_4), (a, \sigma_5), (b, \sigma_6)\}
\]

and “3” is represented by the relation

\[
\{(c, \sigma_1), (b, \sigma_2), (c, \sigma_3), (a, \sigma_4), (b, \sigma_5), (a, \sigma_6)\}
\]

How can we represent “4”? To do this, we need to expand each of the orderings \( \sigma_i \), \( 1 \leq i \leq 6 \), with one new “largest” element, using tagging. Thereto, consider the tag

\[\text{eoo}(x_\Sigma, y_\langle, y_\rangle) \equiv x_\Sigma \cap y_\rangle\]

Clearly, \( \text{eoo}(r_\Sigma, s_\langle, s_\rangle) = r_\Sigma \cap s_\rangle \) contains the largest element of each ordering (“end of ordering”). Let \( q \) be an arbitrary relation. The tags introduced in \( \text{eoo}(r_\Sigma, s_\langle, s_\rangle)^q \) and \( \text{eoo}(r_\Sigma, s_\langle, s_\rangle)_q^\circ \) can serve as new “largest” elements for each of the orderings, their novelty following from the injectivity of the tagging function \( \tau \). In order to show that the binary relations \( r_\Sigma, s_\langle \) and \( s_\rangle \) can be modified accordingly, consider the following assignment query.

**query** expand \((x_\Sigma, y_\langle, y_\rangle, u)\);

**begin**

\[
x_\Sigma := x_\Sigma \cup \text{eoo}(x_\Sigma, y_\langle, y_\rangle)^q_u;
\]

\[
y_\langle := y_\langle \cup \text{eoo}(x_\Sigma, y_\langle, y_\rangle)^q_u \cdot (\text{eoo}(x_\Sigma, y_\langle, y_\rangle)^q_u)^{-1};
\]

\[
y_\rangle := (y_\rangle - \text{eoo}(x_\Sigma, y_\langle, y_\rangle)) \cup \text{eoo}(x_\Sigma, y_\langle, y_\rangle)^q_u
\]

\[
\quad \cup \text{eoo}(x_\Sigma, y_\langle, y_\rangle) \cdot (\text{eoo}(x_\Sigma, y_\langle, y_\rangle)^q_u)^{-1};
\]

**return** \((x_\Sigma, y_\langle, y_\rangle)\)

**end**

Let \( (r_\Sigma', s_\langle', s_\rangle') = \text{expand}(r_\Sigma, s_\langle, s_\rangle, q) \). Obviously, the binary relations \( r_\Sigma', s_\langle' \) and \( s_\rangle' \), shown in Figure 19, are the correct expansions of \( r_\Sigma, s_\langle \) and \( s_\rangle \), respectively.

The number “4” is now represented by the relation

\[
\{(25, \sigma_1), (26, \sigma_2), (27, \sigma_3), (28, \sigma_4), (29, \sigma_5), (30, \sigma_6)\}
\]

i.e., by the newly introduced tag values as “largest” values of their respective orderings.

The above technique shows that we can always expand the relations \( r_\Sigma, s_\langle, \) and \( s_\rangle \) to represent an arbitrary large natural number. However, there is a small caveat. If \( I \) is the
Figure 19: An expansion of the relations $r_\Sigma$, $s_<$ and $s_>$
empty database, then we need to guarantee that the three binary relations representing $\Sigma(I)$ are all nonempty. Otherwise, it is impossible to introduce new values by tagging. This can be accommodated by assuming that there is one special constant, say "♣", and that $r_\Sigma$, $s_<$, and $s_>$ are initialized to the singleton $\{(♣, ♣)\}$.4

To prove that the extended Tarski algebra can simulate counter machines we need to show that

1. the contents of a counter can be incremented by one,
2. the contents of a counter with a strictly positive value can be decremented by one, and
3. a while-loop program, which loops until a control counter becomes zero, can be simulated.

Assume that counter $ctr-k$ has value $j$. As counters are constructed from the three binary relations $r_\Sigma$, $s_<$ and $s>$, we may assume that these relations already allow the representation of $j$. In order to increment $ctr-k$ by one, we first need to check whether or not $j$ is the largest number of which $r_\Sigma$, $s_<$ and $s>$ allow the representation. If this is the case, then we first have to expand $r_\Sigma$, $s_<$ and $s>$ before we can actually increment the counter. This results in the following extended Tarski algebra expression:

```tarski
query increment(x_\Sigma, y_<, y>, z_k, u);
begin
  if $z_k - eoo(x_\Sigma, y_<, y>) = \emptyset$
    then
        $(x_\Sigma, y_<, y>) := expand(x_\Sigma, y_<, y>, u);
        z_k := y_\geq \cdot (x_\Sigma \cap y_< \cdot z_k);
        return (x_\Sigma, y_<, y>, z_k)
  end
```

If $q$ is an arbitrary relation and if $(r'_\Sigma, s'_<, s'_>) = increment(r_\Sigma, s_<, s>, ctr-k, q)$, then the value of $ctr-k'$ is the value of $ctr-k$ incremented by one. Furthermore, we have either $(r'_\Sigma, s'_<, s'_>) = (r_\Sigma, s_<, s>)$ or $(r'_\Sigma, s'_<, s'_>) = expand(r_\Sigma, s_<, s>, q)$, depending on whether or not an expansion was necessary.

Similarly, it is possible to write a Tarski expression to decrement a counter. Finally since the extended Tarski algebra has while loops, we can also meet Condition 3 in the list of requirements to simulate counter machines.

So, all we have left to show is that it is possible to write an extended Tarski expression to compute the initial contents of $r_\Sigma$, $s<$, and $s>$, as, e.g., in Figure 18 for a database $I$ with domain $\{a, b, c\}$.

The first step in this computation is to initialize the relations $r_\Sigma$, $s<$, and $s>$ and to compute the domain $dom$. Let us assume that the input database $I$ consists of the binary

---

4This assumption requires that we slightly adjust the notion of $\tau$-genericity by defining it only relative to the permutations on $D$ which keep $♣$ fixed.
relations \((r_1, \ldots, r_m)\) and consider the assignment query:

\[
\text{query } \text{initialize}(x_1, \ldots, x_m);
\]

\[
\begin{align*}
  & \text{begin} \\
  & \quad x_{\Sigma} := \{(♣, ♣)\}; \\
  & \quad y_< := \{(♣, ♣)\}; \\
  & \quad y_> := \{(♣, ♣)\}; \\
  & \quad x_{\text{dom}} := x_1^{\text{id}} \cup \ldots \cup x_m^{\text{id}}; \\
  & \quad \text{return } (x_{\Sigma}, y_<, y_>, x_{\text{dom}}) \\
  & \text{end}
\end{align*}
\]

The initialization is realized by putting \((r_{\Sigma}, s_<, s_>, \text{dom}) = \text{initialize}(r_1, \ldots, r_m)\).

Next, one incrementally builds the (partial) orders of size 1, 2, \ldots, up to the orders of size \(|\text{dom}|\) (i.e., the complete orders). In our example, where \(\text{dom} = \{(a, a), (b, b), (c, c)\}\), one builds in the first iteration the orders of size one, i.e., \((a, a), (b, b), (c, c)\), in the second iteration the orders of size two, i.e., \((a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\), and finally, in the third iteration, the orders of size three, i.e., \((a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\). So the algorithm has the following structure:

\[
\text{query } \text{build-up}(x_1, \ldots, x_m, u_1, u_2);
\]

\[
\begin{align*}
  & \text{begin} \\
  & \quad (x_{\Sigma}, y_<, y_>, x_{\text{dom}}) := \text{initialize}(x_1, \ldots, x_m); \\
  & \quad \text{notused} := x_{\Sigma} \cdot x_{\Sigma} \cup x_{\text{dom}} \cdot x_{\text{dom}}; \\
  & \quad \text{while notused } \neq \emptyset \\
  & \quad \quad \text{do} \\
  & \quad \quad \quad \text{min} := x_{\Sigma} \cap y_<; \\
  & \quad \quad \quad \text{max} := x_{\Sigma} \cap y_>; \\
  & \quad \quad \quad \text{temp} := \text{notused}^u_1 \cdot x_{\Sigma}^{-1} \cup \text{notused}^u_1; \\
  & \quad \quad \quad (x_{\Sigma}, y_<, y_>) := \text{extend}(x_{\Sigma}, y_<, y_>, x_{\text{dom}}, \text{min}, \text{max}, \text{temp}, u_2); \\
  & \quad \quad \quad \text{used} := x_{\Sigma}^{-1} \cdot y_< \cdot x_{\text{dom}}; \\
  & \quad \quad \quad \text{notused} := \text{used} \cdot x_{\text{dom}}; \\
  & \quad \quad \text{od}; \\
  & \quad \text{return } (x_{\Sigma}, y_<, y_>) \\
  & \text{end}
\end{align*}
\]

In \text{build-up}, \text{notused} is initialized to \(\{(♣, v) \mid v \in I\}\) to indicate that no value has yet been used in any ordering. Inside the while-loop, the query \text{extend} transforms the existing partial orders to partial orders of size one larger, using the values of \text{min}, \text{max} and \text{temp}.5 To the variable \text{used}, the relation is assigned consisting of all pairs \((\sigma, v)\) with \(\sigma\) an existing partial order and \(v\) a domain value occurring in that order. Finally, \text{notused} will consist of all pairs \((\sigma, w)\) with \(\sigma\) an existing partial order and \(w\) a domain value not occurring in \(\sigma\).

What remains to be explained is how \text{extend} is written. To follow the appropriate computations, it is useful to first consider Figure 20.

In Figure 20, the triples of relations \((r_i^\Sigma, s_<^i, s_>_^i), 0 \leq i \leq 2\), are the states of the relations \(r_{\Sigma}, s_<,\) and \(s_>\) just before the \((i + 1)\)th iteration of the main while loop in \text{build-up}.

\[5\text{Observe in particular that min and max compute the “first” respectively the “last” tuple of each ordering.}\]
Figure 20: The states of the relations $r_\Sigma$, $s_<$, and $s_>$ just before the first, second and third iterations of build-up.
The first step in the extension involves the computation of the tags for the new partial orderings. Remember that the variable \textit{notused} lists for each current order, say \(\sigma\), the domain elements not used in \(\omega\). Each such element therefore determines a different extension of \(\sigma\). Therefore, obtaining tags for the extended orderings can be achieved by tagging \(\textit{notused}\).

Once the tags are obtained for the new (partial) orderings, it is possible to compute the new values for \(r_{\Sigma}, s_<\) and \(s_>\). We first consider \(r_{\Sigma}\):

\[
\text{query new-}(x_{\Sigma}, temp, u_2); \\
\text{begin} \\
x_{\Sigma} := temp^\sigma_{u_2}; \\
\text{return } x_{\Sigma} \\
\text{end}
\]

Computing the new values for \(s_<\) and \(s_>\) is more involved. In order to write down the corresponding etaes more easily, we define for a relation \(r\) the left identity operator:

\[
r^{\text{idl}} = \{(v, v) \mid \exists w \in V : (v, w) \in r\}
\]

Since \(r^{\text{idl}} = (r^{s_})^{-1} \cdot r^s\) with \(s\) an arbitrary relation, the left identity operator can be expressed in the Tarski algebra. The expression computing the new value for \(s_<\) is:

\[
\text{query new-}(x_{\Sigma}, y_<, x_{\text{dom}}, \text{min}, temp, u_2); \\
\text{begin} \\
sm\text{-first} := (\text{temp}^p_{u_2} \cdot \text{min}^{\text{idl}})^{\text{idl}} \cdot x_{\Sigma}; \\
sm\text{-last} := \text{temp}^p_{u_2} \cdot x_{\text{dom}}; \\
sm\text{-temp} := x^{\text{idl}}_{\Sigma} - (\text{sm\text{-first}} \cup \text{sm\text{-last}})^{\text{idl}}; \\
sm\text{-rest} := \text{sm\text{-temp}} \cdot \text{temp}^p_{u_2} \cdot y_<; \\
y_< := \text{sm\text{-first}} \cup \text{sm\text{-last}} \cup \text{sm\text{-rest}}; \\
\text{return } y_< \\
\text{end}
\]

The new value of \(s_<\) is obtained in three parts: \(sm\text{-first}\) contains the “first” pair of each ordering, \(sm\text{-last}\) contains the “last” pair of each ordering and \(sm\text{-rest}\) is used to compute the remaining pair. Similarly, the new value of \(s_>\) is also computed in three parts:

\[
\text{query new->}(x_{\Sigma}, y_>, x_{\text{dom}}, \text{max}, temp, u_2); \\
\text{begin} \\
gr\text{-last} := (\text{temp}^p_{u_2} \cdot x_{\text{dom}})^{\text{idl}} \cdot x_{\Sigma}; \\
gr\text{-onebutlast} := (\text{temp}^p_{u_2} \cdot \text{max}^{\text{idl}})^{\text{idl}} \cdot x_{\Sigma} \cdot \text{temp}; \\
gr\text{-temp} := x^{\text{idl}}_{\Sigma} - (gr\text{-last} \cup gr\text{-onebutlast})^{\text{idl}}; \\
gr\text{-rest} := gr\text{-temp} \cdot \text{temp}^p_{u_2} \cdot y_>; \\
y_> := gr\text{-last} \cup gr\text{-onebutlast} \cup gr\text{-rest}; \\
\text{return } y_> \\
\text{end}
\]
Using the etaes introduced above, we can write the following assignment query for expand:

```
query extend(x_Σ, y_<, y_>, x_dom, min, max, temp, u_2);
begin
  x_Σ := new-Σ(x_Σ, temp, u_2);
  y_< := new-<(x_Σ, y_<, min, temp, u_2);
  y_> := new->(x_Σ, y_>, max, temp, u_2);
return (x_Σ, y_<, y_>)
end
```

which concludes this sketch of the completeness proof.

We want to end this section with two final remarks concerning the proof above. Firstly, it should be mentioned that only the contexts explicitly used in the etaes above were mentioned in the parameter lists of these etaes. The use of the identity and left identity operators implicitly involves contexts too. Secondly, all contexts—whether explicit or implicit—can be seen as parameters of the final etae $E_Q$ simulating $Q$ and must not be computed from other data. As a consequence, each of three tagging strategies discussed in Subsection 3.3 (tuple tagging, relation tagging and universal tagging) already suffices to guarantee completeness.

### 6 Directions for future research

We conclude this article with a (non-exhaustive) list of topics not addressed in this paper, but clearly relevant for tagged-based database models in general and for the Tarski database model in particular. Specifically, we will discuss:

- **tagging of complex objects** other than ordered pairs;
- **constraints** such as functional dependencies in the Tarski database model; and
- **implementation considerations** for the Tarski database model and the Tarski algebra.

Turning to the first item, it should be emphasized that in this article, we limited ourselves (deliberately) to the use of tagging functions which associate tag values to ordered pairs. In Subsection 4.2 we showed that, even within this limited setting, it is possible to interpret certain tags as representations for finite sets. We also indicated, however, that, unlike for ordered pairs, we cannot guarantee that a finite set of objects is represented by a unique tag. Thus we are confronted with the problem of multiple tags (copies) for the same complex object. A simple way to overcome this problem is to admit tagging functions defined over domains of complex objects other than ordered pairs. We could for example consider a tagging function associating a unique tag to each finite set of values (i.e., data values or tag values). Such a tagging function would allow the definition of finite-set tagging operators in addition to the (ordered-pair) tagging operators already considered in this paper. This approach would of course overcome the multiple-copies problem and would yield a cleaner language to simulate the nested relational algebra. It should however be emphasized that we would still remain within the realm of binary (flat) relations.
An important database design tool for relational database we would like to discuss here in the context of tagging is the theory of constraints. The best-known examples of such constraints for Codd-relational databases are functional dependencies. Clearly, it is reasonable to assume that such constraints play as pivotal a role in Tarski-relational databases. The fact we wish to elicit here is that it is possible to use Tarski algebra expressions to formulate such constraints, as was already pointed out by Tarski himself in his original paper on the calculus of (binary) relations [28].

The underlying idea is quite simple. Suppose that we have a binary relation \( f \). In general \( f \) will not define a function. However, if we postulate that \( f \) satisfy the constraint

\[
 f^{-1} \cdot f \subseteq f^{id}
\]

then \( f \) clearly defines a function. This observation is the key to the problem of formulating (general) functional dependencies.

Of course, other types of constraints can equally well be expressed in this way. Suppose for example that we wish to state that \( r \) is a transitively closed binary relation. This can be formulated with the constraint \( r \cdot r \subseteq r \).

We wish to conclude this article with some observations concerning the (possible) implementation of the Tarski database model and the Tarski algebra. As it turns out, various researchers have proposed storage architectures to facilitate such implementations. We are thinking in particular about the so called decomposed storage model [12, 21]. In the decomposed storage model, the database is internally stored as a collection of binary relations. So-called surrogates (our tags) are used to conceptually connect arbitrary relations. Thus, in a strong sense, the data is highly partitioned.

As discussed convincingly in [12], however, this storage model offers many advantages. The most striking characteristic is the simplicity and uniformity of the data structures needed to store and access data. This leads in turn to simpler access data methods. In particular, the set of access operations is small and straightforward to implement. On the other hand, due to the fragmentation of the data, access methods typically translate in complex sequences of access operations. The developers of the decomposed storage model justifiably make the analogy to RISC technology for computer hardware. It is in such a sense that we think Tarski algebra expressions can best be viewed. Although each individual operation in such an expression is simple, the complexity resides in the entire expression.

The authors of [21] also hint at the potential to support the decomposed storage structure on parallel machines. In summary, there appears evidence in the literature of database systems that the Tarski database model can be effectively and efficiently supported. In an age wherein it is believed that the majority of improvements in performance will come through parallel computation, the Tarski database model presents itself as a serious competitor to the relational database model as a conceptual model to improve the performance of general purpose database management systems.

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